hep-th/0606244
TIFR/TH/06-15
HRI-P-06-06-002

Rotating Attractors

Dumitru Astefanesei[a], Kevin Goldstein[b], Rudra P. Jena[b],
Ashoke Sen[a] and Sandip P. Trivedi[b]

[a] *Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad 211019, India*

and

[b] *Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India*

Abstract

We prove that, in a general higher derivative theory of gravity coupled to abelian gauge fields and neutral scalar fields, the entropy and the near horizon background of a rotating extremal black hole is obtained by extremizing an entropy function which depends only on the parameters labeling the near horizon background and the electric and magnetic charges and angular momentum carried by the black hole. If the entropy function has a unique extremum then this extremum must be independent of the asymptotic values of the moduli scalar fields and the solution exhibits attractor behaviour. If the entropy function has flat directions then the near horizon background is not uniquely determined by the extremization equations and could depend on the asymptotic data on the moduli fields, but the value of the entropy is still independent of this asymptotic data. We illustrate these results in the context of two derivative theories of gravity in several examples. These include Kerr black hole, Kerr-Newman black hole, black holes in Kaluza-Klein theory, and black holes in toroidally compactified heterotic string theory.

Contents

1 Introduction and Summary

The attractor mechanism has played an important role in recent studies of black holes in string theory [1, 2, 3]. According to this the geometry and other field configurations of an extremal black hole near its horizon is to a large extent insensitive to the asymptotic data on the scalar fields of the theory. More precisely, if the theory contains a set of massless scalars with flat potential — known as the moduli fields — then the black hole entropy and often the near horizon field configuration is independent of the asymptotic values of these scalar fields.

Although initial studies of the attractor mechanism were carried out in the context of spherically symmetric supersymmetric extremal black holes in supergravity theories in 3+1 dimensions with two derivative action, by now it has been generalized to many other cases. These examples include non-supersymmetric theories, actions with higher derivative corrections, extremal black holes in higher dimensions etc.[4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. In particular it has been shown that in an arbitrary theory of gravity coupled to abelian gauge fields, neutral scalar fields and p-form gauge fields with a gauge and general coordinate invariant local Lagrangian density, the entropy of a spherically symmetric extremal black hole remains invariant under continuous deformation of the asymptotic data for the moduli fields [30, 31], although occasional discrete jumps are not ruled out. In a generic situation the complete near horizon background is independent of this asymptotic data and depends only on the charges carried by the black hole, but in special cases (which happen to be quite generic in supersymmetric string theories) there may be some dependence of the near horizon background on this asymptotic data.

Most of the studies on the attractor mechanism however have been carried out in the context of spherically symmetric black holes — for some exceptions see [40, 41, 34, 42]. The goal of this paper is to remedy this situation and generalize the study of the attractor mechanism to rotating black hole solutions. Our starting point is an observation made in [43] that the near horizon geometries of extremal Kerr and Kerr-Newman black holes have SO(2,1)×U(1) isometry. Armed with this observation we prove a general result that is as powerful as its non-rotating counterpart. In the context of 3+1 dimensional theories, our analysis shows that *in an arbitrary theory of gravity coupled to abelian gauge fields and neutral scalar fields with a gauge and general coordinate invariant local Lagrangian density, the entropy of a rotating extremal black hole remains invariant, except for occasional jumps, under continuous deformation of the asymptotic data for the moduli fields if an extremal black hole is defined to be the one whose near horizon field configuration has* $SO(2,1)\times U(1)$ *isometry.* In a generic situation the complete near horizon background is independent of this asymptotic data and depends only on the charges carried by the black hole, but in special cases there may be some dependence of the near horizon background on this asymptotic data.

The strategy for obtaining this result, elaborated in detail in section 2, is to use the entropy function formalism [30, 31]. As in the case of non-rotating black holes we find that the near horizon background of a rotating extremal black hole is obtained by extremizing a functional of the background fields on the horizon, and that Wald's entropy [44, 45, 46, 47] is given by precisely the same functional evaluated at its extremum. Thus if this functional has a unique extremum with no flat directions then the near horizon field configuration is determined completely in terms of the charges and angular momentum, with no possibility of any dependence on the asymptotic data on the moduli fields. On the other hand if the functional has flat directions so that the extremization equations do not determine the near horizon background completely, then there can be some dependence of this background on the asymptotic data, but the entropy, being equal to the value of the functional at the extremum, is still independent of this data. Finally, if the functional has several extrema at which it takes different values, then for different ranges of asymptotic values of the moduli fields the near horizon geometry could correspond to different extrema. In this case as we move in the space of asymptotic data the entropy would change discontinuously as we cross the boundary between two different domains of attraction, although within a given domain it stays fixed. As in the case of non-rotating black holes, these results are valid given the existence of a black hole solution with SO(2,1)$\times$U(1) symmetric near horizon geometry, but our analysis by itself does not tell us whether a solution of this form exists. For this, one needs to carry out a more detailed analysis of the full solution along the lines of [4].

Although in this paper we focus our attention on four dimensional rotating black holes with horizons of spherical topology, the strategy outlined above is valid for extremal black holes in any dimension with horizon of any compact topology, provided we define an extremal black hole to be the one whose near horizon geometry has an SO(2,1) isometry. The analysis is also valid for extremal black holes in asymptotically anti de-Sitter space as long as Wald's formula for black hole entropy continues to hold. In particular the proof that the entropy of an extremal rotating black hole in any higher derivative theory of gravity does not change, except for occasional jumps, under continuous variation of the asymptotic values of the moduli fields is valid in this general context. All that changes is that when we try to explicitly solve the differential equations which arise out of the extremization conditions, we need to use boundary conditions which are appropriate to the horizon of a given topology. Equivalently if we carry out the analysis by expanding various functions describing the near horizon background in a complete set of basis functions, then we must use basis functions which are appropriate to that given topology. We should note however that as we vary the asymptotic values of the moduli fields, we must hold fixed all the conserved charges appropriate to the particular near horizon geometry of the black hole. This point has been elaborated further in footnote 2.

In section 3 we explore this formalism in detail in the context of an arbitrary two derivative theory of gravity coupled to scalar and abelian vector fields. The extremization conditions now reduce to a set of second order differential equations with parameters and boundary conditions which depend only on the charges and the angular momentum. Thus the only ambiguity in the solution to these differential equations arise from undetermined integration constants. We prove explicitly that in a generic situation all the integration constants are fixed once we impose the appropriate boundary conditions and smoothness requirement on the solutions. We also show that even in a non-generic situation where some of the integration constants are not fixed (and hence could depend on the asymptotic data on the moduli fields), the value of the entropy is independent of these undetermined integration constants.

In section 4 we specialize even further to a class of black holes for which all the scalar fields are constant on the horizon. This, of course, happens automatically in theories without any scalar fields, but also happens for purely electrically charged black holes in theories without any $F\widetilde{F}$ type coupling in the Lagrangian density. In this case we can solve all the differential equations explicitly and determine the near horizon background completely, with the constant values of the scalar fields being determined by extremizing an effective potential — the same potential that appears in the determination of the attractor values in the case of non-rotating black holes [4]. We use these general results to compute the entropy and near horizon geometry of extremal Kerr as well as extremal Kerr-Newman black holes, and reproduce the known results in these cases.

In section 5 we use a different strategy for testing our general results. Here we take some of the known extremal rotating black hole solutions in two derivative theories of gravity coupled to matter, and study their near horizon geometry to determine if they exhibit attractor behaviour. We focus on two particular classes of examples — the Kaluza-Klein black holes studied in [48, 49, 50] and black holes in toroidally compactified heterotic string theory studied in [51] (see, also, [52] for a restricted class of such black holes).[1] In both these examples, we find two kinds of extremal limits. One of these branches,

[1]Both types of black holes are special cases of general black hole solutions in toroidally compactified heterotic string theory and, as we show, various formulæ involving entropy and near horizon metric can be regarded as special cases of general duality invariant formulæ for these quantities.

corresponding to the surface W in [48], does not have an ergo-sphere and can exist only for angular momentum of magnitude less than a certain upper bound. We call this the ergo-free branch. The other branch, corresponding to the surface S in [48], does have an ergo-sphere and can exist for angular momentum of magnitude larger than a certain lower bound. We call this the ergo-branch. On both branches the entropy turns out to be independent of the asymptotic values of the moduli fields, in accordance with our general arguments. We find however that while on the ergo-free branch the scalar and all other background fields at the horizon are independent of the asymptotic data on the moduli fields, this is not the case for the ergo-branch. Thus on the ergo-free branch we have the full attractor behaviour, whereas on the ergo-branch only the entropy is attracted to a fixed value independent of the asymptotic data. On general grounds we expect that once higher derivative corrections originating at tree, loop, and non-perturbative level are taken into account these flat directions of the entropy function will be lifted and we shall get a unique near horizon background even on the ergo-branch.

2 General Analysis

We begin by considering a general four dimensional theory of gravity coupled to a set of abelian gauge fields $A_\mu^{(i)}$ and neutral scalar fields $\{\phi_s\}$ with action

$$\mathcal{S} = \int d^4x\, \sqrt{-\det g}\, \mathcal{L}\,, \tag{2.1}$$

where $\sqrt{-\det g}\,\mathcal{L}$ is the lagrangian density, expressed as a function of the metric $g_{\mu\nu}$, the scalar fields $\{\Phi_s\}$, the gauge field strengths $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$, and covariant derivatives of these fields. In general $\mathcal{L}$ will contain terms with more than two derivatives. We consider a rotating extremal black hole solution whose near horizon geometry has the symmetries of $AdS_2 \times S^1$. The most general field configuration consistent with the $SO(2,1)\times U(1)$ symmetry of $AdS_2 \times S^1$ is of the form:

$$\begin{aligned}
& ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = v_1(\theta)\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \beta^2\, d\theta^2 + \beta^2\, v_2(\theta)(d\phi - \alpha r dt)^2 \\
& \Phi_s = u_s(\theta) \\
& \frac{1}{2} F^{(i)}_{\mu\nu} dx^\mu \wedge dx^\nu = (e_i - \alpha b_i(\theta)) dr\wedge dt + \partial_\theta b_i(\theta) d\theta \wedge (d\phi - \alpha r dt)\,,
\end{aligned} \tag{2.2}$$

where α, β and e_i are constants, and v_1, v_2, u_s and b_i are functions of θ. Here ϕ is a periodic coordinate with period 2π and θ takes value in the range $0 \le \theta \le \pi$. The SO(2,1) isometry of AdS_2 is generated by the Killing vectors[43]:

$$L_1 = \partial_t, \qquad L_0 = t\partial_t - r\partial_r, \qquad L_{-1} = (1/2)(1/r^2 + t^2)\partial_t - (tr)\partial_r + (\alpha/r)\partial_\phi \,. \tag{2.3}$$

The form of the metric given in (2.2) implies that the black hole has zero temperature.

We shall assume that the deformed horizon, labelled by the coordinates θ and ϕ, is a smooth deformation of the sphere.[2] This requires

$$\begin{aligned} v_2(\theta) &= \theta^2 + \mathcal{O}(\theta^4) \quad \text{for } \theta \simeq 0 \\ &= (\pi-\theta)^2 + \mathcal{O}((\pi-\theta)^4) \quad \text{for } \theta \simeq \pi \,. \end{aligned} \tag{2.4}$$

For the configuration given in (2.2) the magnetic charge associated with the ith gauge field is given by

$$p_i = \int d\theta d\phi F^{(i)}_{\theta\phi} = 2\pi(b_i(\pi) - b_i(0)) \,. \tag{2.5}$$

Since an additive constant in b_i can be absorbed into the parameters e_i, we can set $b_i(0) = -p_i/4\pi$. This, together with (2.5), now gives

$$b_i(0) = -\frac{p_i}{4\pi}, \qquad b_i(\pi) = \frac{p_i}{4\pi} \,. \tag{2.6}$$

Requiring that the gauge field strength is smooth at the north and the south poles we get

$$\begin{aligned} b_i(\theta) &= -\frac{p_i}{4\pi} + \mathcal{O}(\theta^2) \quad \text{for } \theta \simeq 0 \\ &= \frac{p_i}{4\pi} + \mathcal{O}((\pi-\theta)^2) \quad \text{for } \theta \simeq \pi \,. \end{aligned} \tag{2.7}$$

Finally requiring that the near horizon scalar fields are smooth at the poles gives

$$\begin{aligned} u_s(\theta) &= u_s(0) + \mathcal{O}(\theta^2) \quad \text{for } \theta \simeq 0 \\ &= u_s(\pi) + \mathcal{O}((\pi-\theta)^2) \quad \text{for } \theta \simeq \pi \,. \end{aligned} \tag{2.8}$$

[2]Although in two derivative theories the horizon of a four dimensional black hole is known to have spherical topology, once higher derivative terms are added to the action there may be other possibilities. Our analysis can be easily generalized to the case where the horizon has the topology of a torus rather than a sphere. All we need is to take the θ coordinate to be a periodic variable with period 2π and expand the various functions in the basis of periodic functions of θ. However if the near horizon geometry is invariant under both ϕ and θ translations, then in the expression for L_{-1} given in (2.3) we could add a term of the form $-(\gamma/r)\partial_\theta$, and the entropy could have an additional dependence on the charge conjugate to the variable γ. This represents the Noether charge associated with θ translation, but does not correspond to a physical charge from the point of view of the asymptotic observer since the full solution is not invariant under θ translation.

Note that the smoothness of the background requires the Taylor series expansion around $\theta = 0, \pi$ to contain only even powers of θ and $(\pi - \theta)$ respectively.

A simple way to see the $SO(2,1) \times U(1)$ symmetry of the configuration (2.2) is as follows. The $U(1)$ transformation acts as a translation of ϕ and is clearly a symmetry of this configuration. In order to see the SO(2,1) symmetry of this background we regard ϕ as a compact direction and interprete this as a theory in three dimensions labelled by coordinates $\{x^m\} \equiv (r, \theta, t)$ with metric $\hat{g}_{mn}$, vectors $a^{(i)}_m$ and a_m (coming from the ϕ-m component of the metric) and scalar fields Φ_s, $\psi \equiv g_{\phi\phi}$ and $\chi_i \equiv A^{(i)}_\phi$. If we denote by $f^{(i)}_{mn}$ and f_{mn} the field strengths associated with the three dimensional gauge fields $a^{(i)}_m$ and a_m respectively, then the background (2.2) can be interpreted as the following three dimensional background:

$$\begin{aligned} &\widehat{ds}^2 \equiv \hat{g}_{mn} dx^m dx^n = v_1(\theta)\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + \beta^2\, d\theta^2 \\ &\Phi_s = u_s(\theta), \qquad \psi = \beta^2\, v_2(\theta), \quad \chi_i = b_i(\theta)\,, \\ &\frac{1}{2} f^{(i)}_{mn} dx^m \wedge dx^n = e_i\, dr \wedge dt, \qquad \frac{1}{2} f_{mn} dx^m \wedge dx^n = -\alpha dr \wedge dt\,. \end{aligned} \tag{2.9}$$

The (r, t) coordinates now describe an AdS_2 space and this background is manifestly $SO(2,1)$ invariant. In this description the Killing vectors take the standard form

$$L_1 = \partial_t, \qquad L_0 = t\partial_t - r\partial_r, \qquad L_{-1} = (1/2)(1/r^2 + t^2)\partial_t - (tr)\partial_r\,. \tag{2.10}$$

Eq.(2.9) and hence (2.2) describes the most general field configuration consistent with the $SO(2,1) \times U(1)$ symmetry. Thus in order to derive the equations of motion we can evaluate the action on this background and then extremize the resulting expression with respect to the parameters labelling the background (2.2). The only exception to this are the parameters e_i and α labelling the field strengths. The variation of the action with respect to these parameters do not vanish, but give the corresponding conserved electric charges q_i and the angular momentum J (which can be regarded as the electric charge associated with the three dimensional gauge field a_m.)

