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Walls of Marginal Stability and Dyon Spectrum in $\mathcal{N} = 4$ Supersymmetric String Theories

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Abstract

The spectrum of quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric string theories can change as the asymptotic moduli cross walls of marginal stability on which the dyon can break apart into a pair of half BPS states. In this paper we classify these marginal stability walls and examine this phenomenon in the context of exact dyon spectrum found in a class of $\mathcal{N} = 4$ supersymmetric string theories. We argue that the dyon partition functions in different domains separated by marginal stability walls are the same, but the choice of integration contour needed for extracting the degeneracies from the partition function differ in these different regions. We also find that in the limit of large charges the change in the degeneracy is exponentially suppressed compared to the leading contribution. This is consistent with the fact that in the computation of black hole entropy we do not encounter any change as the asymptotic moduli fields move across the walls of marginal stability. Finally we carry out some tests of S-duality invariance in the theory.

Contents

1 Introduction and Summary

In a series of papers we computed the exact degeneracy of quarter BPS dyons in a class of $\mathcal{N} = 4$ supersymmetric string compactifications in an appropriate corner of the moduli space of these theories [1–3] verifying and generalizing earlier conjectures [4–6]. Alternative approaches to this problem leading to similar results have also been developed [4,7–10]. The result for the degeneracy takes the form of integration over a three real dimensional subspace (a contour) of the Siegel upper half plane parametrizing genus two Riemann surfaces, and the integrand involves inverse of a certain meromorphic modular form of a subgroup of the Siegel modular group.

It is well known however that for a dyon with a given set of charges the moduli space of $\mathcal{N} = 4$ supersymmetric string theory contains subspaces on which the original dyon becomes marginally unstable against decay into a pair of other dyons [11,12]. In particular the moduli space contains walls of marginal stability – codimension one subspaces – on which the mass of the quarter BPS dyon becomes equal to the sum of masses of a pair of half BPS dyons whose charges add up to that of the original quarter BPS dyon. On this subspace the original dyon becomes marginally unstable against decay into this pair of half BPS dyons, and typically the spectrum of the original dyon changes discontinuously as we move through these marginal stability walls in the moduli space [11, 12]. Thus an important question is: how does the dyon spectrum computed in [1–3] change as we move away from the particular corner of the moduli space in which the degeneracy was computed?

A glimpse of this issue was already seen in the analysis of [1–3] where it was found that even in the corner of the moduli space where the result was computed, – in a weakly coupled type IIB string theory compactified on a certain orbifold, – the result for the degeneracy changes discontinuously as the angle between certain pair of circles of the compact manifold passes through zero. This change could be attributed to the fact that precisely at this point the system under consideration became marginally stable against decay into a pair of half BPS states. However the change was such that the expression for the degeneracy continues to be given by a similar integral with identical integrand, but the contour over which the integral is to be performed gets changed. If we try to deform the new contour into the original contour we encounter a pole of the integrand and hence the two contributions differ by the residue at the pole. One can also interpret the result by saying that the dyon partition function formally remains the same as we move through the marginal stability wall, but the point in the Siegel moduli space around which we should series expand the partition function to extract the degeneracies changes as we move through these walls.

In this paper we classify these marginal stability walls in the moduli space of the theory and explore what happens when we move across these walls. Although the walls are complicated codimension one surfaces in the moduli space, one gets a simpler picture by regarding them as curves in the axion-dilaton moduli space for fixed values of the other moduli. We find that these curves are circles and straight lines, and could intersect on the real axis or at $i\infty$, but have no intersection in the interior of the upper half plane. Furthermore although the slopes of the lines and the radii and the centres of the circles depend on the charges and other moduli, the points where they intersect the real axis and each other are universal. As a result a region bounded by these curves has universal vertices but boundaries which depend on the other moduli and charges. One such region has been displayed in Fig.1 in §3. As we move from one of these regions to another, we cross marginal stability walls and as a result the dyon spectrum could change.

Typically most of these walls lie outside the domain in which the approximation made in our computation of the degeneracy can be trusted. However using T- and S-duality invariance of the theory we can extract useful information about these walls. A useful input in reaching this conclusion is the observation made in [13] that under an S-duality transformation the degeneracy formula does not remain invariant, but there is a change in the integration contour. This can be attributed to the fact that a duality transformation acts both on the charges as well as the asymptotic values of the moduli fields, whereas the degeneracy formula is computed for different charges but in the same region of the moduli space, in a sense that will be made precise in §4. The apparent lack of duality symmetry of the degeneracy formula is due to the fact that the region in which the degeneracy formula is calculated and the region obtained after a duality transformation are typically separated by walls of marginal stability. Thus the knowledge of how the contour changes under a duality transformation can be used to extract information about how it changes from one region of the moduli space to another as we move through these marginal stability walls. This way the change in the spectrum across any wall of marginal stability can be encoded as the result of changing the integration contour leaving the integrand unchanged, or equivalently as shifting the point around which we expand the partition function to extract the degeneracy.

One can try to find a physical interpretation of these changes in the degeneracy by explicitly evaluating the residues at the poles picked up by the contour as we move it from the position associated with one side of a wall of marginal stability to that on the other side of the wall. We can do this easily in the context of the marginal stability wall found in the original analysis of [1]. We find that the change in the degeneracy is proportional to the total degeneracy of the half-BPS electric and magnetic states into which the original dyon can decay on this particular marginal stability wall.

Given these results, it is natural to ask what happens in the large charge limit where the statistical entropy given by the logarithm of the degeneracy can be compared with the black hole entropy. We find that in this limit the change in the degeneracy as we move across a wall of marginal stability, – encoded in the residues at the poles which we encounter while deforming the new contour to the old one, – is exponentially suppressed relative to the leading contribution to the degeneracy. As a result the change in the statistical entropy as we move across the wall of marginal stability is exponentially suppressed. This is consistent with the fact that the entropy of the corresponding black hole is controlled solely by the charges and is independent of the asymptotic moduli due to attractor mechanism. Thus as the asymptotic values of the moduli fields move across the wall of marginal stability the answer for the black hole entropy does not change. Our results indicate however that if we are able to incorporate non-perturbative (in inverse charges) corrections in the computation of the black hole entropy then we should see a dependence of the entropy on the asymptotic values of the moduli fields, possibly along the line of [14, 15].

Since we use duality invariance to find how the contour should be deformed as we move across various walls of marginal stability, one could ask if there is any non-trivial test of duality that one could perform. If we could identify duality transformations which leave a region invariant, – in a sense that will be made precise in §4, – then the expression for the degeneracy should not change under such a duality transformation. This requires that under such a duality transformation either the contour should remain unchanged or it should move to a new position such that in deforming it from the new to the old position we do not encounter a pole in the integrand. This can then be explicitly tested. There is also a possibility that a duality transformation maps one region to another such that the degeneracy in each of these regions can be computed directly. In this case we have an independent result of how the contour should change as we move from the first region to the second and this can then be compared with the predictions coming from duality. We identify some duality transformations of these types and carry out the required consistency checks.

The rest of the paper is organized as follows. In §2 we review the results of [1–3] about the degeneracy of quarter BPS dyons in a class of $\mathcal{N} = 4$ supersymmetric string theories. In §3 we determine the locations of the walls of marginal stability in $\mathcal{N} = 4$ supersymmetric string theories. In §4 we determine how the different domains of the moduli space, bounded by the marginal stability walls, are mapped to each other under T- and S-duality symmetries of the theory and use this information to determine how the spectrum should change as we pass through a particular marginal stability wall. In §5 we show that the change in the statistical entropy as we move across the marginal stability walls is non-leading compared to the full entropy. In §6 we perform some tests of S-duality invariance of the theory.

Finally we would like to remind the reader that our analysis will focus on the marginal stability walls associated with decay of the dyon into a pair of half-BPS states. It will be worth exploring if there are interesting phenomena associated with decay into a pair of quarter BPS states.

Some related issues have been discussed in [13].

2 Dyon Spectrum in a Class of $\mathcal{N} = 4$ Supersymmetric Models

In this section we shall review the dyon spectrum in a class of $\mathcal{N} = 4$ supersymmetric models analyzed in [1–3, 5, 6]. Somewhat different approaches leading to similar results have been developed in [4, 7–10].

These theories are constructed by taking a $\mathbb{Z}_N$ orbifold of type IIB string theory on $\mathcal{M} \times S^1 \times \widetilde{S}^1$ where $\mathcal{M}$ is either K3 or T^4. The generator g of the $\mathbb{Z}_N$ group involves $1/N$ unit of shift along the circle S^1 together with an order N transformation $\widetilde{g}$ in $\mathcal{M}$. $\widetilde{g}$ is chosen so that it commutes with an $\mathcal{N} = 4$ supersymmetry algebra of the parent theory. Thus the final theory has $\mathcal{N} = 4$ supersymmetry.

The description of the theory given above will be referred to as the first description of the theory. Another useful description is obtained by a series of duality transformations. We first make an S-duality transformation in the type IIB theory. Next we make an $R \to 1/R$ duality on the circle $\widetilde{S}^1$ that takes type IIB string theory on $\mathcal{M} \times S^1 \times \widetilde{S}^1$ to type IIA string $\mathcal{M} \times S^1 \times \widehat{S}^1$ where $\widehat{S}^1$ is the circle dual to $\widetilde{S}^1$. Finally using six dimensional string-string duality we relate this to a heterotic string theory on $T^4 \times S^1 \times \widehat{S}^1$ for $\mathcal{M} = K3$ and type IIA string theory on $T^4 \times S^1 \times \widehat{S}^1$ for $\mathcal{M} = T^4$. Under this duality the transformation $\widetilde{g}$ gets mapped to a transformation $\widehat{g}$ that acts only as a shift on the right-moving degrees of freedom on the world-sheet and as a shift plus rotation on the left-moving degrees of freedom. In the final theory, obtained by taking the orbifold of heterotic or type IIA string theory on $T^4 \times S^1 \times \widehat{S}^1$ by a $1/N$ unit of shift along S^1 together with the transformation $\widehat{g}$, all the space-time supersymmetries come from the right-moving sector of the world-sheet. We shall call this the second description of the theory.

These theories typically have several moduli fields. In the second description one complex modulus scalar arises from the axion (a) - dilaton (ϕ) combination, where the axion by definition is the scalar field obtained by dualizing the NSNS sector 2-form field. We shall denote by τ the combination $a + iS$ where $S = e^{-2\phi}$. Other moduli fields arise from the $\widehat{g}$ invariant components of the metric, antisymmetric tensor field and gauge fields (in case of heterotic string theory) along the six dimensional internal torus and may be encoded in an $r \times r$ matrix valued field M satisfying

$$MLM^T = M, \quad M^T = M\,, \tag{2.1}$$

where L is a matrix with 6 eigenvalues $+1$ and $(r-6)$ eigenvalues -1. Here r is the rank of the gauge group and depends on the specific model being considered. These U(1) gauge fields arise from the $\widehat{g}$ invariant ten dimensional gauge fields (in case of heterotic string theory) as well as $\widehat{g}$ invariant components of the metric and the 2-form field with one leg along the internal torus and one leg along the non-compact directions.