To implement this procedure we define:

$$f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] = \int d\theta d\phi \sqrt{-\det g}\, \mathcal{L}\,. \tag{2.11}$$

Note that f is a function of α, β, e_i and a functional of $v_1(\theta)$, $v_2(\theta)$, $u_s(\theta)$ and $b_i(\theta)$. The equations of motion now correspond to[3]

$$\frac{\partial f}{\partial \alpha} = J, \quad \frac{\partial f}{\partial \beta} = 0, \quad \frac{\partial f}{\partial e_i} = q_i \,, \quad \frac{\delta f}{\delta v_1(\theta)} = 0 \,, \quad \frac{\delta f}{\delta v_2(\theta)} = 0, \quad \frac{\delta f}{\delta u_s(\theta)} = 0, \quad \frac{\delta f}{\delta b_i(\theta)} = 0 \,. \tag{2.12}$$

Equivalently, if we define:

$$\mathcal{E}[J, \vec{q}, \alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] = 2\pi \left(J\alpha + \vec{q}\cdot\vec{e} - f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] \right) \,, \tag{2.13}$$

then the equations of motion take the form:

$$\frac{\partial \mathcal{E}}{\partial \alpha} = 0, \quad \frac{\partial \mathcal{E}}{\partial \beta} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0 \,, \quad \frac{\delta \mathcal{E}}{\delta v_1(\theta)} = 0 \,, \quad \frac{\delta \mathcal{E}}{\delta v_2(\theta)} = 0, \quad \frac{\delta \mathcal{E}}{\delta u_s(\theta)} = 0, \quad \frac{\delta \mathcal{E}}{\delta b_i(\theta)} = 0 \,. \tag{2.14}$$

These equations are subject to the boundary conditions (2.4), (2.7), (2.8). For formal arguments it will be useful to express the various functions of θ appearing here by expanding them as a linear combination of appropriate basis states which make the constraints (2.4), (2.7) manifest, and then varying $\mathcal{E}$ with respect to the coefficients appearing in this expansion. The natural functions in terms of which we can expand an arbitrary ϕ-independent function on a sphere are the Legendre polynomials $P_l(\cos\theta)$. We take

$$\begin{aligned} v_1(\theta) &= \sum_{l=0}^{\infty} \widetilde{v}_1(l)\, P_l(\cos\theta) \,, \quad v_2(\theta) = \sin^2\theta + \sin^4\theta \sum_{l=0}^{\infty} \widetilde{v}_2(l)\, P_l(\cos\theta) \,, \\ u_s(\theta) &= \sum_{l=0}^{\infty} \widetilde{u}_s(l)\, P_l(\cos\theta) \,, \quad b_i(\theta) = -\frac{p_i}{4\pi}\cos\theta + \sin^2\theta \sum_{l=0}^{\infty} \widetilde{b}_i(l)\, P_l(\cos\theta) \,. \end{aligned} \tag{2.15}$$

This expansion explicitly implements the constraints (2.4), (2.7) and (2.8). Substituting this into (2.13) gives $\mathcal{E}$ as a function of J, q_i, α, β, e_i, $\widetilde{v}_1(l)$, $\widetilde{v}_2(l)$, $\widetilde{u}_s(l)$ and $\widetilde{b}_i(l)$. Thus the equations (2.14) may now be reexpressed as

$$\frac{\partial \mathcal{E}}{\partial \alpha} = 0, \quad \frac{\partial \mathcal{E}}{\partial \beta} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0 \,, \quad \frac{\partial \mathcal{E}}{\partial \widetilde{v}_1(l)} = 0 \,, \quad \frac{\partial \mathcal{E}}{\partial \widetilde{v}_2(l)} = 0, \quad \frac{\partial \mathcal{E}}{\partial \widetilde{u}_s(l)} = 0, \quad \frac{\partial \mathcal{E}}{\partial \widetilde{b}_i(l)} = 0 \,. \tag{2.16}$$

Let us now turn to the analysis of the entropy associated with this black hole. For this it will be most convenient to regard this configuration as a two dimensional extremal black hole by regarding the θ and ϕ directions as compact. In this interpretation the

[3]Our definition of the angular momentum differs from the standard one by a $-$ sign.

zero mode of the metric $\widehat{g}_{\alpha\beta}$ given in (2.9), with $\alpha, \beta = r, t$, is interpreted as the two dimensional metric $h_{\alpha\beta}$:

$$h_{\alpha\beta} = \frac{1}{2}\int_0^\pi d\theta \, \sin\theta \, \widehat{g}_{\alpha\beta} \, , \tag{2.17}$$

whereas all the non-zero modes of $\widehat{g}_{\alpha\beta}$ are interpreted as massive symmetric rank two tensor fields. This gives

$$h_{\alpha\beta} dx^\alpha dx^\beta = v_1(-r^2 dt^2 + dr^2/r^2) \, , \qquad v_1 = \widetilde{v}_1(0) \, . \tag{2.18}$$

Thus the near horizon configuration, regarded from two dimensions, involves AdS_2 metric, accompanied by background electric fields $f^{(i)}_{\alpha\beta}$ and $f_{\alpha\beta}$, a set of massless and massive scalar fields originating from the fields $u_s(\theta)$, $v_2(\theta)$ and $b_i(\theta)$, and a set of massive symmetric rank two tensor fields originating from $v_1(\theta)$. According to the general results derived in [44, 45, 46, 47], the entropy of this black hole is given by:

$$S_{BH} = -8\pi \, \frac{\delta \mathcal{S}^{(2)}}{\delta R^{(2)}_{rtrt}} \sqrt{-h_{rr} \, h_{tt}} \, , \tag{2.19}$$

where $R^{(2)}_{\alpha\beta\gamma\delta}$ is the two dimensional Riemann tensor associated with the metric $h_{\alpha\beta}$, and $\mathcal{S}^{(2)}$ is the general coordinate invariant action of this two dimensional field theory. In taking the functional derivative with respect to $R_{\alpha\beta\gamma\delta}$ in (2.19) we need to express all multiple covariant derivatives in terms of symmetrized covariant derivatives and the Riemann tensor, and then regard the components of the Riemann tensor as independent variables.

We now note that for this two dimensional configuration that we have, the electric field strengths $f^{(i)}_{\alpha\beta}$ and $f_{\alpha\beta}$ are proportional to the volume form on AdS_2, the scalar fields are constants and the tensor fields are proportional to the AdS_2 metric. Thus the covariant derivatives of all gauge and generally covariant tensors which one can construct out of these two dimensional fields vanish. In this case (2.19) simplifies to:

$$S_{BH} = -8\pi \sqrt{-\det h} \, \frac{\partial \mathcal{L}^{(2)}}{\partial R^{(2)}_{rtrt}} \sqrt{-h_{rr} \, h_{tt}} \tag{2.20}$$

where $\sqrt{-\det h} \, \mathcal{L}^{(2)}$ is the two dimensional Lagrangian density, related to the four dimensional Lagrangian density via the formula:

$$\sqrt{-\det h} \, \mathcal{L}^{(2)} = \int d\theta d\phi \sqrt{-\det g} \, \mathcal{L} \, . \tag{2.21}$$

Also while computing (2.20) we set to zero all terms in $\mathcal{L}^{(2)}$ which involve covariant derivatives of the Riemann tensor and other gauge and general coordinate covariant combinations of fields.

We can now proceed in a manner identical to that in [30] to show that the right hand side of (2.20) is the entropy function at its extremum. First of all from (2.18) it follows that

$$R^{(2)}_{rtrt} = v_1 = \sqrt{-h_{rr}h_{tt}}\,. \tag{2.22}$$

Using this we can express (2.20) as

$$S_{BH} = -8\pi\,\sqrt{-\det h}\,\frac{\partial\mathcal{L}^{(2)}}{\partial R^{(2)}_{rtrt}} R^{(2)}_{rtrt}\,. \tag{2.23}$$

Let us denote by $\mathcal{L}^{(2)}_\lambda$ a deformation of $\mathcal{L}^{(2)}$ in which we replace all factors of $R^{(2)}_{\alpha\beta\gamma\delta}$ for $\alpha,\beta,\gamma,\delta = r,t$ by $\lambda R^{(2)}_{\alpha\beta\gamma\delta}$, and define

$$f^{(2)}_\lambda \equiv \sqrt{-\det h}\,\mathcal{L}^{(2)}_\lambda\,, \tag{2.24}$$

evaluated on the near horizon geometry. Then

$$\lambda\frac{\partial f^{(2)}_\lambda}{\partial\lambda} = \sqrt{-\det h}\,R^{(2)}_{\alpha\beta\gamma\delta}\frac{\partial\mathcal{L}^{(2)}}{\delta R^{(2)}_{\alpha\beta\gamma\delta}} = 4\,\sqrt{-\det h}\,R^{(2)}_{rtrt}\frac{\partial\mathcal{L}^{(2)}}{\partial R^{(2)}_{rtrt}}\,. \tag{2.25}$$

Using this (2.23) may be rewritten as

$$S_{BH} = -2\pi\lambda\frac{\partial f^{(2)}_\lambda}{\partial\lambda}\bigg|_{\lambda=1}\,. \tag{2.26}$$

Let us now consider the effect of the scaling

$$\lambda\to s\lambda, \quad e_i\to se_i, \quad \alpha\to s\alpha, \quad \widetilde{v}_1(l)\to s\widetilde{v}_1(l) \quad \text{for} \quad 0\le l<\infty\,, \tag{2.27}$$

under which $\lambda\, R^{(2)}_{\alpha\beta\gamma\delta}\to s^2\,\lambda\, R^{(2)}_{\alpha\beta\gamma\delta}$. Now since $\mathcal{L}^{(2)}$ does not involve any explicit covariant derivatives, all indices of $h^{\alpha\beta}$ must contract with the indices in $f^{(i)}_{\alpha\beta}$, $f_{\alpha\beta}$, $R^{(2)}_{\alpha\beta\gamma\delta}$ or the indices of the rank two symmetric tensor fields whose near horizon values are given by the parameters $\widetilde{v}_1(l)$. From this and the definition of the parameters e_i, $\widetilde{v}_1(l)$, and α it follows that $\mathcal{L}^{(2)}_\lambda$ remains invariant under this scaling, and hence $f^{(2)}_\lambda$ transforms to $sf^{(2)}_\lambda$, with the overall factor of s coming from the $\sqrt{-\det h}$ factor in the definition of $f^{(2)}_\lambda$. Thus we have:

$$\lambda\frac{\partial f^{(2)}_\lambda}{\partial\lambda} + e_i\frac{\partial f^{(2)}_\lambda}{\partial e_i} + \alpha\frac{\partial f^{(2)}_\lambda}{\partial\alpha} + \sum_{l=0}^{\infty}\widetilde{v}_1(l)\frac{\partial f^{(2)}_\lambda}{\partial\widetilde{v}_1(l)} = f^{(2)}_\lambda\,. \tag{2.28}$$

Now it follows from (2.11), (2.21) and (2.24) that

$$f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] = f^{(2)}_{\lambda=1} \,. \tag{2.29}$$

Thus the extremization equations (2.12) implies that

$$\frac{\partial f^{(2)}_\lambda}{\partial e_i} = q_i, \quad \frac{\partial f^{(2)}_\lambda}{\partial \alpha} = J, \quad \frac{\partial f^{(2)}_\lambda}{\partial \widetilde{v}_1(l)} = 0 \,, \qquad \text{at } \lambda = 1 \,. \tag{2.30}$$

Hence setting $\lambda = 1$ in (2.28) we get

$$\lambda \frac{\partial f^{(2)}_\lambda}{\partial \lambda}\bigg|_{\lambda=1} = -e_i q_i - J\alpha + f^{(2)}_{\lambda=1} = -e_i q_i - J\alpha + f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] \,. \tag{2.31}$$

Eqs.(2.26) and the definition (2.13) of the entropy function now gives

$$S_{BH} = \mathcal{E} \tag{2.32}$$

at its extremum.

Using the fact that the black hole entropy is equal to the value of the entropy function at its extremum, we can derive some useful results following the analysis of [30, 31]. If the entropy function has a unique extremum with no flat directions then the extremization equations (2.16) determine the near horizon field configuration completely and the entropy as well as the near horizon field configuration is independent of the asymptotic moduli since the entropy function depends only on the near horizon quantities. On the other hand if the entropy function has flat directions then the extremization equations do not determine all the near horizon parameters, and these undetermined parameters could depend on the asymptotic values of the moduli fields. However even in this case the entropy, being independent of the flat directions, will be independent of the asymptotic values of the moduli fields.

Although expanding various θ-dependent functions in the basis of Legendre polynomials is useful for general argument leading to attractor behaviour, for practical computation it is often more convenient to directly solve the differential equation in θ. For this we shall need to carefully take into account the effect of the boundary terms. We shall see this while studying explicit examples.

3 Extremal Rotating Black Hole in General Two Derivative Theory

We now consider a four dimensional theory of gravity coupled to a set of scalar fields $\{\Phi_s\}$ and gauge fields $A_\mu^{(i)}$ with a general two derivative action of the form:[4]

$$\mathcal{S} = \int d^4x\, \sqrt{-\det g}\, \mathcal{L}\,, \tag{3.1}$$

$$\mathcal{L} = R - h_{rs}(\vec{\Phi}) g^{\mu\nu} \partial_\mu \Phi_s \partial_\nu \Phi_r - f_{ij}(\vec{\Phi}) g^{\mu\rho} g^{\nu\sigma} F^{(i)}_{\mu\nu} F^{(j)}_{\rho\sigma} - \frac{1}{2} \widetilde{f}_{ij}(\vec{\Phi})\, (\sqrt{-\det g})^{-1} \epsilon^{\mu\nu\rho\sigma} F^{(i)}_{\mu\nu} F^{(j)}_{\rho\sigma}\,, \tag{3.2}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally anti-symmetric symbol with $\epsilon^{tr\theta\phi} = 1$ and h_{rs}, f_{ij} and $\widetilde{f}_{ij}$ are fixed functions of the scalar fields $\{\Phi_s\}$. We use the following ansatz for the near horizon configuration of the scalar and gauge fields[5]

$$\begin{aligned}
& ds^2 = \Omega(\theta)^2 e^{2\psi(\theta)} (-r^2 dt^2 + dr^2/r^2 + \beta^2 d\theta^2) + e^{-2\psi(\theta)} (d\phi - \alpha r dt)^2 \\
& \Phi_s = u_s(\theta) \\
& \frac{1}{2} F^{(i)}_{\mu\nu} dx^\mu \wedge dx^\nu = (e_i - \alpha b_i(\theta)) dr \wedge dt + \partial_\theta b_i(\theta) d\theta \wedge (d\phi - \alpha r dt)\,,
\end{aligned} \tag{3.3}$$

with $0 \le \phi < 2\pi$, $0 \le \theta \le \pi$. Regularity at $\theta = 0$ and $\theta = \pi$ requires that

$$\Omega(\theta) e^{\psi(\theta)} \to \text{constant as } \theta \to 0, \pi\,, \tag{3.4}$$

and

$$\beta \Omega(\theta) e^{2\psi(\theta)} \sin\theta \to 1 \qquad \text{as } \theta \to 0, \pi\,. \tag{3.5}$$

This gives

$$\begin{aligned}
& \Omega(\theta) \to a_0 \sin\theta, \quad e^{\psi(\theta)} \to \frac{1}{\sqrt{\beta a_0} \sin\theta}, \quad \text{as } \theta \to 0, \\
& \Omega(\theta) \to a_\pi \sin\theta, \quad e^{\psi(\theta)} \to \frac{1}{\sqrt{\beta a_\pi} \sin\theta}, \quad \text{as } \theta \to \pi\,,
\end{aligned} \tag{3.6}$$

where a_0 and a_π are arbitrary constants. In the next two sections we shall describe examples of rotating extremal black holes in various two derivative theories of gravity with near horizon geometry of the form described above. However none of these black

[4]In the rest of the paper we shall be using the normalization of the Einstein-Hilbert term as given in eq.(3.2). This corresponds to choosing the Newton's constant G_N to be $1/16\pi$.