Following the chain of dualities relating the first description to the second description one can work out the origin of the various fields in the first description. In particular the modulus $\tau = a + iS$ can be shown to correspond to the complex structure modulus of the torus spanned by the S^1 and $\widetilde{S}^1$ directions.

We shall now give the precise relationship between some of the moduli fields and the geometric quantities associated with the compactification in the second description and also give precise expression for some of the charges in terms of physical quantum numbers carried by a state. To do this we need to fix a normalization convention for the coordinates along S^1 and $\widehat{S}^1$. We choose x^4 and x^5 to be the coordinates along $\widehat{S}^1$ and S^1 respectively and choose their periods before orbifolding to be $2\pi\sqrt{\alpha'}$ and $2\pi N\sqrt{\alpha'}$ respectively. Thus after orbifolding both can be regarded as having period $2\pi\sqrt{\alpha'}$; however since periodicity under a $2\pi\sqrt{\alpha'}$ translation action S^1 also involves an action of the generator $\widehat{g}$ of the internal $\mathbb{Z}_N$ symmetry group, the momentum along S^1, measured in units of $1/\sqrt{\alpha'}$, could be fractional, – some multiple of $1/N$, – if $\widehat{g}$ acts non-trivially on the state. A string will be said to carry one unit of winding along S^1 (or $\widetilde{S}^1$) if, as we go once around the string, the x^4 (x^5) coordinate shifts by $2\pi\sqrt{\alpha'}$. Thus an untwisted sector state whose coordinate along S^1 changes by multiples of $2\pi N\sqrt{\alpha'}$ will carry winding charge along S^1 in multiples of N, but twisted sector states can carry generic integer winding charges. A single H-monopole associated with S^1 will correspond to an array of NS 5-branes wrapped on $\widehat{S}^1 \times T^4$ and placed at intervals of $2\pi\sqrt{\alpha'}$ along S^1. Finally the original Kaluza-Klein monopole associated with S^1, represented by a Taub-NUT space with an asymptotic circle of radius $2\pi N\sqrt{\alpha'}$ along x^4, will develop a $\mathbb{Z}_N$ singularity at its centre after the orbifolding and has to be regarded as carrying N units of Kaluza-Klein monopole charge associated with S^1. Thus the Kaluza-Klein monopole charge associated with S^1 will be quantized in units of N. Similar definition can be given for the H and Kaluza-Klein monopole charges associated with $\widehat{S}^1$, but in this case the normalization is straightforward and both the charges are allowed to take arbitrary integer values.

Let x^μ ($0 \le \mu \le 3$) denote the coordinates along the non-compact coordinates. For our analysis it will be useful to study in detail a subsector of the theory in which we include only those gauge fields which are associated with the 4μ and 5μ components of the metric and the anti-symmetric tensor field, only those components of M which encode information about the components of the metric and the anti-symmetric tensor field along $S^1 \times \widehat{S}^1$, the axion-dilaton field, and the four dimensional metric. In this subsector there are four gauge fields $A^{(i)}_\mu$ $(1 \le i \le 4)$ and a 4×4 matrix valued field M satisfying

$$M^T = M, \quad MLM^T = L, \quad L \equiv \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} . \tag{2.2}$$

The fields $A^{(i)}_\mu$ and M are related to the ten dimensional string metric G_{MN} and 2-form field B_{MN} via the relations [16, 17]:

$$\begin{aligned}
&\widehat{G}_{mn} \equiv G^{(10)}_{mn}, \quad \widehat{B}_{mn} \equiv B^{(10)}_{mn}, \qquad m, n = 4, 5, \\
&M = \begin{pmatrix} \widehat{G}^{-1} & \widehat{G}^{-1} B \\ -\widehat{B}\widehat{G}^{-1} & \widehat{G} - \widehat{B}\widehat{G}^{-1}\widehat{B} \end{pmatrix} \\
&A^{(m-3)}_\mu = \frac{1}{2} (\widehat{G}^{-1})^{mn} G^{(10)}_{m\mu}, \quad A^{(m-1)}_\mu = \frac{1}{2} B^{(10)}_{m\mu} - \widehat{B}_{mn} A^{(m-3)}_\mu, \\
&\quad 4 \le m, n \le 5, \quad 0 \le \mu, \nu \le 3 .
\end{aligned} \tag{2.3}$$

A general dyonic state in the theory is characterized by an r dimensional electric charge vector $\vec{Q}$ and an r dimensional magnetic charge vector $\vec{P}$. However if we consider a dyon that is charged only under the gauge fields $A^{(i)}_\mu$ $(1 \le i \le 4)$ introduced in (2.3), the corresponding charge vectors can be taken to be four dimensional. If we consider a state with momentum $\widehat{n}$ and winding $-\widehat{w}$ along $\widehat{S}^1$, momentum n' and winding $-w'$ along S^1, Kaluza-Klein monopole charge $\widehat{N}$ and H-monopole charge $-\widehat{W}$ associated with $\widehat{S}^1$ and Kaluza-Klein monopole charge N' and H-monopole charge $-W'$ associated with S^1, then we define the four dimensional electric charge vector $\vec{Q}$ and magnetic charge vector $\vec{P}$ characterizing the state to be

$$Q = \begin{pmatrix} \widehat{n} \\ n' \\ \widehat{w} \\ w' \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{W} \\ W' \\ \widehat{N} \\ N' \end{pmatrix} . \tag{2.4}$$

From our earlier discussion it follows that n' is quantized in units of $1/N$, N' is quantized in units of N and all other quantum numbers appearing in (2.4) are quantized in integer units. The precise relation between Q, P and the electric and magnetic charges associated with the gauge fields $A^{(i)}_\mu$ has been derived in [18] by working in the $\alpha' = 16$ unit.