[5]This is related to the ansatz (2.2) by a reparametrization of the θ coordinate.

holes will be supersymmetric even though many of them will be found in supersymmetric theories.

Using (3.2), (3.3) and (3.5) we get

$$
\begin{aligned}
\mathcal{E} &\equiv 2\pi(J\alpha + \vec{q}\cdot\vec{e} - \int d\theta d\phi \sqrt{-\det g}\,\mathcal{L}) \\
&= 2\pi J\alpha + 2\pi\vec{q}\cdot\vec{e} - 4\pi^2 \int d\theta \Big[2\Omega(\theta)^{-1}\beta^{-1}(\Omega'(\theta))^2 - 2\Omega(\theta)\beta - 2\Omega(\theta)\beta^{-1}(\psi'(\theta))^2 \\
&\quad + \frac{1}{2}\alpha^2\Omega(\theta)^{-1}\beta e^{-4\psi(\theta)} - \beta^{-1}\Omega(\theta)h_{rs}(\vec{u}(\theta))u_r'(\theta)u_s'(\theta) + 4\widetilde{f}_{ij}(\vec{u}(\theta))(e_i - \alpha b_i(\theta))b_j'(\theta) \\
&\quad + 2f_{ij}(\vec{u}(\theta))\left\{\beta\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_i - \alpha b_i(\theta))(e_j - \alpha b_j(\theta)) - \beta^{-1}\Omega(\theta)e^{2\psi(\theta)}b_i'(\theta)b_j'(\theta)\right\}\Big] \\
&\quad + 8\pi^2\left[\Omega(\theta)^2 e^{2\psi(\theta)}\sin\theta(\psi'(\theta) + 2\Omega'(\theta)/\Omega(\theta))\right]_{\theta=0}^{\theta=\pi}\,.
\end{aligned}
\tag{3.7}
$$

The boundary terms in the last line of (3.7) arise from integration by parts in $\int\sqrt{-\det g}\mathcal{L}$. Eq.(3.7) has the property that under a variation of Ω for which $\delta\Omega/\Omega$ does not vanish at the boundary and/or a variation of ψ for which $\delta\psi$ does not vanish at the boundary, the boundary terms in $\delta\mathcal{E}$ cancel if (3.6) is satisfied. This ensures that once the $\mathcal{E}$ is extremized under variations of ψ and Ω for which $\delta\psi$ and $\delta\Omega$ vanish at the boundary, it is also extremized with respect to the constants a_0 and a_π appearing in (3.6) which changes the boundary values of Ω and ψ. Also due to this property we can now extremize the entropy function with respect to β without worrying about the constraint (3.5) since the additional term that comes from the compensating variation in Ω and/or ψ will vanish due to Ω and/or ψ equations of motion.

The equations of motion of various fields may now be obtained by extremizing the entropy function $\mathcal{E}$ with respect to the functions $\Omega(\theta)$, $\psi(\theta)$, $u_s(\theta)$, $b_i(\theta)$ and the parameters e_i, α, β labelling the near horizon geometry. This gives

$$
\begin{aligned}
&-4\beta^{-1}\Omega''(\theta)/\Omega(\theta) + 2\beta^{-1}(\Omega'(\theta)/\Omega(\theta))^2 - 2\beta - 2\beta^{-1}(\psi'(\theta))^2 - \frac{1}{2}\alpha^2\Omega(\theta)^{-2}\beta e^{-4\psi(\theta)} \\
&-\beta^{-1}h_{rs}(\vec{u}(\theta))u_r'(\theta)u_s'(\theta) \\
&+2f_{ij}(\vec{u}(\theta))\left\{-\beta\Omega(\theta)^{-2}e^{-2\psi(\theta)}(e_i - \alpha b_i(\theta))(e_j - \alpha b_j(\theta)) - \beta^{-1}e^{2\psi(\theta)}b_i'(\theta)b_j'(\theta)\right\} \\
&= 0\,,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&4\beta^{-1}\Omega(\theta)\psi''(\theta) + 4\beta^{-1}\Omega'(\theta)\psi'(\theta) - 2\alpha^2\Omega(\theta)^{-1}\beta e^{-4\psi(\theta)} \\
&+2f_{ij}(\vec{u}(\theta))\left\{-2\beta\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_i - \alpha b_i(\theta))(e_j - \alpha b_j(\theta)) - 2\beta^{-1}\Omega(\theta)e^{2\psi(\theta)}b_i'(\theta)b_j'(\theta)\right\} \\
&= 0\,,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&2\left(\beta^{-1}\Omega(\theta)h_{rs}(\vec{u}(\theta))u_s'(\theta)\right)' - \beta^{-1}\Omega(\theta)\partial_r h_{ts}(\vec{u}(\theta))u_t'(\theta)u_s'(\theta) \\
&+2\partial_r f_{ij}(\vec{u}(\theta))\left\{\beta\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_i-\alpha b_i(\theta))(e_j-\alpha b_j(\theta)) - \beta^{-1}\Omega(\theta)e^{2\psi(\theta)}b_i'(\theta)b_j'(\theta)\right\} \\
&+4\partial_r \widetilde{f}_{ij}(\vec{u}(\theta))(e_i-\alpha b_i(\theta))b_j'(\theta) \\
&= 0\,, \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
&-4\alpha\beta f_{ij}(\vec{u}(\theta))\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_j-\alpha b_j(\theta)) + 4\beta^{-1}\left(f_{ij}(\vec{u}(\theta))\Omega(\theta)e^{2\psi(\theta)}b_j'(\theta)\right)' \\
&\quad -4\partial_r\widetilde{f}_{ij}(\vec{u}(\theta))u_r'(\theta)(e_j-\alpha b_j(\theta)) = 0\,, \qquad (3.11)
\end{aligned}
$$

$$
q_i = 8\pi\int d\theta\,\left[f_{ij}(\vec{u}(\theta))\beta\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_j-\alpha b_j(\theta)) + \widetilde{f}_{ij}(\vec{u}(\theta))b_j'(\theta)\right]\,, \qquad (3.12)
$$

$$
\begin{aligned}
J &= 2\pi\int_0^\pi d\theta\,\Big[\alpha\Omega(\theta)^{-1}\beta e^{-4\psi(\theta)} - 4\beta f_{ij}(\vec{u}(\theta))\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_i-\alpha b_i(\theta))b_j(\theta) \\
&\qquad -4\widetilde{f}_{ij}(\vec{u}(\theta))b_i(\theta)b_j'(\theta)\Big]\,, \qquad (3.13)
\end{aligned}
$$

$$
\int d\theta\, I(\theta) = 0\,, \qquad (3.14)
$$

$$
\begin{aligned}
I(\theta) &\equiv -2\Omega(\theta)^{-1}\beta^{-2}(\Omega'(\theta))^2 - 2\Omega(\theta) + 2\Omega(\theta)\beta^{-2}(\psi'(\theta))^2 + \frac{1}{2}\alpha^2\Omega(\theta)^{-1}e^{-4\psi(\theta)} \\
&\quad +\beta^{-2}\Omega(\theta)h_{rs}(\vec{u}(\theta))u_r'(\theta)u_s'(\theta) \\
&\quad +2f_{ij}(\vec{u}(\theta))\left\{\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_i-\alpha b_i(\theta))(e_j-\alpha b_j(\theta)) + \beta^{-2}\Omega(\theta)e^{2\psi(\theta)}b_i'(\theta)b_j'(\theta)\right\}\,. \qquad (3.15)
\end{aligned}
$$

Here $'$ denotes derivative with respect to θ. The required boundary conditions, following from the requirement of the regularity of the solution at $\theta = 0$, π, and that the magnetic charge vector be $\vec{p}$, are:

$$
b_i(0) = -\frac{p_i}{4\pi}, \qquad b_i(\pi) = \frac{p_i}{4\pi}\,, \qquad (3.16)
$$

$$
\Omega(\theta)e^{\psi(\theta)} \to \text{constant as } \theta \to 0, \pi\,, \qquad (3.17)
$$

$$
\beta\Omega(\theta)e^{2\psi(\theta)}\sin\theta \to 1 \quad \text{as } \theta \to 0, \pi\,. \qquad (3.18)
$$

$$
u_s(\theta) \to \text{constant as } \theta \to 0, \pi\,. \qquad (3.19)
$$

Using eqs.(3.8)-(3.11) one can show that

$$
I'(\theta) = 0\,. \qquad (3.20)
$$

Thus $I(\theta)$ is independent of θ. As a consequence of eq.(3.14) we now have

$$I(\theta) = 0\,. \tag{3.21}$$

Combining eqs.(3.8) and (3.21) we get

$$\Omega'' + \beta^2 \Omega = 0\,. \tag{3.22}$$

A general solution to this equation is of the form

$$\Omega = a \sin(\beta\theta + b)\,, \tag{3.23}$$

where a and b are integration constants. In order that Ω has the behaviour given in (3.6) for θ near 0 and π, and not vanish at any other value of θ, we must have

$$b = 0, \qquad \beta = 1\,, \tag{3.24}$$

and hence

$$\Omega(\theta) = a \sin\theta\,. \tag{3.25}$$

In order to analyze the rest of the equations, it will be useful to consider the Taylor series expansion of $u_r(\theta)$ and $b_i(\theta)$ around $\theta = 0, \pi$

$$\begin{aligned} u_r(\theta) &= u_r(0) + \frac{1}{2}\theta^2 u_r''(0) + \cdots \\ u_r(\theta) &= u_r(\pi) + \frac{1}{2}(\theta-\pi)^2 u_r''(\pi) + \cdots \\ b_i(\theta) &= b_i(0) + \frac{1}{2}\theta^2 b_i''(0) + \cdots \\ b_i(\theta) &= b_i(\pi) + \frac{1}{2}(\theta-\pi)^2 b_i''(\pi) + \cdots\,, \end{aligned} \tag{3.26}$$

where we have made use of (2.7), (2.8). We now substitute (3.26) into (3.11) and study the equation near $\theta = 0$ by expanding the left hand side of the equation in powers of θ and using the boundary conditions (3.6). Only odd powers of θ are non-zero. The first non-trivial equation, appearing as the coefficient of the order θ term, involves $b_i(0)$, $b_i''(0)$ and $b_i''''(0)$ and can be used to determine $b_i''''(0)$ in terms of $b_i(0)$ and $b_i''(0)$. Higher order terms determine higher derivatives of b_i at $\theta = 0$ in terms of $b_i(0)$ and $b_i''(0)$. As a result $b_i''(0)$ is not determined in terms of $b_i(0)$ by solving the equations of motion near $\theta = 0$ and we can choose $b_i(0)$ and $b_i''(0)$ as the two independent integration constants of this equation. Of these $b_i(0)$ is determined directly from (3.16). On the other hand for a given configuration of the other fields, $b_i''(0)$ is also determined from (3.16) indirectly by requiring that $b_i(\pi)$ be $p_i/4\pi$. Thus we expect that generically the integration constants associated with the solutions to eqs.(3.11) are fixed by the boundary conditions (3.16).

Let us now analyze eqs.(3.10) and (3.21) together, – eq.(3.9) holds automatically when the other equations are satisfied. For this it will be useful to introduce a new variable

$$\tau = \ln \tan \frac{\theta}{2}\,, \tag{3.27}$$

satisfying

$$\frac{d\tau}{d\theta} = \frac{1}{\sin\theta}\,. \tag{3.28}$$

As θ varies from 0 to π, τ varies from $-\infty$ to ∞. We denote by $\cdot$ derivative with respect to τ and rewrite eqs.(3.10) and (3.21) in this variable. This gives

$$\begin{aligned} &2a^2(h_{rs}(\vec{u})\dot{u}_s)^{\cdot} - a^2\partial_t h_{rs}(\vec{u})\dot{u}_t\dot{u}_s + 4a\partial_r \widetilde{f}_{ij}(\vec{u})(e_i - \alpha b_i)\dot{b}_j \\ &\quad +2\partial_r f_{ij}(\vec{u})\left\{e^{-2\psi}(e_i-\alpha b_i)(e_j-\alpha b_j) - a^2 e^{2\psi}\dot{b}_i\dot{b}_j\right\} = 0\,, \end{aligned} \tag{3.29}$$

and

$$-2a^2+2a^2\dot{\psi}^2+\frac{1}{2}\alpha^2 e^{-4\psi}+a^2 h_{rs}(\vec{u})\dot{u}_r\dot{u}_s+2f_{ij}(\vec{u})\left\{e^{-2\psi}(e_i-\alpha b_i)(e_j-\alpha b_j)+a^2e^{2\psi}\dot{b}_i\dot{b}_j\right\} = 0\,. \tag{3.30}$$

If we denote by m the number of scalars then we have a set of m second order differential equations and one first order differential equation, giving altogether $2m+1$ constants of integration. We want to see in a generic situation how many of these constants are fixed by the required boundary conditions on $\vec{u}$ and ψ. We shall do this by requiring that the equations and the boundary conditions are consistent. Thus for example if ψ, $\{b_i\}$ and $\{u_s\}$ satisfy their required boundary conditions then we can express the equations near $\theta = 0$ (or $\theta = \pi$) as:

$$2a^2(\hat{h}_{rs}\dot{u}_s)^{\cdot} \simeq 0\,, \tag{3.31}$$

and

$$-2a^2 + a^2\hat{h}_{rs}\dot{u}_r\dot{u}_s + 2a^2\dot{\psi}^2 \simeq 0\,. \tag{3.32}$$

Here $\hat{h}_{rs}$ are constants giving the value of $h_{rs}(\vec{u})$ at $\vec{u} = \vec{u}(0)$ (or $\vec{u} = \vec{u}(\pi)$). Note that we have used the boundary conditions to set some of the terms to zero but have kept the terms containing highest derivatives of ψ and u_r even if they are required to vanish due to the boundary conditions. The general solutions to these equations near $\theta = 0$ are

$$u_s(\theta) \simeq c_s + v_s\tau\,, \quad \psi(\theta) \simeq c - \tau\sqrt{1 - \frac{1}{2}\hat{h}_{rs}v_s v_s}\,. \tag{3.33}$$

where c_s, v_s and c are the $2m+1$ integration constants. Since $\tau \to -\infty$ as $\theta \to 0$, in order that u_s approaches a constant value $u_s(0)$ as $\theta \to 0$, we must require all the v_s to vanish. On the other hand requiring that ψ satisfies the boundary condition (3.18) determines c to be $-\ln(2\sqrt{a})$. This gives altogether $m+1$ conditions on the $(2m+1)$ integration constants. Carrying out the same analysis near $\theta = \pi$ gives another $(m+1)$ conditions among the integration constants. Thus the boundary conditions on $\vec{u}$ and ψ not only determine all $(2m+1)$ integration constants of (3.29), (3.30), but give an additional condition among the as yet unknown parameters a, α and e_i entering the equations.

This constraint, together with the remaining equations (3.12) and (3.13), gives altogether $n+2$ constraints on the $n+2$ variables e_i, a and α, where n is the number of $U(1)$ gauge fields. Since generically $(n+2)$ equations in $(n+2)$ variables have only a discrete number of solutions we expect that generically the solution to eqs.(3.8)-(3.19) has no continuous parameters.