Restricted to this subspace the T-duality transformation of the theory is parametrized by a 4×4 matrix Ω preserving the charge lattice and satisfying

$$\Omega^T L \Omega = L \,. \tag{2.5}$$

The action of T-duality takes us from a charge vector $(\vec{Q}, \vec{P})$ to another charge vector $(\vec{Q}', \vec{P}')$ given by

$$\vec{P}' = (\Omega^T)^{-1} \vec{P}, \qquad \vec{Q}' = (\Omega^T)^{-1} \vec{Q} \,, \tag{2.6}$$

and the moduli field M to M' given by

$$M' = \Omega M \Omega^T \,, \tag{2.7}$$

leaving τ unchanged. From eqs.(2.5), (2.6) it follows that the following inner products are invariant under T-duality transformation

$$\begin{aligned} & Q^2 \equiv Q^T L Q = 2(\widehat{n}\widehat{w} + n'w'), \qquad P^2 \equiv P^T L P = 2(\widehat{N}\widehat{W} + N'W'), \\ & P \cdot Q \equiv P^T L Q = \widehat{N}\widehat{n} + \widehat{W}\widehat{w} + N'n' + W'w' \,. \end{aligned} \tag{2.8}$$

These combinations are independent of the moduli fields M. Using the moduli field we can construct more general T-duality invariant combinations like $Q^T M Q$, $P^T M P$ and $Q^T M P$. We shall make use of these quantities in §3.

In the full theory $\vec{Q}$ and $\vec{P}$ are r dimensional vectors and Ω and M are $r \times r$ matrices, and L is also an $r \times r$ matrix with 6 eigenvalues 1 and $(r-6)$ eigenvalues -1. However the form of eqs.(2.5)-(2.7) remains the same.

The theories under consideration also have S-duality symmetry which leaves the field M unchanged, changes the vector $(\vec{Q}, \vec{P})$ to another vector $(\vec{Q}'', \vec{P}'')$ via the formula

$$\begin{pmatrix} \vec{Q}'' \\ \vec{P}'' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1(N) \,, \tag{2.9}$$

and transforms $\tau = a + iS$ to

$$\tau'' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \,, \tag{2.10}$$

where the $\Gamma_1(N)$ group of matrices is defined by the conditions

$$\alpha\delta - \beta\gamma = 1, \qquad \alpha, \delta = 1 \text{ mod } N, \qquad \gamma = 0 \text{ mod } N, \qquad \beta \in \mathbb{Z} \,. \tag{2.11}$$

We now consider in the first description of this theory a configuration with a single D5-brane wrapped on $\mathcal{M} \times S^1$, Q_1 D1-branes wrapped on S^1, a single Kaluza-Klein monopole associated with the circle $\widetilde{S}^1$, momentum $-n/N$ along S^1 and momentum J along $\widetilde{S}^1$ [7]. Since a D5-brane wrapped on $\mathcal{M}$ carries, besides the D5-brane charge, $-\zeta$ units of D1-brane charge with ζ given by the Euler character of $\mathcal{M}$ divided by 24 [19], the net D1-brane charge carried by the system is $Q_1 - \zeta$. By following the chain of dualities described earlier and a suitable sign convention for the charges in the first description, we can map this to a configuration in the second description with momentum $-n/N$ along S^1, a single Kaluza-Klein monopole associated with $\widehat{S}^1$, $(-Q_1 + \zeta)$ NS 5-brane charge wrapped along $T^4 \times S^1$, J NS 5-brane charge wrapped along $T^4 \times \widehat{S}^1$ and unit fundamental string winding charge along S^1 [1]. In particular the Kaluza-Klein monopole charge associated with $\widetilde{S}^1$ in the first description gets mapped to the fundamental string winding number along S^1 in the second description and the D5-brane wrapped on $\mathcal{M} \times S^1$ in the first description gets mapped to Kaluza-Klein monopole charge associated with $\widehat{S}^1$ in the second description. Using (2.4) we see that this corresponds to the charge vectors[1]

$$Q = \begin{pmatrix} 0 \\ -n/N \\ 0 \\ -1 \end{pmatrix}, \quad P = \begin{pmatrix} Q_1 - \zeta \\ -J \\ 1 \\ 0 \end{pmatrix}. \tag{2.12}$$