In special cases however some of the integration constants may remain undetermined, reflecting a family of solutions corresponding to the same set of charges. As discussed in section 2, these represent flat directions of the entropy function and hence the entropy associated with all members of this family will have identical values. We shall now give a more direct argument to this effect. Suppose as we go from one member of the family to a neighbouring member, each scalar field changes to

$$u_r(\theta) \to u_r(\theta) + \delta u_r(\theta), \tag{3.34}$$

and suppose all the other fields and parameters change in response, keeping the electric charges q_i, magnetic charges p_i and angular momentum fixed:

$$\begin{aligned} \Omega \to \Omega + \delta\Omega, \quad \psi \to \psi + \delta\psi, \quad b_i \to b_i + \delta b_i, \\ e_i \to e_i + \delta e_i, \quad \alpha \to \alpha + \delta\alpha, \quad \beta \to \beta + \delta\beta\,. \end{aligned} \tag{3.35}$$

Let us calculate the resulting change in the entropy $\mathcal{E}$. The changes in e_i, α, β do not contribute to any change in $\mathcal{E}$, since $\partial_{e_i}\mathcal{E} = 0$, $\partial_\alpha\mathcal{E} = 0$ and $\partial_\beta\mathcal{E} = 0$. The only possible contributions from varying Ω, ψ, b_i, u_r can come from boundary terms, since the bulk equations are satisfied. Varying $\mathcal{E}$ subject to the equations of motion, one finds the following boundary terms at the poles:

$$\begin{aligned}\delta\mathcal{E} &= 8\pi^2\Big[\beta^{-1}\Omega h_{rs}u_r'\delta u_s - 2\tilde{f}_{ij}(e_i-\alpha b_i)\delta b_j + 2f_{ij}\left\{\beta^{-1}\Omega e^{2\psi}b_i'\right\}\delta b_j \\ &\quad +\beta^{-1}\left(-2\Omega^{-1}\Omega'\delta\Omega + 2\Omega\psi'\delta\psi + \delta(\Omega\psi'+2\Omega')\right)\Big]_{\theta=0}^{\theta=\pi}\,. \end{aligned} \tag{3.36}$$

Terms involving δb_i at the boundary vanish since the boundary conditions (3.16), (3.26) imply that for fixed magnetic charges δb_i and b_i' must vanish at $\theta=0$ and $\theta=\pi$. Our boundary conditions imply that variations of Ω and ψ at the poles are not independent. From the boundary condition (3.5) it follows that

$$\delta\Omega = -2\Omega\delta\psi \tag{3.37}$$

at $\theta=0$, π, while from (3.6) one can see that at the poles

$$\delta\psi' = 0\,. \tag{3.38}$$

Combining the previous two equations gives

$$\delta\Omega' = -2\Omega'\delta\psi \tag{3.39}$$

at the poles. If we vary just Ω and ψ one finds

$$\begin{aligned}\delta_{\{\Omega,\psi\}}\mathcal{E} &= 8\pi^2\beta^{-1}\left[-2\Omega^{-1}\Omega'\delta\Omega + 2\Omega\psi'\delta\psi + \delta(\Omega\psi'+2\Omega')\right]_{\theta=0}^{\theta=\pi} \\ &= 8\pi^2\beta^{-1}\left[4\Omega'\delta\psi + 2\Omega\psi'\delta\psi + \psi'\delta\Omega + 2\delta\Omega'\right]_{\theta=0}^{\theta=\pi} \\ &= 0\,. \end{aligned} \tag{3.40}$$

Finally, the boundary terms proportional to δu_r go like,

$$\delta_{\vec{u}}\mathcal{E} \propto \left[\Omega h_{rs}u_r'\delta u_s\right]_0^\pi. \tag{3.41}$$

Since $\Omega\to 0$ as $\theta\to 0,\pi$, these too vanish. Thus we learn that the entropy is independent of any undetermined constant of integration.

Before concluding this section we would like to note that using the equations of motion for various fields we can express the charges q_i, the angular momentum J as well as the black hole entropy, i.e. the value of the entropy function at its extremum, as boundary terms evaluated at $\theta = 0$ and $\theta = \pi$. For example using (3.11) we can express (3.12) as

$$q_i = \frac{8\pi}{\alpha}\Big[f_{ij}\Omega e^{2\psi} b_j' - \tilde{f}_{ij}(e_j - \alpha b_j)\Big]_{\theta=0}^{\theta=\pi} \tag{3.42}$$

Similarly using (3.9) and (3.11) we can express (3.13) as

$$J = \frac{4\pi}{\alpha}\Big[\Omega\psi' - \Omega f_{ij} e^{2\psi} b_i b_j' + \tilde{f}_{ij} b_i (e_j - \alpha b_j)\Big]_{\theta=0}^{\theta=\pi} - \frac{q_i e_i}{2\alpha} \tag{3.43}$$

Finally using (3.8), (3.9) we can express the entropy function $\mathcal{E}$ given in (3.7) as

$$\mathcal{E} = 8\pi^2 \left[-2\Omega' + \Omega^2 e^{2\psi} \sin\theta \left(\psi' + 2\frac{\Omega'}{\Omega}\right)\right]_{\theta=0}^{\theta=\pi} \tag{3.44}$$

Using eq.(3.25) and the boundary conditions (3.6) this gives,

$$\mathcal{E} = 16\pi^2 a \tag{3.45}$$

Using eqs. (3.3) and (3.25) it is easy to see that $\mathcal{E} = A/4G_N$ where A is the area of the event horizon. (Note that in our conventions $G_N = 1/16\pi$). This is the expected result for theories with two derivative action.

4 Solutions with Constant Scalars

In this section we shall solve the equations derived in section 3 in special cases where there are no scalars or where the scalars $u_s(\theta)$ are constants:

$$\vec{u}(\theta) = \vec{u}_0\,. \tag{4.1}$$

In this case we can combine (3.9), (3.21), (3.24) and (3.25) to get

$$\sin^2\theta(\psi'' + (\psi')^2) + \sin\theta\cos\theta\psi' - \frac{\alpha^2}{4a^2} e^{-4\psi} - 1 = 0\,. \tag{4.2}$$

The unique solution to this equation subject to the boundary conditions (3.18) is:

$$e^{-2\psi(\theta)} = \frac{2a\sin^2\theta}{2 - (1 - \sqrt{1-\alpha^2})\sin^2\theta}\,. \tag{4.3}$$

We now define the coordinate ξ through the relation:

$$\xi = -\frac{2}{\alpha}\tan^{-1}\left(\frac{\alpha}{1+\sqrt{1-\alpha^2}}\cos\theta\right), \tag{4.4}$$

so that

$$d\xi = \frac{d\theta}{\Omega(\theta)e^{2\psi(\theta)}} \,. \tag{4.5}$$

As θ varies from 0 to π, ξ varies from $-\xi_0$ to ξ_0, with ξ_0 given by

$$\xi_0 = \frac{1}{\alpha}\sin^{-1}\alpha \,. \tag{4.6}$$

In terms of this new coordinate ξ, (3.11) takes the form:

$$\frac{d^2}{d\xi^2}(e_i - \alpha b_i(\theta)) + \alpha^2(e_i - \alpha b_i(\theta)) = 0 \,. \tag{4.7}$$

This has solution:

$$(e_i - \alpha b_i(\theta)) = A_i \sin\left(\alpha\xi + B_i\right) \,, \tag{4.8}$$

where A_i and B_i are integration constants. These can be determined using the boundary condition (3.16):

$$A_i \sin(-\alpha\xi_0 + B_i) = e_i + \alpha\frac{p_i}{4\pi} \,, \qquad A_i \sin(\alpha\xi_0 + B_i) = e_i - \alpha\frac{p_i}{4\pi} \,. \tag{4.9}$$

This gives

$$\begin{aligned} B_i &= \tan^{-1}\left(-\frac{4\pi e_i}{\alpha p_i}\tan(\alpha\xi_0)\right) = \tan^{-1}\left(-\frac{4\pi e_i}{p_i\sqrt{1-\alpha^2}}\right) \,, \\ A_i &= \left(\frac{e_i^2}{\cos^2(\alpha\xi_0)} + \frac{\alpha^2 p_i^2}{16\pi^2\sin^2(\alpha\xi_0)}\right)^{1/2} = \left(\frac{e_i^2}{1-\alpha^2} + \frac{p_i^2}{16\pi^2}\right)^{1/2} \,. \end{aligned} \tag{4.10}$$

Using (3.42) we now get:

$$q_i = 16\pi \sum_j \left(f_{ij}(\vec{u}_0)\,\sin B_j - \widetilde{f}_{ij}(\vec{u}_0)\cos B_j\right)\, A_j = 16\,\pi\,\sum_j \left\{ f_{ij}(\vec{u}_0)\frac{e_j}{\sqrt{1-\alpha^2}} + \widetilde{f}_{ij}(\vec{u}_0)\,\frac{p_j}{4\pi}\right\} \,. \tag{4.11}$$

This gives A_i, B_i and e_i in terms of a, α, $\vec{u}_0$ and the charges $\vec{q}$, $\vec{p}$, J.

Substituting the known solutions for $\Omega(\theta)$, $\psi(\theta)$ and $b_i(\theta)$ into eq.(3.21) and evaluating the left hand side of this equation at $\theta = \pi/2$ we get

$$a\sqrt{1-\alpha^2} = \sum_{i,j} f_{ij}(\vec{u}_0)A_iA_j\cos(B_i - B_j) = \sum_{i,j} f_{ij}(\vec{u}_0)\left\{\frac{p_ip_j}{16\pi^2} + \frac{e_ie_j}{1-\alpha^2}\right\} \,. \tag{4.12}$$

On the other hand (3.43) gives

$$J = 8\pi a\alpha \,. \tag{4.13}$$

Since A_i, B_i and e_i are known in terms of a, α, $\vec{u}_0$ and $\vec{q}$, $\vec{p}$, J, we can use (4.12) and (4.13) to solve for α and a in terms of $\vec{u}_0$, $\vec{q}$, $\vec{p}$ and J. (3.45) then gives the black hole entropy in terms of $\vec{u}_0$, $\vec{q}$, $\vec{p}$ and J. The final results are:

$$\alpha = \frac{J}{\sqrt{J^2 + V_{eff}(\vec{u}_0, \vec{q}, \vec{p})^2}}, \qquad a = \frac{\sqrt{J^2 + V_{eff}(\vec{u}_0, \vec{q}, \vec{p})^2}}{8\pi}, \tag{4.14}$$

and

$$S_{BH} = 2\pi \sqrt{J^2 + V_{eff}(\vec{u}_0, \vec{q}, \vec{p})^2}, \tag{4.15}$$

where

$$V_{eff}(\vec{u}_0, \vec{q}, \vec{p}) = \frac{1}{32\pi} f^{ij}(\vec{u}_0) \widehat{q}_i \widehat{q}_j + \frac{1}{2\pi} f_{ij}(\vec{u}_0) p_i p_j \tag{4.16}$$

is the effective potential introduced in [4]. Here $f^{ij}(\vec{u}_0)$ is the matrix inverse of $f_{ij}(\vec{u}_0)$ and

$$\widehat{q}_i \equiv q_i - 4\, \widetilde{f}_{ij}(\vec{u}_0)\, p_j \,. \tag{4.17}$$

Finally we turn to the determination of $\vec{u}_0$. If there are no scalars present in the theory then of course there are no further equations to be solved. In the presence of scalars we need to solve the remaining set of equations (3.10). In the special case when all the f_{ij} and $\widetilde{f}_{ij}$ are independent of $\vec{u}$ these equations are satisfied by any constant $\vec{u} = \vec{u}_0$. Thus $\vec{u}_0$ is undetermined and represent flat directions of the entropy function. However if f_{ij} and $\widetilde{f}_{ij}$ depend on $\vec{u}$ then there will be constraints on $\vec{u}_0$. First of all note that since the entropy must be extremized with respect to all possible deformations consistent with the $SO(2,1) \times U(1)$ symmetry, it must be extremized with respect to $\vec{u}_0$. This in turn requires that $\vec{u}_0$ be an extremum of $V_{eff}(\vec{u}_0, \vec{q}, \vec{p})$ as in [4]. In this case however there are further conditions coming from (3.10) since the entropy function must also be extremized with respect to variations for which the scalar fields are not constant on the horizon. In fact in the generic situation it is almost impossible to satisfy (3.10) with constant $\vec{u}(\theta)$. We shall now discuss a special case where it is possible to satisfy these equations, – this happens for purely electrically charged black holes when there are no $F\widetilde{F}$ coupling in the theory (i.e. when $\widetilde{f}_{ij}(\vec{u}) = 0$).[6] In this case (4.10) gives

$$B_i = \frac{\pi}{2}, \qquad A_i = \frac{e_i}{\cos(\alpha\xi_0)} = \frac{e_i}{\sqrt{1-\alpha^2}}, \tag{4.18}$$

[6]Clearly there are other examples with non-vanishing p_i and/or $\widetilde{f}_{ij}$ related to this one by electric-magnetic duality rotation.

and eqs.(4.11), (4.8) give, respectively,

$$A_i = \frac{1}{16\pi} f^{ij}(\vec{u}_0) q_j, \qquad e_i = \frac{\sqrt{1-\alpha^2}}{16\pi} f^{ij}(\vec{u}_0) q_j \,, \tag{4.19}$$

$$\begin{aligned} (e_i - \alpha b_i(\theta)) &= A_i \cos(\alpha\xi) = \frac{1}{16\pi} f^{ij}(\vec{u}_0) q_j \cos(\alpha\xi) \\ &= \frac{1}{16\pi} f^{ij}(\vec{u}_0) q_j \, \frac{2\sqrt{1-\alpha^2} + (1-\sqrt{1-\alpha^2})\sin^2\theta}{2 - (1-\sqrt{1-\alpha^2})\sin^2\theta} \,. \end{aligned} \tag{4.20}$$

If following (4.16) we now define:

$$V_{eff}(\vec{u}, \vec{q}) = \frac{1}{32\pi} \, f^{ij}(\vec{u}) q_i q_j \,, \tag{4.21}$$

then substituting the known solutions for Ω and ψ into eq.(3.10) and using (4.20) we can see that (3.10) is satisfied if the scalars are at an extremum $\vec{u}_0$ of V_{eff}, i.e.

$$\partial_r V_{eff}(\vec{u}_0, \vec{q}) = 0 \,. \tag{4.22}$$

With the help of (4.19), eq.(4.12) now takes the form:

$$a\sqrt{1-\alpha^2} = \frac{1}{256\pi^2} f^{ij}(\vec{u}_0, \vec{q}) q_i q_j = \frac{1}{8\pi} \, V_{eff}(\vec{u}_0, \vec{q}) \,, \tag{4.23}$$

Using (4.13), (4.23) we get

$$\alpha = \frac{J}{\sqrt{J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2}} \,, \qquad a = \frac{\sqrt{J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2}}{8\pi} \,, \tag{4.24}$$

$$\begin{aligned} \Omega &= \frac{\sqrt{J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2}}{8\pi} \sin\theta, \\ e^{-2\psi} &= \frac{1}{4\pi} \, \frac{\left(J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2\right)\sin^2\theta}{(1+\cos^2\theta)\sqrt{J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2} + V_{eff}(\vec{u}_0, \vec{q})\sin^2\theta} \,, \end{aligned} \tag{4.25}$$

$$(e_i - b_i(\theta)) = \frac{1}{16\pi} f^{ij}(\vec{u}_0) q_j \, \frac{2V_{eff} + (\sqrt{J^2 + V_{eff}^2} - V_{eff})\sin^2\theta}{2\sqrt{J^2 + V_{eff}^2} - (\sqrt{J^2 + V_{eff}^2} - V_{eff})\sin^2\theta} \tag{4.26}$$

Eq.(3.45) now gives the black hole entropy to be

$$S_{BH} = 2\pi\sqrt{J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2} \,. \tag{4.27}$$

We shall now illustrate the results using explicit examples of extremal Kerr black hole and extremal Kerr-Newman black hole.