This gives

$$Q^2 = 2n/N, \qquad P^2 = 2(Q_1 - \zeta), \qquad Q \cdot P = J\,. \tag{2.13}$$

The counting of states of this system in the weak coupling region of the first description was carried out in [1–3]. We shall now summarize the results of this analysis. We denote by $d(\vec{Q}, \vec{P})$ the number of bosonic minus fermionic quarter BPS supermultiplets carrying a given set of charges $(\vec{Q}, \vec{P})$, a supermultiplet being considered bosonic (fermionic) if it is obtained by tensoring the basic 64 dimensional quarter BPS supermultiplet, with helicity ranging from $-\frac{3}{2}$ to $\frac{3}{2}$, with a supersymmetry singlet bosonic (fermionic) state. Our result for $d(\vec{Q}, \vec{P})$ is

$$d(\vec{Q}, \vec{P}) = \frac{1}{N} \int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i (N\widetilde{\rho} Q^2 + \widetilde{\sigma} P^2/N + 2\widetilde{v} Q\cdot P)} \frac{1}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})}, \tag{2.14}$$

where $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space

[1]Recall that an NS 5-brane wrapped on $T^4 \times S^1$ represents an H-monopole associated with $\widehat{S}^1$ and an NS 5-brane wrapped on $T^4 \times \widehat{S}^1$ represents an H-monopole associated with S^1.

labelled by $(\widetilde{\rho}=\widetilde{\rho}_1+i\widetilde{\rho}_2, \widetilde{\sigma}=\widetilde{\sigma}_1+i\widetilde{\sigma}_2, \widetilde{v}=\widetilde{v}_1+i\widetilde{v}_2)$. $\mathcal{C}$ corresponds to the subspace:[2]

$$\begin{aligned} \widetilde{\rho}_2 = M_1, \quad \widetilde{\sigma}_2 = M_2, \quad \widetilde{v}_2 = -M_3, \\ 0 \le \widetilde{\rho}_1 \le 1, \quad 0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1\,, \end{aligned} \tag{2.15}$$

M_1, M_2 and M_3 being large but fixed positive numbers with $M_3 << M_1, M_2$, and

$$\begin{aligned} \widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) &= e^{2\pi i(\widetilde{\alpha}\widetilde{\rho}+\widetilde{\gamma}\widetilde{\sigma}+\widetilde{v})} \\ &\times \prod_{b=0}^{1}\prod_{r=0}^{N-1}\prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},l\in\mathbb{Z},j\in 2\mathbb{Z}+b\\ k',l\ge 0,j<0 \text{ for } k'=l=0}} \left(1-\exp\left(2\pi i(k'\widetilde{\sigma}+l\widetilde{\rho}+j\widetilde{v})\right)\right)^{\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c_b^{(r,s)}(4k'l-j^2)}\,. \end{aligned} \tag{2.16}$$

The coefficients $c_b^{(r,s)}(u)$, $\widetilde{\alpha}$, $\widetilde{\gamma}$ encode information about the spectrum of two dimensional superconformal σ-model with target space $\mathcal{M}$ and are defined as follows. First we define

$$F^{(r,s)}(\tau,z) \equiv \frac{1}{N} Tr_{RR;\widetilde{g}^r}\left(\widetilde{g}^s(-1)^{F_L+F_R} e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\bar{L}_0} e^{2\pi i F_L z}\right), \qquad 0\le r,s\le N-1\,, \tag{2.17}$$

where $Tr_{RR;\widetilde{g}^r}$ denotes trace over all the Ramond-Ramond (RR) sector states twisted by $\widetilde{g}^r$ in the SCFT described above, L_n, $\bar{L}_n$ denote the left- and right-moving Virasoro generators and F_L and F_R denote the world-sheet fermion numbers associated with left and right-moving sectors in this SCFT. In defining L_0 and $\bar{L}_0$ of a state we subtract $c_L/24$ and $c_R/24$ from the conformal weights of the corresponding operators so that the RR sector ground state has $L_0=\bar{L}_0=0$. Due to the insertion of $(-1)^{F_R}$ factor in the trace the contribution to $F^{(r,s)}$ comes only from the $\bar{L}_0=0$ states. As a result $F^{(r,s)}$ does not depend on $\bar{\tau}$. Furthermore, using the existence of an $SU(2)_L\times SU(2)_R$ R-symmetry current algebra in this theory one can show that $F^{(r,s)}(\tau,z)$ have expansions of the form

$$F^{(r,s)}(\tau,z) = \sum_{b=0}^{1}\sum_{\substack{j\in 2\mathbb{Z}+b, n\in\mathbb{Z}/N\\ 4n-j^2\ge -b^2}} c_b^{(r,s)}(4n-j^2) e^{2\pi i n\tau+2\pi i j z}\,, \tag{2.18}$$

[2]The sign of $\widetilde{v}_2$ chosen here differs from the ones used in [1–3]. As will be discussed later, this choice of sign is valid for a specific choice of sign of the axion field; for the other choice the sign of $\widetilde{v}_2$ needs to be reversed. The other difference is that we have put the $M_3 << M_1, M_2$ condition to make explicit the fact that while interpreting $d(\vec{Q},\vec{P})$ as the coefficients of the Fourier expansion of $1/\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$ as in eqs.(2.21), (2.22) we need to first carry out the expansion in powers of $e^{2\pi i\widetilde{\rho}}$ and $e^{2\pi i\widetilde{\sigma}/N}$ and then carry out the expansion in powers of $e^{-2\pi i\widetilde{v}}$. This was implicit in the results of [1–3] but was not stated explicitly.

for some coefficients $c_b^{(r,s)}(u)$. This defines the coefficients $c_b^{(r,s)}(u)$. We now define

$$Q_{r,s} = N \left(c_0^{(r,s)}(0) + 2\, c_1^{(r,s)}(-1) \right) , \tag{2.19}$$

$$\widetilde{\alpha} = \frac{1}{24N}\, Q_{0,0} - \frac{1}{2N} \sum_{s=1}^{N-1} Q_{0,s}\, \frac{e^{-2\pi i s/N}}{(1-e^{-2\pi i s/N})^2} , \qquad \widetilde{\gamma} = \frac{1}{24N}\, Q_{0,0} . \tag{2.20}$$

This defines all the coefficients appearing in (2.14).