4.1 Extremal Kerr Black Hole in Einstein Gravity

We consider ordinary Einstein gravity in four dimensions with action

$$\mathcal{S} = \int d^4x \sqrt{-\det g}\, \mathcal{L}, \qquad \mathcal{L} = R\,. \tag{4.28}$$

In this case since there are no matter fields we have $V_{eff}(\vec{u}_0, \vec{q}) = 0$. Let us for definiteness consider the case where $J > 0$. It then follows from the general results derived earlier that

$$\alpha = 1, \qquad a = \frac{J}{8\pi}\,, \tag{4.29}$$

$$\Omega = \frac{J}{8\pi} \sin\theta, \qquad e^{-2\psi} = \frac{J}{4\pi} \frac{\sin^2\theta}{1+\cos^2\theta}\,, \tag{4.30}$$

and

$$S_{BH} = 2\pi J\,. \tag{4.31}$$

Thus determines the near horizon geometry and the entropy of an extremal Kerr black hole and agrees with the results of [43].

4.2 Extremal Kerr-Newman Black Hole in Einstein-Maxwell Theory

Here we consider Einstein gravity in four dimensions coupled to a single Maxwell field:

$$\mathcal{S} = \int d^4x \sqrt{-\det g}\, \mathcal{L}, \qquad \mathcal{L} = R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\,. \tag{4.32}$$

In this case we have $f_{11} = \frac{1}{4}$. Hence $f^{11} = 4$ and

$$V_{eff}(\vec{u}_0, \vec{q}) = \frac{q^2}{8\pi}\,. \tag{4.33}$$

Thus we have

$$\alpha = \frac{J}{\sqrt{J^2 + (q^2/8\pi)^2}}\,, \qquad a = \frac{\sqrt{J^2 + (q^2/8\pi)^2}}{8\pi}\,. \tag{4.34}$$

$$\Omega = a \sin\theta, \qquad e^{-2\psi} = \frac{2a\sin^2\theta}{1+\cos^2\theta + q^2\sin^2\theta / \left(8\pi\sqrt{J^2+(q^2/8\pi)^2}\right)}\,, \tag{4.35}$$

and

$$S_{BH} = 2\pi\sqrt{J^2 + (q^2/8\pi)^2}\,. \tag{4.36}$$

The near horizon geometry given in (4.34), (4.35) agrees with the results of [43].

Comparing (4.24)-(4.27) with (4.34)-(4.36) we see that the results for the general case of constant scalar field background is obtained from the results for extremal Kerr-Newman black hole carrying electric charge q via the replacement of q by q_{eff} where

$$q_{eff} = \sqrt{8\pi \, V_{eff}(\vec{u}_0, \vec{q})} \,. \tag{4.37}$$

5 Examples of Attractor Behaviour in Full Black Hole Solutions

The set of equations (3.8)-(3.13) and (3.21) are difficult to solve explicitly in the general case. However there are many known examples of rotating extremal black hole solutions in a variety of two derivative theories of gravity. In this section we shall examine the near horizon geometry of these solutions and check that they obey the consequences of the generalized attractor mechanism discussed in sections 2 and 3.

5.1 Rotating Kaluza-Klein Black Holes

In this section we consider the four dimensional theory obtained by dimensional reduction of the five dimensional pure gravity theory on a circle. The relevant four dimensional fields include the metric $g_{\mu\nu}$, a scalar field Φ associated with the radius of the fifth dimension and a U(1) gauge field A_μ. The lagrangian density is given by

$$\mathcal{L} = R - 2g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - e^{2\sqrt{3}\Phi} g^{\mu\rho} g^{\nu\sigma} \, F_{\mu\nu} F_{\rho\sigma} \,. \tag{5.1}$$

Identifying Φ as Φ_1 and A_μ as $A^{(1)}_\mu$ and comparing (3.2) and (5.1) we see that we have in this example

$$h_{11} = 2, \quad f_{11} = e^{2\sqrt{3}\Phi} \,. \tag{5.2}$$

Suppose we have an extremal rotating black hole solution in this theory with near horizon geometry of the form given in (3.3). Let us define $\tau = \ln \tan(\theta/2)$ as in (3.27), denote by $\cdot$ derivative with respect to τ and define

$$\chi(\theta) = e - \alpha b(\theta) \,. \tag{5.3}$$

Using (3.24) and (3.25) we can now express appropriate linear combinations of eqs.(3.9) - (3.11) and (3.21) as

$$\ddot{\psi} = \frac{\alpha^2}{4a^2} e^{-4\psi} + 1 - \dot{\psi}^2 - \dot{\Phi}^2 \tag{5.4}$$

$$\ddot{\Phi} + \sqrt{3}e^{2\sqrt{3}\Phi}\left\{e^{-2\psi}a^{-2}\chi^2 - \alpha^{-2}e^{2\psi}\dot{\chi}^2\right\} = 0 \tag{5.5}$$

$$\alpha^2 a^{-2} e^{2\sqrt{3}\Phi-2\psi}\chi + \left(e^{2\sqrt{3}\Phi+2\psi}\dot{\chi}\right)^{\cdot} = 0\,. \tag{5.6}$$

$$-2a^2 + 2a^2\dot{\psi}^2 + \frac{1}{2}\alpha^2 e^{-4\psi} + 2a^2\dot{\Phi}^2 + 2\left\{e^{2\sqrt{3}\Phi-2\psi}\chi^2 + a^2\alpha^{-2}e^{2\sqrt{3}\Phi+2\psi}\dot{\chi}^2\right\} = 0\,. \tag{5.7}$$

Refs.[48, 49, 50] explicitly constructed rotating charged black hole solutions in this theory. Later we shall analyze the near horizon geometry of these black holes in extremal limit and verify that they satisfy eqs.(5.4)-(5.7).

Next we note that the lagrangian density (5.1) has a scaling symmetry:

$$\Phi \to \Phi + \lambda, \qquad F_{\mu\nu} \to e^{-\sqrt{3}\lambda}F_{\mu\nu}\,. \tag{5.8}$$

Since the magnetic and electric charges p and q are proportional to $F_{\theta\phi}$ and $\partial\mathcal{L}/\partial F_{rt}$ respectively, we see that under the transformation (5.8), q and p transforms to $e^{\sqrt{3}\lambda}q$ and $e^{-\sqrt{3}\lambda}p$ respectively. Thus if we want to keep the electric and the magnetic charges fixed, we need to make a compensating transformation of the parameters labelling the electric and magnetic charges of the solution. This shows that we can generate a one parameter family of solutions carrying fixed electric and magnetic charges by using the transformation:

$$\Phi \to \Phi + \lambda, \quad F_{\mu\nu} \to e^{-\sqrt{3}\lambda}F_{\mu\nu}, \quad Q \to e^{-\sqrt{3}\lambda}Q, \quad P \to e^{\sqrt{3}\lambda}P\,, \tag{5.9}$$

where Q and P are electric and magnetic charges labelling the original solution. This transformation will change the asymptotic value of the scalar field Φ leaving the electric and magnetic charges fixed. Thus according to the general arguments given in section 2, the entropy associated with the solution should not change under the deformation (5.9). On the other hand since (5.8) is a symmetry of the theory, the entropy is also invariant under this transformation. Combining these two results we see that the entropy must be invariant under

$$Q \to e^{-\sqrt{3}\lambda}Q, \quad P \to e^{\sqrt{3}\lambda}P\,. \tag{5.10}$$

Furthermore if the entropy function has no flat direction so that the near horizon geometry is fixed completely by extremizing the entropy function then the near horizon geometry, including the scalar field configuration, should be invariant under the transformation (5.9).

5.1.1 The black hole solution

We now turn to the black hole solution described in [48, 49, 50]. The metric associated with this solution is given by

$$ds^2 = -\frac{\tilde{\Delta}}{\sqrt{f_p f_q}}(dt - w d\phi)^2 + \frac{\sqrt{f_p f_q}}{\Delta} dr^2 + \sqrt{f_p f_q} d\theta^2 + \frac{\Delta\sqrt{f_p f_q}}{\tilde{\Delta}} \sin^2\theta d\phi^2 \tag{5.11}$$

where

$$\begin{aligned}
f_p &= r^2 + a_K^2\cos^2\theta + r(\tilde{p} - 2M_K) + \frac{\tilde{p}}{\tilde{p}+\tilde{q}}\frac{(\tilde{p}-2M_K)(\tilde{q}-2M_K)}{2} \\
&\quad - \frac{\tilde{p}\sqrt{(\tilde{p}^2 - 4M_K^2)(\tilde{q}^2 - 4M_K^2)}}{2(\tilde{p}+\tilde{q})}\frac{a_K}{M_K}\cos\theta & (5.12)\\
f_q &= r^2 + a_K^2\cos^2\theta + r(\tilde{q} - 2M_K) + \frac{\tilde{q}}{\tilde{p}+\tilde{q}}\frac{(\tilde{p}-2M_K)(\tilde{q}-2M_K)}{2} \\
&\quad + \frac{\tilde{q}\sqrt{(\tilde{p}^2 - 4M_K^2)(\tilde{q}^2 - 4M_K^2)}}{2(\tilde{p}+\tilde{q})}\frac{a_K}{M_K}\cos\theta & (5.13)\\
w &= \sqrt{\tilde{p}\tilde{q}}\frac{(\tilde{p}\tilde{q} + 4M_K^2)r - M_K(\tilde{p}-2M_K)(\tilde{q}-2M_K)}{2(\tilde{p}+\tilde{q})\tilde{\Delta}}\frac{a_K}{M_K}\sin^2\theta & (5.14)\\
\Delta &= r^2 - 2M_K r + a_K^2 & (5.15)\\
\tilde{\Delta} &= r^2 - 2M_K r + a_K^2\cos^2\theta\,. & (5.16)
\end{aligned}$$

M_K, a_K, $\tilde{p}$ and $\tilde{q}$ are four parameters labelling the solution. The solution for the dilaton is of the form

$$\exp(-4\Phi/\sqrt{3}) = \frac{f_p}{f_q}\,. \tag{5.17}$$

The dilaton has been asymptotically set to 0, but this can be changed using the transformation (5.9). Finally, the gauge field is given by

$$A_t = -f_q^{-1}\left(\frac{Q}{4\sqrt{\pi}}\left(r + \frac{\tilde{p}-2M_K}{2}\right) + \frac{1}{2}\frac{a_K}{M_K}\sqrt{\frac{\tilde{q}^3(\tilde{p}^2 - 4M_K^2)}{4(\tilde{p}+\tilde{q})}}\cos\theta\right) \tag{5.18}$$

$$\begin{aligned}
A_\phi &= -\frac{P}{4\sqrt{\pi}}\cos\theta - f_q^{-1}\frac{P}{4\sqrt{\pi}}a_K^2\sin^2\theta\cos\theta \\
&\quad - \frac{1}{2}f_q^{-1}\sin^2\theta\frac{a_K}{M_K}\sqrt{\frac{\tilde{p}(\tilde{q}^2 - 4M_K^2)}{4(\tilde{p}+\tilde{q})^3}}\left[(\tilde{p}+\tilde{q})(\tilde{p}r - M_K(\tilde{p}-2M_K)) + \tilde{q}(\tilde{p}^2 - 4M_K^2)\right]
\end{aligned} \tag{5.19}$$

where Q and P, labelling the electric and magnetic charges of the black hole, are given by,

$$Q^2 = 4\pi \frac{\tilde{q}(\tilde{q}^2 - 4M_K^2)}{(\tilde{p}+\tilde{q})} \tag{5.20}$$

$$P^2 = 4\pi \frac{\tilde{p}(\tilde{p}^2 - 4M_K^2)}{(\tilde{p}+\tilde{q})}\,. \tag{5.21}$$

The mass and angular momentum of the black hole can be expressed in terms of M_K, a_K, $\tilde{p}$ and $\tilde{q}$ as follows:[7]

$$M = 4\pi\,(\tilde{q}+\tilde{p}) \tag{5.22}$$

$$J = 4\pi\, a_K\, (\tilde{p}\tilde{q})^{1/2}\, \frac{\tilde{p}\tilde{q} + 4M_K^2}{M_K(\tilde{p}+\tilde{q})}\,. \tag{5.23}$$

5.1.2 Extremal limit: The ergo-free branch

As first discussed in [48], in this case the moduli space of extremal black holes consist of two branches. Let us first concentrate on one of these branches corresponding to the surface W in [48]. We consider the limit: $M_K, a_K \to 0$ with a_K/M_K, $\tilde{q}$ and $\tilde{p}$ held finite. In this limit $\tilde{q}$, $\tilde{p}$ and a_K/M_K can be taken as the independent parameters labelling the solution. Then (5.20-5.23) become

$$M = 4\pi\,(\tilde{q}+\tilde{p}) \tag{5.24}$$

$$Q^2 = 4\pi \frac{\tilde{q}^3}{(\tilde{q}+\tilde{p})} \tag{5.25}$$

$$P^2 = 4\pi \frac{\tilde{p}^3}{(\tilde{q}+\tilde{p})} \tag{5.26}$$

$$J = 4\pi \frac{a_K}{M_K} \frac{(\tilde{p}\tilde{q})^{3/2}}{\tilde{p}+\tilde{q}} = \frac{a_K}{M_K}|PQ|\,. \tag{5.27}$$

For definiteness we shall take P and Q to be positive from now on.

In this limit Δ, $\widetilde{\Delta}$, f_p, f_q, w and A_μ become

$$\Delta = \widetilde{\Delta} = r^2 \tag{5.28}$$

[7] In defining the mass and angular momentum we have taken into account the fact that we have $G_N = 1/16\pi$. At present the normalization of the charges Q and P have been chosen arbitrarily, but later we shall relate them to the charges q and p introduced in section 3.

$$f_p = r^2 + \tilde{p}r + \frac{\tilde{p}^2\tilde{q}}{2(\tilde{p}+\tilde{q})}\left(1 - \frac{a_K}{M_K}\cos\theta\right) \tag{5.29}$$

$$f_q = r^2 + \tilde{q}r + \frac{\tilde{q}^2\tilde{p}}{2(\tilde{p}+\tilde{q})}\left(1 + \frac{a_K}{M_K}\cos\theta\right) \tag{5.30}$$

$$w = \frac{(\tilde{p}\tilde{q})^{\frac{3}{2}}}{2(\tilde{p}+\tilde{q})}\frac{a_K}{M_K}\frac{\sin^2\theta}{r} = \frac{J}{8\pi}\frac{\sin^2\theta}{r} \tag{5.31}$$

$$A_t = -\frac{Q}{4\sqrt{\pi}}f_q^{-1}\left(\left(r + \frac{\tilde{p}}{2}\right) + \frac{1}{2}\left(\frac{a_K}{M_K}\right)\tilde{p}\cos\theta\right) \tag{5.32}$$

$$A_\phi = -\frac{P}{4\sqrt{\pi}}\left[\cos\theta + \frac{1}{2}f_q^{-1}\sin^2\theta\left(\frac{a_K}{M_K}\right)\frac{\tilde{q}}{(\tilde{p}+\tilde{q})}\left((\tilde{p}+\tilde{q})r + \tilde{q}\tilde{p}\right)\right] \tag{5.33}$$

In order that the scalar field configuration is well defined everywhere outside the horizon, we need f_p/f_q to be positive in this region. This gives

$$a_K \leq M_K\,. \tag{5.34}$$

This in turn implies that the coefficient of g_{tt}, being proportional to $\widetilde{\Delta}/\sqrt{f_p f_q}$ remains positive everywhere outside the horizon. Thus there is no ergo-sphere for this black hole. We call this branch of solutions the ergo-free branch.