As an alternative to (2.14), (2.15) we can express $d(\vec{Q}, \vec{P})$ as

$$d(\vec{Q}, \vec{P}) = g\left(\frac{N}{2} Q^2, \frac{1}{2\,N}\, P^2, Q\cdot P \right) , \tag{2.21}$$

where $g(m,n,p)$ are the coefficients of Fourier expansion of the function $1/\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$:

$$\frac{1}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})} = \sum_{m,n,p} g(m,n,p)\, e^{2\pi i (m\,\widetilde{\rho} + n\,\widetilde{\sigma} + p\,\widetilde{v})} . \tag{2.22}$$

Let us denote by G the group of 4×4 matrices generated by:

$$\begin{aligned}
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad ad - bc = 1, \quad c = 0 \bmod N, \quad a, d = 1 \bmod N \\
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \lambda, \mu \in \mathbb{Z},
\end{aligned} \tag{2.23}$$

and let $\widetilde{G}$ denote the group of 4×4 matrices satisfying the requirement that:

$$\widetilde{g} \in \widetilde{G} \quad \text{iff} \quad U^{-1}\,\widetilde{g} U \in G , \tag{2.24}$$

where[3]

$$U = \begin{pmatrix} 0 & 0 & 0 & -1/\sqrt{N} \\ -\sqrt{N} & 0 & 0 & 0 \\ 0 & \sqrt{N} & 0 & 0 \\ 0 & 0 & -1/\sqrt{N} & 0 \end{pmatrix} . \tag{2.25}$$

[3]Even though U contains factors of $\sqrt{N}$ the elements of $\widetilde{G}$ are integers due to the fact that for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, the elements of C are multiples of N [5].

An element $\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} \in \widetilde{G}$ induces a transformation on $(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ via the relations:

$$\begin{pmatrix} \widetilde{\rho}' & \widetilde{v}' \\ \widetilde{v}' & \widetilde{\sigma}' \end{pmatrix} = (\widetilde{A}\widetilde{\Omega} + \widetilde{B})(\widetilde{C}\widetilde{\Omega} + \widetilde{D})^{-1}, \qquad \widetilde{\Omega} \equiv \begin{pmatrix} \widetilde{\rho} & \widetilde{v} \\ \widetilde{v} & \widetilde{\sigma} \end{pmatrix}, \tag{2.26}$$

and one can show that $\widetilde{\Phi}$ transforms as a Siegel modular form of weight k under this transformation [3]:

$$\widetilde{\Phi}(\widetilde{\rho}', \widetilde{\sigma}', \widetilde{v}') = \det(\widetilde{C}\widetilde{\Omega} + \widetilde{D})^k \, \widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}), \tag{2.27}$$

where

$$k = \frac{1}{2} \sum_{s=0}^{N-1} c_0^{(0,s)}(0). \tag{2.28}$$

This finishes our review of the main results. However two points about the degeneracy formula given above need special mention. Eqs.(2.14) and (2.21) are equivalent only if the sum over m, n, p in (2.22) are convergent on the contour $\mathcal{C}$. For the choice of $\mathcal{C}$ given in (2.15) this requires that $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})^{-1}$ has a power series expansion in positive powers of $e^{2\pi i\widetilde{\rho}}$ and $e^{2\pi i\widetilde{\sigma}/N}$ except possibly for a finite number of negative powers, and that the coefficient of any given term in this double power series expansion has an expansion in positive powers of $e^{-2\pi i\widetilde{v}}$ except possibly for a finite number of negative powers. This in particular requires that the sum over m and n in (2.22) are bounded from below, and that for fixed m and n the sum over p is bounded from above. By examining the formula (2.16) for $\widetilde{\Phi}$ and the result that the coefficients $c_b^{(r,s)}(u)$ are non-zero only for $u \geq -b^2$, we can verify that with the exception of the contribution from the $k' = l = 0$ term in this product, the other terms, when expanded in a power series expansion in $e^{2\pi i\widetilde{\rho}}$, $e^{2\pi i\widetilde{\sigma}}$ and $e^{2\pi i\widetilde{v}}$, does have the form of (2.22) with p bounded from above (and below) for fixed m, n. However for the $k' = l = 0$ term, which gives a contribution $e^{-2\pi i\widetilde{v}}/(1-e^{-2\pi i\widetilde{v}})^2$, there is an ambiguity in carrying out the series expansion. We could either use the form given above and expand the denominator in a series expansion in $e^{-2\pi i\widetilde{v}}$ so that the criterion described above is satisfied, or express it as $e^{2\pi i\widetilde{v}}/(1-e^{2\pi i\widetilde{v}})^2$ and expand it in a series expansion in $e^{2\pi i\widetilde{v}}$ in which case the sum over p will be bounded from below rather than from above. One finds that depending on the angle between S^1 and $\widetilde{S}^1$, only one of these expansions produces the degeneracy correctly via (2.22) [1]. The physical spectrum actually changes as this angle passes through 90° since at this point the system is only marginally stable. On the other hand our degeneracy formula (2.14), (2.15) implicitly assumes that we have expanded this factor in powers of $e^{-2\pi i\widetilde{v}}$ since only in this case the sum over p in (2.22) is bounded from above for fixed m, n. Thus as it stands the expression for $d(\vec{Q},\vec{P})$ given in (2.14), (2.15) is valid for a specific range of values of the angle between S^1 and $\widetilde{S}^1$, which, in the second description of the system, corresponds to the sign of the axion field. For the other sign of the axion we need to carry out the integral over a different contour $\widehat{\mathcal{C}}$ defined as