5.1.3 Near horizon behaviour

In our coordinate system the horizon is at $r = 0$. To find the near horizon geometry, we consider the limit

$$r \to sr, \qquad t \to s^{-1}t \qquad s \to 0\,. \tag{5.35}$$

Metric The near horizon behaviour of the metric is given by:

$$ds^2 = -\frac{r^2}{v_1(\theta)}(dt - \frac{b}{r}d\phi)^2 + v_1(\theta)\left(\frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta d\phi^2\right) \tag{5.36}$$

with

$$v_1(\theta) = \lim_{r\to 0}\sqrt{f_p f_q} = \frac{1}{8\pi}\sqrt{P^2Q^2 - J^2\cos^2\theta}, \qquad b = \frac{J}{8\pi}\sin^2\theta\,. \tag{5.37}$$

By straightforward algebraic manipulation this metric can be rewritten as

$$ds^2 = \frac{a^2\sin^2\theta}{v_1(\theta)}\left(d\phi - \alpha r dt'\right)^2 + v_1(\theta)\left(-r^2dt'^2 + \frac{dr^2}{r^2} + d\theta^2\right) \tag{5.38}$$

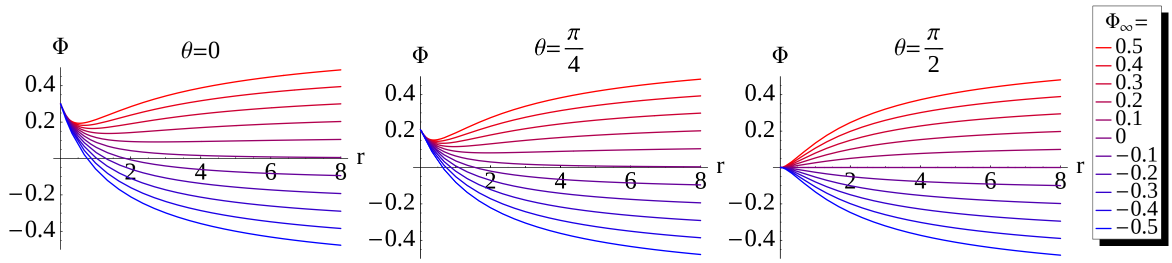

Figure 1: Radial evolution of the scalar field starting with different asymptotic values at three different values of θ. We take $P = Q = 4\sqrt{\pi}$, $J = 16\pi/3$ for $\Phi_\infty = 0$, and then change Φ_∞ and P, Q using the transformation (5.9).

with

$$t' = t/a\,, \tag{5.39}$$

$$a = \frac{1}{8\pi}\sqrt{P^2Q^2 - J^2}\,, \tag{5.40}$$

$$\alpha = -J/\sqrt{P^2Q^2 - J^2}\,. \tag{5.41}$$

Gauge fields Near the horizon the gauge fields behave like

$$\frac{1}{2}\,F_{\mu\nu}dx^\mu dx^\nu = \left[\frac{2a\sqrt{\pi}}{Q}\frac{1}{(1+\mu\cos\theta)}dr\wedge dt' + \frac{1}{4\sqrt{\pi}}\,P\sin\theta\frac{(1-\mu^2)}{(1+\mu\cos\theta)^2}d\theta\wedge(d\phi - \alpha r dt')\right]\,, \tag{5.42}$$

where

$$\mu = \frac{J}{PQ}\,. \tag{5.43}$$

Scalar Field In the near horizon limit the scalar field becomes

$$e^{-4\Phi/\sqrt{3}}\Big|_{r=M} \quad = \quad \left(\frac{P}{Q}\right)^{\frac{2}{3}}\frac{PQ - J\cos\theta}{PQ + J\cos\theta} \tag{5.44}$$

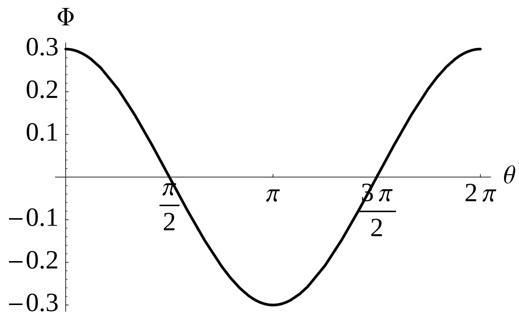

Figure 2: Scalar field profile at the horizon of the Kaluza-Klein black hole. We take $P = Q = 4\sqrt{\pi}$, $J = 16\pi/3$ for $\Phi_\infty = 0$, and then change Φ_∞ and P, Q using the transformation (5.9). The figure shows that the scalar field profile at the horizon is independent of Φ_∞.

Entropy Finally the entropy associated with this solution is given by

$$S_{BH} = 4\pi \int d\theta d\phi \sqrt{g_{\theta\theta}\, g_{\phi\phi}} = 16\pi^2 a = 2\pi\sqrt{P^2Q^2 - J^2}\,. \tag{5.45}$$

We now see that the entropy is invariant under (5.10) and the near horizon background, including the scalar field configuration given in (5.44), is invariant under the transformation (5.9).[8] This shows that the near horizon field configuration is independent of the asymptotic value of the modulus field Φ. This can also be seen explicitly by studying the radial evolution of Φ for various asymptotic values of Φ; numerical results for this evolution have been plotted in fig.1. Fig.2 shows the plot of $\Phi(\theta)$ vs. θ at the horizon of the black hole.

5.1.4 Entropy function analysis

The analysis of section 5.1.3 shows that the near horizon field configuration is precisely of the form described in eq.(3.3) with

$$\Omega(\theta) = a\sin\theta, \quad e^{-2\psi(\theta)} = \frac{8\pi\, a^2 \sin^2\theta}{\sqrt{P^2Q^2 - J^2\cos^2\theta}}\,, \quad e - \alpha b(\theta) = \frac{2\sqrt{\pi}\, a}{Q}\frac{1}{(1+\mu\cos\theta)}\,,$$

[8] As described in eqs.(5.48), (5.49), the charges q, p are related to the parameters Q, P by some normalization factors. These factors do not affect the transformation laws of the charges given in (5.9), (5.10).

$$e^{-4\Phi/\sqrt{3}} = \left(\frac{P}{Q}\right)^{\frac{2}{3}} \frac{PQ - J\cos\theta}{PQ + J\cos\theta}\,, \quad a = \frac{1}{8\pi}\sqrt{P^2Q^2 - J^2}, \quad \alpha = -\frac{J}{\sqrt{P^2Q^2 - J^2}}\,. \tag{5.46}$$

We can easily verify that this configuration satisfies eqs.(5.4)-(5.7) obtained by extremizing the entropy function.

Using eq.(3.16) with values of h_{11} and f_{11} given in (5.2) we get

$$e = \frac{1}{2}\left[(e - \alpha b(\pi)) + (e - \alpha b(0))\right] = \frac{P^2 Q}{4\sqrt{\pi}\sqrt{P^2Q^2 - J^2}}\,, \tag{5.47}$$

and

$$p = -\frac{2\pi}{\alpha}\left[(e - \alpha b(\pi)) - (e - \alpha b(0))\right] = \sqrt{\pi}\,P\,. \tag{5.48}$$

Eq.(3.42) now gives

$$q = \frac{8\pi}{\alpha}\left[\frac{e^{2\sqrt{3}\Phi} b'}{\sin\theta}\right]_0^\pi = 4\sqrt{\pi}\,Q\,. \tag{5.49}$$

Finally the right hand side of eq.(3.43) evaluated for the background (5.46) gives the answer J showing that we have correctly identified the parameter J as the angular momentum carried by the black hole.

5.1.5 The ergo-branch

The extremal limit on this branch, corresponding to the surface S in [48], amounts to taking

$$a_K = M_K \tag{5.50}$$

in the black hole solution. Thus we have the relations

$$Q^2 = 4\pi\,\frac{\tilde{q}(\tilde{q}^2 - 4M_K^2)}{(\tilde{p} + \tilde{q})}, \quad P^2 = 4\pi\,\frac{\tilde{p}(\tilde{p}^2 - 4M_K^2)}{(\tilde{p} + \tilde{q})}, \quad J = 4\pi\,\sqrt{\tilde{p}\tilde{q}}\,\frac{\tilde{p}\tilde{q} + 4M_K^2}{(\tilde{p} + \tilde{q})}\,. \tag{5.51}$$

In order to take the near horizon limit of this solution we first let

$$r \to r + M_K \tag{5.52}$$

which shifts the horizon to $r = 0$. Near the horizon Δ, $\tilde{\Delta}$ and w become

$$\Delta = r^2 \tag{5.53}$$

$$\tilde{\Delta} = -M_K^2 \sin^2\theta + \mathcal{O}(r^2) \tag{5.54}$$

$$w = -\sqrt{\tilde{q}\tilde{p}}\,(1 + \bar{w}r) + \mathcal{O}(r^2) \tag{5.55}$$

with

$$\bar{w} = \frac{\tilde{p}\tilde{q} + 4M_K^2}{2(\tilde{p}+\tilde{q})M_K^2}\,. \tag{5.56}$$

Note that $\widetilde{\Delta}$ changes from being positive at large distance to negative at the horizon. Thus g_{tt} changes sign as we go from the asymptotic region to the horizon and the solution has an ergo-sphere. We call this branch of solutions the ergo-branch. Using eqs.(5.53)-(5.56) we can write the metric as

$$ds^2 = \frac{M_K^2 \sin^2\theta}{\sqrt{f_p f_q}}\left(dt + \sqrt{\tilde{q}\tilde{p}}(1+\bar{w}r)d\phi\right)^2 + \sqrt{f_p f_q}\left(\frac{dr^2}{r^2} + d\theta^2 - \frac{r^2}{M_K^2}d\phi^2\right) + \cdots \tag{5.57}$$

where $\cdots$ denote terms which will eventually vanish in the near horizon limit that we are going to describe below. After letting

$$\phi \to \phi - t/\sqrt{\tilde{q}\tilde{p}} \tag{5.58}$$

and taking the near horizon limit

$$r \to s\, r, \qquad t \to s^{-1}\, t, \qquad s \to 0\,, \tag{5.59}$$

the metric becomes

$$ds^2 = \frac{M_K^2 \sin^2\theta}{v_1(\theta)}(\sqrt{\tilde{q}\tilde{p}}d\phi - \bar{w}r dt)^2 + v_1(\theta)\left(\frac{dr^2}{r^2} + d\theta^2 - \frac{r^2}{M_K^2\,\tilde{q}\tilde{p}}dt^2\right) \tag{5.60}$$

where

$$v_1(\theta) = \lim_{r\to 0}\sqrt{f_p\, f_q}\,. \tag{5.61}$$

Finally rescaling

$$t \to M_K\sqrt{\tilde{q}\tilde{p}}\; t \tag{5.62}$$

the metric becomes of the form given in (3.3) with

$$\Omega = M_K\sqrt{\tilde{p}\tilde{q}}\,\sin\theta, \quad e^{-2\psi} = \frac{M_K^2\tilde{p}\tilde{q}\sin^2\theta}{v_1(\theta)}, \quad \alpha = M_K\;\bar{w}\,. \tag{5.63}$$

Using eqs.(5.56) and (5.51) we find that

$$\alpha = \frac{J}{\sqrt{J^2 - P^2Q^2}}, \quad \Omega = \frac{1}{8\pi}\sqrt{J^2 - P^2Q^2}\,\sin\theta, \quad e^{-2\psi} = \frac{(J^2 - P^2Q^2)\sin^2\theta}{64\pi^2 v_1(\theta)}\,. \tag{5.64}$$

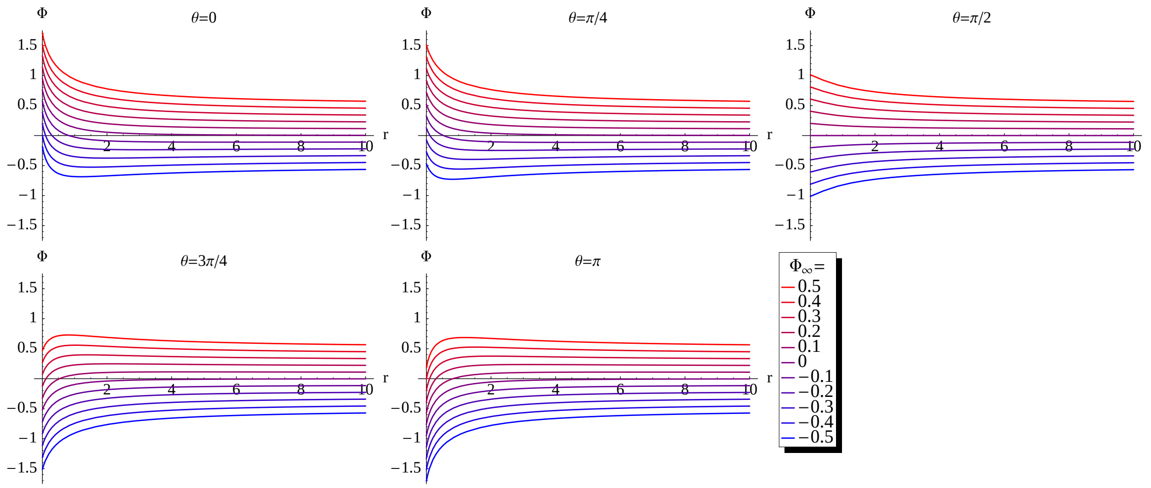

Figure 3: Radial evolution of the scalar field for an ergo-branch black hole starting with different asymptotic values at five different values of θ. We take $P = Q = 2\sqrt{\pi}$ and $J = 4\pi\sqrt{2}$ for $\Phi_\infty = 0$, and then change Φ_∞ and P, Q using the transformation (5.9).