$$\begin{aligned} \widetilde{\rho}_2 = M_1, \quad \widetilde{\sigma}_2 = M_2, \quad \widetilde{v}_2 = M_3, \\ 0 \le \widetilde{\rho}_1 \le 1, \quad 0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1\,, \end{aligned} \tag{2.29}$$

where M_1, M_2 and M_3 are large positive numbers with $M_3 << M_1, M_2$. In the convention that we shall be using in this paper, we need to use the contour $\mathcal{C}$ for positive sign of the axion and the contour $\widehat{\mathcal{C}}$ for negative sign of the axion.

It turns out that walls of marginal stability, – codimension one subspaces of the asymptotic moduli space on which the BPS mass of the system becomes equal to the sum of masses of two or more other states carrying the same total charge, – are quite generic for quarter BPS states in $\mathcal{N}=4$ supersymmetric string theories [11], and we expect the spectrum to change discontinuously as the asymptotic moduli fields pass through any of the walls of marginal stability.[4] Thus the expression for the degeneracy given in this section holds only in a finite region of the moduli space, bounded by the walls of marginal stability. This will be discussed in more detail in §3.

Another point about the formula (2.14) is that although it was derived for special charge vectors $\vec{Q}$, $\vec{P}$ described in (2.12), it has been expressed as a function of the T-duality invariant combinations P^2, Q^2 and $Q\cdot P$. Even though we expect T-duality to be a symmetry of the theory, it is not guaranteed that the formula written in this fashion hold for all charge vectors. First of all a T-duality transformation acts not only on the charges but also on the asymptotic moduli. Had the spectrum been independent of the asymptotic moduli, we could have demanded that the spectrum remains invariant under T-duality transformation of the charges. However if a T-duality transformation takes the asymptotic moduli fields across a wall of marginal stability, then all we can say is that the spectrum remains unchanged under a simultaneous T-duality transformation of the moduli fields and the charges, but if we are sitting at a fixed point in the moduli space then the spectrum is not invariant under T-duality transformation on the charges.

[4] If a state becomes marginally stable on a surface of codimension ≥ 2, then we can always move around this subspace in going from one point to another and hence the spectrum cannot change discontinuously.

Second point is that even if we ignore the issues related to the walls of marginal stability, two charge vectors carrying the same values of P^2, Q^2 and $Q \cdot P$ may not necessarily be related by a T-duality transformation.[5] In that case the degeneracy of states for these two charge vectors could be different. An example of this is that a state that carries fractional momentum along S^1 can never be related to a state carrying integer momentum along S^1, although they may carry same values of Q^2, P^2 and $Q \cdot P$. Both these issues, together with the S-duality transformation properties of the degeneracy formula, will be discussed in §4.

3 Walls of Marginal Stability

As has been briefly mentioned in §2, the degeneracy formula given in (2.14), (2.15) is expected to be valid within a certain region of the moduli space bounded by codimension one subspaces on which the BPS state under consideration becomes marginally stable. As we cross this subspace of the moduli space, the spectrum can change discontinuously. In this section we shall study in some detail the locations of these walls of marginal stability so that we can identify the region within which our degeneracy formula will remain valid.

Let us consider a state carrying electric charge $\vec{Q}$ and magnetic charge $\vec{P}$ and examine under what condition it can decay into a pair of half-BPS states. This happens when its mass is equal to the sum of the masses of a pair of half BPS states whose electric and magnetic charges add up to $\vec{Q}$ and $\vec{P}$ respectively. Since for half BPS states the electric and magnetic charges must be parallel, these pair of states must have charge vectors of the form $(a\vec{M}, c\vec{M})$ and $(b\vec{N}, d\vec{N})$ for some constants a, b, c, d and a pair of r-dimensional vectors $\vec{M}$, $\vec{N}$. We shall normalize $\vec{M}$, $\vec{N}$ such that

$$ad - bc = 1\,. \tag{3.1}$$

Then the requirement that the charges add up to $(\vec{Q}, \vec{P})$ gives

$$\vec{M} = d\vec{Q} - b\vec{P}, \qquad \vec{N} = -c\vec{Q} + a\vec{P}\,. \tag{3.2}$$

Thus the charges of the decay products are given by

$$(ad\vec{Q} - ab\vec{P}, cd\vec{Q} - cb\vec{P}) \quad \text{and} \quad (-bc\vec{Q} + ab\vec{P}, -cd\vec{Q} + ad\vec{P})\,. \tag{3.3}$$

[5] Generically they are related by a continuous T-duality transformation but only a discrete subgroup of this is a genuine symmetry of the theory.

Note that under the scale transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tag{3.4}$$

eqs.(3.1) and (3.3) remain unchanged. There is another discrete transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.5}$$

which leaves (3.1) unchanged and exchanges the two decay products in (3.3). A pair of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ related by (3.4) or (3.5) describe identical decay channels.