The scalar field Φ becomes in this limit

$$e^{-4\Phi/\sqrt{3}} = \frac{f_p}{f_q}\,, \tag{5.65}$$

where f_p and f_q now refer to the functions f_p and f_q at the horizon:

$$f_p = -M_K^2 \sin^2\theta + M_K\tilde{p} + \frac{\tilde{p}}{\tilde{p}+\tilde{q}}\frac{(\tilde{p}-2M_K)(\tilde{q}-2M_K)}{2} - \frac{\tilde{p}\sqrt{(\tilde{p}^2-4M_K^2)(\tilde{q}^2-4M_K^2)}}{2(\tilde{p}+\tilde{q})}\cos\theta \tag{5.66}$$

$$f_q = -M_K^2 \sin^2\theta + M_K\tilde{q} + \frac{\tilde{q}}{\tilde{p}+\tilde{q}}\frac{(\tilde{p}-2M_K)(\tilde{q}-2M_K)}{2} + \frac{\tilde{q}\sqrt{(\tilde{p}^2-4M_K^2)(\tilde{q}^2-4M_K^2)}}{2(\tilde{p}+\tilde{q})}\cos\theta\,. \tag{5.67}$$

The near horizon gauge field can also be calculated by a tedious but straightforward procedure after taking into account the change in coordinates described above. The final result is of the form given in (3.3) with

$$e - \alpha b(\theta) = \frac{M_K\sqrt{\tilde{p}\tilde{q}}}{4\sqrt{\pi}\,f_q}\left(\frac{1}{2}\,\frac{\tilde{p}}{\tilde{q}}\,Q\,\sin^2\theta + P\sqrt{\frac{\tilde{q}}{\tilde{p}}}\,\cos\theta\right). \tag{5.68}$$

This gives

$$e = \frac{1}{2}\left[(e-\alpha b(\pi)) + (e-\alpha b(0))\right] = -\frac{P^2 Q}{4\sqrt{\pi}\sqrt{J^2-P^2Q^2}}\,, \tag{5.69}$$

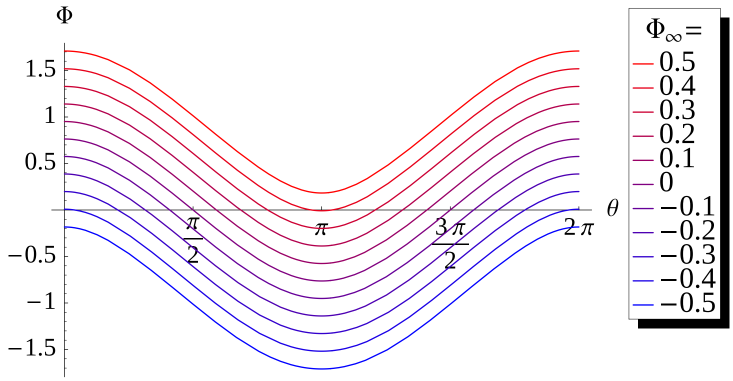

Figure 4: Scalar field profile at the horizon for a black hole on the ergo-branch for different asymptotic values of Φ. We take $P = Q = 2\sqrt{\pi}$ and $J = 4\pi\sqrt{2}$ for $\Phi_\infty = 0$, and then change Φ_∞ and P, Q using the transformation (5.9). Clearly the scalar field profile at the horizon depends on its asymptotic value.

$$p = -\frac{2\pi}{\alpha}\left[(e - \alpha b(\pi)) - (e - \alpha b(0))\right] = \sqrt{\pi}\, P\,, \tag{5.70}$$

and

$$q = \frac{8\pi}{\alpha}\left[\frac{e^{2\sqrt{3}\Phi} b'}{\sin\theta}\right]_0^\pi = 4\sqrt{\pi}\, Q\,. \tag{5.71}$$

Finally, the entropy associated with this solution can be easily calculated by computing the area of the horizon, and is given by

$$S_{BH} = 2\pi\sqrt{J^2 - P^2 Q^2}\,. \tag{5.72}$$

We have explicitly checked that the near horizon ergo-branch field configurations described above satisfy the differential equations (5.4)-(5.7).

The entropy is clearly invariant under the transformation (5.10). However in this case the near horizon background is not invariant under the transformation (5.9). One way to see this is to note that under the transformation (5.10) the combination $M_K^2 \tilde{p}\tilde{q} = (J^2 - P^2 Q^2)/64\pi^2$ remains invariant. This shows that M_K cannot remain invariant under this transformation, since if M_K had been invariant then $\tilde{p}\tilde{q}$ would be invariant, and the invariance of J given in (5.51) would imply that $\tilde{p} + \tilde{q}$ is also invariant. This in turn would mean that M_K, $\tilde{p}$ and $\tilde{q}$ are all invariant under (5.10) and hence P and Q would be invariant which is clearly a contradiction. Given the fact that M_K is not invariant under this transformation we see that the coefficient of the $\sin^2\theta$ term in f_p and f_q are not invariant under (5.10). This in turn shows that ψ, and hence the background metric, is not invariant under the transformation (5.9). This is also seen from figures 3 and 4 where we have shown respectively the radial evolution of the scalar field and the scalar field profile at the horizon for different asymptotic values of Φ. Nevertheless several components of the near horizon background, *e.g.* $\Omega(\theta)$ and the parameters α and e do remain invariant under this transformation, indicating that at least these components do get attracted towards fixed values as we approach the horizon.

5.2 Black Holes in Toroidally Compactified Heterotic String Theory

The theory under consideration is a four dimensional theory of gravity coupled to a complex scalar $S = S_1 + iS_2$, a 4×4 matrix valued scalar field M satisfying the constraint

$$MLM^T = L, \qquad L = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{5.73}$$

and four U(1) gauge fields $A_\mu^{(i)}$ $(1 \le i \le 4)$.[9] Here I_2 denotes 2×2 identity matrix. The bosonic part of the lagrangian density is

$$\begin{aligned}\mathcal{L} \;=\;& R - \frac{1}{2}\, g^{\mu\nu} S_2^{-2}\partial_\mu \bar{S}\partial_\nu S + \frac{1}{8} g^{\mu\nu} Tr(\partial_\mu M L \partial_\nu M L) \\ & -\frac{1}{4}\, S_2 g^{\mu\rho} g^{\nu\sigma} F^{(i)}_{\mu\nu} (LML)_{ij} F^{(j)}_{\rho\sigma} + \frac{1}{4}\, S_1 g^{\mu\rho} g^{\nu\sigma} F^{(i)}_{\mu\nu} L_{ij} \widetilde{F}^{(j)}_{\rho\sigma}\,,\end{aligned} \tag{5.74}$$

where

$$\widetilde{F}^{(i)\mu\nu} = \frac{1}{2}\,(\sqrt{-\det g})^{-1} \epsilon^{\mu\nu\rho\sigma}\, \widetilde{F}^{(i)}_{\rho\sigma}\,. \tag{5.75}$$

General rotating black solution in this theory, carrying electric charge vector $\vec{q}$ and magnetic charge vector $\vec{p}$, has been constructed in [51]. Before we begin analyzing the solution, we would like to note that the lagrangian density (5.74) is invariant under an SO(2,2) rotation:

$$M \to \Omega M \Omega^T, \quad F^{(i)}_{\mu\nu} \to \Omega_{ij} F^{(j)}_{\mu\nu}\,, \tag{5.76}$$

where Ω is a 4×4 matrix satisfying

$$\Omega L \Omega^T = L\,. \tag{5.77}$$

[9] Actual heterotic string theory has 28 gauge fields and a 28×28 matrix valued scalar field, but the truncated theory discussed here contains all the non-trivial information about the theory.

Thus given a classical solution, we can generate a class of classical solutions using this transformation. Since the magnetic and electric charges p_i and q_i are proportional to $F^{(i)}_{\theta\phi}$ and $\partial\mathcal{L}/\partial F^{(i)}_{rt}$ respectively, we see that under the transformation (5.76), $p_i \to \Omega_{ij} p_j$, $q_i \to (\Omega^T)^{-1}_{ij} q_j$. Thus if we want the new solution to have the same electric and magnetic charges, we must make compensating transformation in the parameters labelling the electric and magnetic charges. This shows that we can generate a family of solutions carrying the same electric and magnetic charges by making the transformation:

$$M \to \Omega M \Omega^T, \quad F^{(i)}_{\mu\nu} \to \Omega_{ij} F^{(j)}_{\mu\nu}, \quad Q_i \to \Omega^T_{ij} Q_j, \quad P_i \to \Omega^{-1}_{ij} P_j \,, \tag{5.78}$$

where $\vec{Q}$ and $\vec{P}$ are the parameters which label electric and magnetic charges in the original solution. This transformation changes the asymptotic value of M leaving the charges unchanged. Thus the general argument of section 2 will imply that the entropy must remain invariant under such a transformation. Invariance of the entropy under the transformation (5.76), which is a symmetry of the theory, will then imply that the entropy must be invariant under

$$Q_i \to \Omega^T_{ij} Q_j, \quad P_i \to \Omega^{-1}_{ij} P_j \,. \tag{5.79}$$

On the other hand if there is a unique background for a given set of charges then the background itself must be invariant under the transformation (5.78).

The equations of motion derived from the lagrangian density (5.74) is also invariant under the electric magnetic duality transformation:

$$S \to \frac{aS+b}{cS+d}\,, \qquad F^{(i)}_{\mu\nu} \to (cS_1 + d) F^{(i)}_{\mu\nu} + cS_2 (ML)_{ij} \widetilde{F}^{(j)}_{\mu\nu}\,, \tag{5.80}$$

where a, b, c, d are real numbers satisfying $ad - bc = 1$. We can use this transformation to generate a family of black hole solutions from a given solution. From the definition of electric and magnetic charges it follows that under this transformation the electric and magnetic charge vectors $\vec{q}$, $\vec{p}$ transform as:

$$\vec{q} \to (a\vec{q} - bL\vec{p}), \qquad \vec{p} \to (-cL\vec{q} + d\vec{p})\,. \tag{5.81}$$

Thus if we want the new solution to have the same charges as the old solution we must perform compensating transformation on the electric and magnetic charge parameters $\vec{Q}$ and $\vec{P}$. We can get a family of solutions with the same electric and magnetic charges but different asymptotic values of the scalar field S by the transformation:

$$S \to \frac{aS+b}{cS+d}, \quad F^{(i)}_{\mu\nu} \to (cS_1 + d) F^{(i)}_{\mu\nu} + cS_2 (ML)_{ij} \widetilde{F}^{(j)}_{\mu\nu}, \quad \vec{Q} \to d\vec{Q} + bL\vec{P}, \quad \vec{P} \to cL\vec{Q} + a\vec{P}\,. \tag{5.82}$$

Arguments similar to the one given for the O(2,2) transformation shows that the entropy must remain invariant under the transformation

$$\vec{Q} \to d\vec{Q} + bL\vec{P}, \quad \vec{P} \to cL\vec{Q} + a\vec{P}. \tag{5.83}$$

Furthermore if the entropy function has a unique extremum then the near horizon field configuration must also remain invariant under the transformation (5.82).

5.2.1 The black hole solution

Ref.[51] constructed rotating black hole solutions in this theory carrying the following charges:

$$Q = \begin{pmatrix} 0 \\ Q_2 \\ 0 \\ Q_4 \end{pmatrix}, \qquad P = \begin{pmatrix} P_1 \\ 0 \\ P_3 \\ 0 \end{pmatrix}. \tag{5.84}$$

These black holes break all the supersymmetries of the theory. In order to describe the solution we parametrize the matrix valued scalar field M as

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{5.85}$$

where G and B are 2×2 matrices of the form

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & B_{12} \\ -B_{12} & 0 \end{pmatrix}. \tag{5.86}$$

Physically G and B represent components of the string metric and the anti-symmetric tensor field along an internal two dimensional torus. The solution is given by

$$\begin{aligned}
G_{11} &= \frac{(r + 2m\sinh^2\delta_4)(r + 2m\sinh^2\delta_2) + l^2\cos^2\theta}{(r + 2m\sinh^2\delta_3)(r + 2m\sinh^2\delta_2) + l^2\cos^2\theta}, \\
G_{12} &= \frac{2ml\cos\theta(\sinh\delta_3\cosh\delta_4\sinh\delta_1\cosh\delta_2 - \cosh\delta_3\sinh\delta_4\cosh\delta_1\sinh\delta_2)}{(r + 2m\sinh^2\delta_3)(r + 2m\sinh^2\delta_2) + l^2\cos^2\theta}, \\
G_{22} &= \frac{(r + 2m\sinh^2\delta_3)(r + 2m\sinh^2\delta_1) + l^2\cos^2\theta}{(r + 2m\sinh^2\delta_3)(r + 2m\sinh^2\delta_2) + l^2\cos^2\theta}, \\
B_{12} &= -\frac{2ml\cos\theta(\sinh\delta_3\cosh\delta_4\cosh\delta_1\sinh\delta_2 - \cosh\delta_3\sinh\delta_4\sinh\delta_1\cosh\delta_2)}{(r + 2m\sinh^2\delta_3)(r + 2m\sinh^2\delta_2) + l^2\cos^2\theta}, \\
Im\,S &= \frac{\Delta^{\frac{1}{2}}}{(r + 2m\sinh^2\delta_3)(r + 2m\sinh^2\delta_4) + l^2\cos^2\theta}, \\
ds^2 &= \Delta^{\frac{1}{2}}[-\frac{r^2 - 2mr + l^2\cos^2\theta}{\Delta}dt^2 + \frac{dr^2}{r^2 - 2mr + l^2} + d\theta^2 + \frac{\sin^2\theta}{\Delta}\{(r + 2m\sinh^2\delta_3)
\end{aligned}$$

$$
\begin{aligned}
\times & \quad (r+2m\sinh^2\delta_4)(r+2m\sinh^2\delta_1)(r+2m\sinh^2\delta_2)+l^2(1+\cos^2\theta)r^2+W \\
+ & \quad 2ml^2 r\sin^2\theta\}d\phi^2 - \frac{4ml}{\Delta}\{(\cosh\delta_3\cosh\delta_4\cosh\delta_1\cosh\delta_2 \\
- & \quad \sinh\delta_3\sinh\delta_4\sinh\delta_1\sinh\delta_2)r + 2m\sinh\delta_3\sinh\delta_4\sinh\delta_1\sinh\delta_2\}\sin^2\theta dt d\phi],
\end{aligned}
\tag{5.87}
$$

where

$$
\begin{aligned}
\Delta & \equiv (r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_4)(r+2m\sinh^2\delta_1)(r+2m\sinh^2\delta_2) \\
& + (2l^2r^2+W)\cos^2\theta, \\
W & \equiv 2ml^2(\sinh^2\delta_3+\sinh^2\delta_4+\sinh^2\delta_1+\sinh^2\delta_2)r \\
& + 4m^2l^2(2\cosh\delta_3\cosh\delta_4\cosh\delta_1\cosh\delta_2\sinh\delta_3\sinh\delta_4\sinh\delta_1\sinh\delta_2 \\
& - 2\sinh^2\delta_3\sinh^2\delta_4\sinh^2\delta_1\sinh^2\delta_2 - \sinh^2\delta_4\sinh^2\delta_1\sinh^2\delta_2 \\
& - \sinh^2\delta_3\sinh^2\delta_1\sinh^2\delta_2 - \sinh^2\delta_3\sinh^2\delta_4\sinh^2\delta_2 - \sinh^2\delta_3\sinh^2\delta_4\sinh^2\delta_1) \\
& + l^4\cos^2\theta.
\end{aligned}
\tag{5.88}
$$

a, m, δ_1, δ_2, δ_3 and δ_4 are parameters labelling the solution. Ref.[51] did not explicitly present the results for $Re\ S$ and the gauge fields.

The ADM mass M, electric and magnetic charges $\{Q_i, P_i\}$, and the angular momentum J are given by:[10]

$$
\begin{aligned}
M &= 8\pi\, m(\cosh^2\delta_1+\cosh^2\delta_2+\cosh^2\delta_3+\cosh^2\delta_4) - 16\pi\, m, \\
Q_2 &= 4\sqrt{2\pi}\, m\cosh\delta_1\sinh\delta_1, \qquad Q_4 = 4\sqrt{2\pi}\, m\cosh\delta_2\sinh\delta_2, \\
P_1 &= 4\sqrt{2\pi}\, m\cosh\delta_3\sinh\delta_3, \qquad P_3 = 4\sqrt{2\pi}\, m\cosh\delta_4\sinh\delta_4, \\
J &= -16\pi\, lm(\cosh\delta_1\cosh\delta_2\cosh\delta_3\cosh\delta_4 - \sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4).
\end{aligned}
\tag{5.89}
$$

The entropy associated with this solution was computed in [51] to be

$$
S_{BH} = 32\pi^2\Big[m^2(\prod_{i=1}^4\cosh\delta_i + \prod_{i=1}^4\sinh\delta_i) + m\sqrt{m^2-l^2}(\prod_{i=1}^4\cosh\delta_i - \prod_{i=1}^4\sinh\delta_i)\Big]. \tag{5.90}
$$

As in the case of Kaluza-Klein black hole this solution also has two different kinds of extremal limit which we shall denote by ergo-branch and ergo-free branch. The ergo-branch was discussed in [51].

[10] In defining M and J we have taken into account our convention $G_N = 16\pi$, and also the fact that our definition of the angular momentum differs from the standard one by a minus sign. Normalizations of $\vec{Q}$ and $\vec{P}$ are arbitrary at this stage.

5.2.2 The ergo-branch

The extremal limit corresponding to the ergo-branch is obtained by taking the limit $l \to m$. In this limit the second term in the expression for the entropy vanishes and the first term gives

$$S_{BH} = 2\pi \sqrt{J^2 + Q_2 Q_4 P_1 P_3}\,. \tag{5.91}$$

Now the most general transformation of the form (5.79) which does not take the charges given in (5.84) outside this family is:

$$\Omega = \begin{pmatrix} e^{\gamma} & 0 & 0 & 0 \\ 0 & e^{\beta} & 0 & 0 \\ 0 & 0 & e^{-\gamma} & 0 \\ 0 & 0 & 0 & e^{-\beta} \end{pmatrix}, \tag{5.92}$$

for real parameters γ, β. This gives

$$P_1 \to e^{-\gamma} P_1, \quad P_3 \to e^{\gamma} P_3, \quad Q_2 \to e^{\beta} Q_2, \quad Q_4 \to e^{-\beta} Q_4\,. \tag{5.93}$$

On the other hand most general transformation of the type (5.83) which keeps the charge vector within the same family is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}. \tag{5.94}$$

This gives

$$P_1 \to a\, P_1, \quad P_3 \to a\, P_3, \quad Q_2 \to a^{-1}\, Q_2, \quad Q_4 \to a^{-1}\, Q_4\,. \tag{5.95}$$

It is easy to see that the entropy given in (5.91) does not change under the transformations (5.93), (5.95).[11]

After some tedious manipulations along the lines described in section 5.1.5, the near horizon metric can be brought into the form given in eq.(3.3) with

$$\begin{aligned} \Omega &= \frac{1}{8\pi}\sqrt{J^2 + Q_2 Q_4 P_1 P_3}\, \sin\theta, \qquad e^{-2\psi} = \frac{1}{64\pi^2}\,(J^2 + Q_2 Q_4 P_1 P_3)\, \sin^2\theta\, \Delta^{-1/2}\,, \\ \alpha &= \frac{J}{\sqrt{J^2 + Q_2 Q_4 P_1 P_3}}\,, \end{aligned} \tag{5.96}$$

where Δ has to be evaluated on the horizon $r = m$. We have found that the near horizon metric and the scalar fields are not invariant under the corresponding transformations

[11] As in (5.48), (5.49), the parameters $\vec{P}$, $\vec{Q}$ are related to the charges $\vec{p}$, $\vec{q}$ by some overall normalization factors. These factors do not affect the transformation laws of the charges given in (5.93), (5.95).

(5.78) and (5.82) generated by the matrices (5.92) and (5.94) respectively, essentially due to the fact that Δ is not invariant under these transformations. This shows that in this case for a fixed set of charges the entropy function has a family of extrema.

5.2.3 The ergo-free branch

The extremal limit in the ergo-free branch is obtained by taking one or three of the δ_i's negative, and then taking the limit $|\delta_i| \to \infty$, $m \to 0$, $l \to 0$ in a way that keeps the Q_i, P_i and J finite. It is easy to see that in this limit the first term in the expression (5.90) for the entropy vanishes and the second term gives[12]

$$S_{BH} = 2\pi \sqrt{-J^2 - Q_2 Q_4 P_1 P_3}\,. \tag{5.97}$$

Again we see that S_{BH} is invariant under the transformations (5.93), (5.95).

On the ergo-free branch the horizon is at $r = 0$. The near horizon background can be computed easily from (5.87) following the approach described in section 5.1.3 and has the following form after appropriate rescaling of the time coordinate:

$$\begin{aligned} ds^2 &= \frac{1}{8\pi}\sqrt{-Q_2 Q_4 P_1 P_3 - J^2 \cos^2\theta}\left(-r^2 dt^2 + \frac{dr^2}{r^2} + d\theta^2\right) \\ &\quad + \frac{1}{8\pi}\frac{-Q_2 Q_4 P_1 P_3 - J^2}{\sqrt{-Q_2 Q_4 P_1 P_3 - J^2 \cos^2\theta}} \sin^2\theta \, (d\phi - \alpha r dt)^2 \,, \end{aligned} \tag{5.98}$$

$$ImS = \sqrt{-\frac{Q_2 Q_4}{P_1 P_3} - \frac{J^2 \cos^2\theta}{(P_1 P_3)^2}}\,, \tag{5.99}$$

$$G_{11} = \left|\frac{P_3}{P_1}\right|, \qquad G_{12} = -\frac{J\cos\theta}{P_1 Q_2}\left|\frac{Q_2}{Q_4}\right|, \qquad G_{22} = \left|\frac{Q_2}{Q_4}\right|, \qquad B_{12} = \frac{J\cos\theta}{P_1 Q_4}\,, \tag{5.100}$$

where

$$\alpha = -J/\sqrt{-Q_2 Q_4 P_1 P_3 - J^2}\,. \tag{5.101}$$

It is easy to see that the background is invariant under (5.78) and (5.82) for transformation matrices of the form described in (5.92) and (5.94).

[12] Note that the product $Q_2 Q_4 P_1 P_3$ is negative due to the fact that an odd number of δ_i's are negative.

5.2.4 Duality invariant form of the entropy

In the theory described here a combination of the charges that is invariant under both transformations (5.79) and (5.83) is

$$D \equiv (Q_1Q_3 + Q_2Q_4)(P_1P_3 + P_2P_4) - \frac{1}{4}(Q_1P_1 + Q_2P_2 + Q_3P_3 + Q_4P_4)^2 \,. \tag{5.102}$$

Thus we expect the entropy to depend on the charges through this combination. Now for the charge vectors given in (5.84) we have

$$D = Q_2Q_4P_1P_3 \,. \tag{5.103}$$

Using this result we can express the entropy formula (5.91) in the ergo-branch in the duality invariant form[51]:

$$S_{BH} = 2\pi\sqrt{J^2 + D} \,. \tag{5.104}$$

On the other hand the formula (5.97) on the ergo-free branch may be expressed as

$$S_{BH} = 2\pi\sqrt{-J^2 - D} \,. \tag{5.105}$$

We now note that the Kaluza-Klein black hole described in section (5.1) also falls into the general class of black holes discussed in this section with charges:

$$Q = \sqrt{2}\begin{pmatrix} Q \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad P = \sqrt{2}\begin{pmatrix} P \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.106}$$

Thus in this case

$$D = -P^2Q^2 \,. \tag{5.107}$$

We can now recognize the entropy formulæ (5.45) and (5.72) as special cases of (5.105) and (5.104) respectively.

Finally we can try to write down the near horizon metric on the ergo-free branch in a form that holds for the black hole solutions analyzed in this as well as in the previous subsection and which makes manifest the invariance of the background under arbitrary transformations of the form described in (5.78), (5.82). This is of the form:

$$\begin{aligned} ds^2 &= \frac{1}{8\pi}\sqrt{-D - J^2\cos^2\theta}\left(-r^2dt^2 + \frac{dr^2}{r^2} + d\theta^2\right) \\ &\quad + \frac{1}{8\pi}\frac{-D - J^2}{\sqrt{-D - J^2\cos^2\theta}}\sin^2\theta\,(d\phi - \alpha r dt)^2 \,, \end{aligned} \tag{5.108}$$

where

$$\alpha = -\frac{J}{\sqrt{-D - J^2}} \,. \tag{5.109}$$

(5.38) and (5.98) are special cases of this equation.

Acknowledgment: We would like to thank A. Dabholkar and S. Minwalla for useful discussions. This work was supported by Department of Atomic Energy, Government of India. SPT acknowledges further support from the Swarnajayanti Fellowship, DST, Government of India. DA and KG would like to thank ICTP, Trieste for hospitality. We thank the people of India for generously supporting research in String Theory.

References

[1] S. Ferrara, R. Kallosh and A. Strominger, "N=2 extremal black holes," Phys. Rev. D **52**, 5412 (1995) [arXiv:hep-th/9508072].

[2] A. Strominger, "Macroscopic Entropy of $N = 2$ Extremal Black Holes," Phys. Lett. B **383**, 39 (1996) [arXiv:hep-th/9602111].

[3] S. Ferrara and R. Kallosh, "Supersymmetry and Attractors," Phys. Rev. D **54**, 1514 (1996) [arXiv:hep-th/9602136].

[4] K. Goldstein, N. Iizuka, R. P. Jena and S. P. Trivedi, "Non-supersymmetric attractors," arXiv:hep-th/0507096.

[5] R. Kallosh, "New attractors," arXiv:hep-th/0510024.

[6] P. K. Tripathy and S. P. Trivedi, "Non-supersymmetric attractors in string theory," arXiv:hep-th/0511117.

[7] A. Giryavets, "New attractors and area codes," arXiv:hep-th/0511215.

[8] K. Goldstein, R. P. Jena, G. Mandal and S. P. Trivedi, "A C-Function For Non-Supersymmetric Attractors," arXiv:hep-th/0512138.

[9] R. Kallosh, N. Sivanandam and M. Soroush, "The non-BPS black hole attractor equation," arXiv:hep-th/0602005.

[10] R. Kallosh, "From BPS to Non-BPS Black Holes Canonically," arXiv:hep-th/0603003.

[11] P. Prester, "Lovelock type gravity and small black holes in heterotic string theory," JHEP **0602**, 039 (2006) [arXiv:hep-th/0511306].

[12] M. Alishahiha and H. Ebrahim, "Non-supersymmetric attractors and entropy function," arXiv:hep-th/0601016.

[13] A. Sinha and N. V. Suryanarayana, "Extremal single-charge small black holes: Entropy function analysis," arXiv:hep-th/0601183.

[14] B. Chandrasekhar, S. Parvizi, A. Tavanfar and H. Yavartanoo, "Non-supersymmetric attractors in R**2 gravities," arXiv:hep-th/0602022.

[15] J. M. Maldacena, A. Strominger and E. Witten, "Black hole entropy in M-theory," JHEP **9712**, 002 (1997) [arXiv:hep-th/9711053].

[16] K. Behrndt, G. Lopes Cardoso, B. de Wit, D. Lust, T. Mohaupt and W. A. Sabra, "Higher-order black-hole solutions in N = 2 supergravity and Calabi-Yau string backgrounds," Phys. Lett. B **429**, 289 (1998) [arXiv:hep-th/9801081].

[17] G. Lopes Cardoso, B. de Wit and T. Mohaupt, "Corrections to macroscopic supersymmetric black-hole entropy," Phys. Lett. B **451**, 309 (1999) [arXiv:hep-th/9812082].

[18] G. Lopes Cardoso, B. de Wit and T. Mohaupt, "Deviations from the area law for supersymmetric black holes," Fortsch. Phys. **48**, 49 (2000) [arXiv:hep-th/9904005].

[19] G. Lopes Cardoso, B. de Wit and T. Mohaupt, "Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes," Nucl. Phys. B **567**, 87 (2000) [arXiv:hep-th/9906094].

[20] G. Lopes Cardoso, B. de Wit and T. Mohaupt, "Area law corrections from state counting and supergravity," Class. Quant. Grav. **17**, 1007 (2000) [arXiv:hep-th/9910179].

[21] T. Mohaupt, "Black hole entropy, special geometry and strings," Fortsch. Phys. **49**, 3 (2001) [arXiv:hep-th/0007195].

[22] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, "Stationary BPS solutions in N = 2 supergravity with R**2 interactions," JHEP **0012**, 019 (2000) [arXiv:hep-th/0009234].

[23] G. L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, "Examples of stationary BPS solutions in N = 2 supergravity theories with R**2-interactions," Fortsch. Phys. **49**, 557 (2001) [arXiv:hep-th/0012232].

[24] A. Dabholkar, "Exact counting of black hole microstates," arXiv:hep-th/0409148.

[25] A. Dabholkar, R. Kallosh and A. Maloney, "A stringy cloak for a classical singularity," arXiv:hep-th/0410076.

[26] A. Sen, "How does a fundamental string stretch its horizon?," arXiv:hep-th/0411255.

[27] V. Hubeny, A. Maloney and M. Rangamani, "String-corrected black holes," arXiv:hep-th/0411272.

[28] D. Bak, S. Kim and S. J. Rey, "Exactly soluble BPS black holes in higher curvature N = 2 supergravity," arXiv:hep-th/0501014.

[29] P. Kraus and F. Larsen, "Microscopic black hole entropy in theories with higher derivatives," arXiv:hep-th/0506176.

[30] A. Sen, "Black hole entropy function and the attractor mechanism in higher derivative gravity," arXiv:hep-th/0506177.

[31] A. Sen, "Entropy function for heterotic black holes," arXiv:hep-th/0508042.

[32] P. Kraus and F. Larsen, arXiv:hep-th/0508218.

[33] B. Sahoo and A. Sen, "Higher derivative corrections to non-supersymmetric extremal black holes in N = 2 supergravity," arXiv:hep-th/0603149.

[34] P. Kraus and F. Larsen, "Attractors and black rings," Phys. Rev. D **72**, 024010 (2005) [arXiv:hep-th/0503219].

[35] B. Sahoo and A. Sen, "BTZ black hole with Chern-Simons and higher derivative terms," arXiv:hep-th/0601228.

[36] S. Parvizi and A. Tavanfar, "Partition function of non-supersymmetric black holes in the supergravity limit," arXiv:hep-th/0602292.

[37] M. Alishahiha and H. Ebrahim, "New attractor, entropy function and black hole partition function," arXiv:hep-th/0605279.

[38] S. Ferrara and M. Gunaydin, "Orbits and Attractors for N=2 Maxwell-Einstein Supergravity Theories in Five Dimensions" arXiv:hep-th/0606108.

[39] A. Ghodsi, "R**4 corrections to D1D5p black hole entropy from entropy function arXiv:hep-th/0604106.

[40] J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, "D-branes and spinning black holes," Phys. Lett. B **391**, 93 (1997) [arXiv:hep-th/9602065].

[41] R. Kallosh, A. Rajaraman and W. K. Wong, "Supersymmetric rotating black holes and attractors," Phys. Rev. D **55**, 3246 (1997) [arXiv:hep-th/9611094].

[42] W. Li and A. Strominger, "Supersymmetric probes in a rotating 5D attractor," arXiv:hep-th/0605139.

[43] J. M. Bardeen and G. T. Horowitz, "The extreme Kerr throat geometry: A vacuum analog of AdS(2) x S(2)," Phys. Rev. D **60**, 104030 (1999) [arXiv:hep-th/9905099].

[44] R. M. Wald, "Black hole entropy in the Noether charge," Phys. Rev. D **48**, 3427 (1993) [arXiv:gr-qc/9307038].

[45] T. Jacobson, G. Kang and R. C. Myers, "On black hole entropy," Phys. Rev. D **49**, 6587 (1994) [arXiv:gr-qc/9312023].

[46] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," Phys. Rev. D **50**, 846 (1994) [arXiv:gr-qc/9403028].

[47] T. Jacobson, G. Kang and R. C. Myers, "Black hole entropy in higher curvature gravity," arXiv:gr-qc/9502009.

[48] D. Rasheed, "The rotating dyonic black holes of kaluza-klein theory," *Nucl. Phys.* **B454** (1995) 379–401, [arXiv:hep-th/9505038].

[49] T. Matos and C. Mora, "Stationary dilatons with arbitrary electromagnetic field," *Class. Quant. Grav.* **14** (1997) 2331–2340, [arXiv:hep-th/9610013].

[50] F. Larsen, "Rotating kaluza-klein black holes," *Nucl. Phys.* **B575** (2000) 211–230, [arXiv:hep-th/9909102].

[51] M. Cvetic and D. Youm, "Entropy of Non-Extreme Charged Rotating Black Holes in String Theory," Phys. Rev. D **54**, 2612 (1996) [arXiv:hep-th/9603147].

[52] D. P. Jatkar, S. Mukherji and S. Panda, "Rotating Dyonic Black Holes in Heterotic String Theory," Phys. Lett. B **384**, 63 (1996) [arXiv:hep-th/9601118].