In order that the charge vectors of the decay products given in (3.3) satisfy the charge quantization rules we must ensure that $a\vec{M} = ad\vec{Q} - ab\vec{P}$ and $b\vec{N} = -bc\vec{Q} + ab\vec{P}$ belong to the lattice of electric charges and that $c\vec{M} = cd\vec{Q} - cb\vec{P}$ and $d\vec{N} = -cd\vec{Q} + ad\vec{P}$ belong to the lattice of magnetic charges. For the charge vectors $\vec{Q}$, $\vec{P}$ given in (2.12) this would require

$$ad, ab, bc \in \mathbb{Z}, \qquad cd \in N\,\mathbb{Z}\,. \tag{3.6}$$

The condition $cd \in N\,\mathbb{Z}$ comes from the requirement that $cd\vec{Q} - cb\vec{P}$ is an allowed magnetic charge. In particular for a $\vec{Q}$ of the form (2.12), a magnetic charge $cd\vec{Q}$ represents a state with Kaluza-Klein monopole charge $-cd$ associated with S^1. Since this charge is quantized in units of N, cd must be a multiple of N. We shall denote by $\mathcal{A}$ the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ subject to the equivalence relations (3.4), (3.5) and satisfying (3.1), (3.6).

It is instructive to determine the structure of the set $\mathcal{A}$. We shall first show that using the scale transformation (3.4) we can always choose a, b, c and d to be integers and furthermore the solution is unique for given ad, ab, bc and cd. Since ad, ab, bc and cd are all integers, we can express them as products of prime factors:

$$|ab| = \prod_i p_i^{r_i}, \qquad |cd| = \prod_i p_i^{s_i}, \qquad |ad| = \prod_i p_i^{u_i}, \qquad |bc| = \prod_i p_i^{v_i}, \tag{3.7}$$

where the product over i runs over the prime numbers p_i and r_i, s_i, u_i and v_i are non-negative integers satisfying

$$r_i + s_i = u_i + v_i \quad \forall\, i\,. \tag{3.8}$$

Furthermore since ad and bc differ by 1, they cannot have a common factor. This shows that either u_i or v_i must vanish:

$$u_i\, v_i = 0 \quad \forall\, i\,. \tag{3.9}$$

Let us now look for integer a, b, c and d satisfying (3.7). For this we use the ansatz:

$$|a| = \prod_i p_i^{a_i}, \qquad |b| = \prod_i p_i^{b_i}, \qquad |c| = \prod_i p_i^{c_i}, \qquad |d| = \prod_i p_i^{d_i}, \tag{3.10}$$

where a_i, b_i, c_i and d_i are non-negative integers. Eq.(3.7) now gives

$$a_i + b_i = r_i, \qquad c_i + d_i = s_i, \qquad a_i + d_i = u_i, \qquad b_i + c_i = v_i\,. \tag{3.11}$$

Now (3.9) tells us that for any given i either u_i or v_i (or both) are zero. If $u_i = 0$ then the only possible solution to (3.11) is

$$a_i = 0, \qquad d_i = 0, \qquad b_i = r_i, \qquad c_i = s_i\,. \tag{3.12}$$

On the other hand if $v_i = 0$ then we must have

$$b_i = 0, \qquad c_i = 0, \qquad a_i = r_i, \qquad d_i = s_i\,. \tag{3.13}$$

This gives a unique expression for $|a|$, $|b|$, $|c|$ and $|d|$ using prime factorization. Up to an overall factor of -1 which can be removed with the help of the scale transformation (3.4), we can determine the signs of a, b, c and d in terms of the signs of ab, cd, ad and bc.

The requirement that cd is a multiple of N implies that the prime factors of N are shared by c and d. In case N is prime either c or d must be a multiple of N. Using the freedom (3.5) we can ensure that c is a multiple of N. In this case the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ describe elements of $\Gamma_0(N)$ modulo multiplication by -1.[6] Since $\Gamma_0(2) = \Gamma_1(2)$, for $N = 2$ the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ may be identified as the elements of $\Gamma_1(N)$ modulo multiplication by -1. On the other hand using the freedom of multiplication by -1 we can convert any $\Gamma_0(3)$ matrix to a $\Gamma_1(3)$ matrix. Thus for $N = 3$ the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ may be chosen to be $\Gamma_1(3)$ matrices.

The case $N = 1$ with $\mathcal{M} = K3$, corresponding to heterotic string theory on T^6 in the second description, is somewhat special. In this case the set $\mathcal{A}$ consists of $PSL(2, \mathbb{Z})$ matrices subject to the equivalence relation (3.5).

We shall now determine the wall of marginal stability corresponding to the decay channel given in (3.3). Our starting point will be the formula for the mass $m(\vec{Q}, \vec{P})$ of a

[6] $\Gamma_0(N)$ contains matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying $ad - bc = 1$, $a, b, d \in \mathbb{Z}$, $c \in N\,\mathbb{Z}$.

BPS state carrying electric charge $\vec{Q}$ and magnetic charge $\vec{P}$ [20, 21]

$$\begin{aligned} m(\vec{Q},\vec{P})^2 &= \frac{1}{S_\infty}(Q-\bar\tau_\infty P)^T(M_\infty+L)(Q-\tau_\infty P) \\ &\quad +2\left[(Q^T(M_\infty+L)Q)(P^T(M_\infty+L)P)-(P^T(M_\infty+L)Q)^2\right]^{1/2}\,, \end{aligned} \tag{3.14}$$

where $\tau = a + iS$ and the subscript ∞ denotes asymptotic values of various fields. This expression is manifestly invariant under the T- and S-duality transformations described in eqs.(2.5)-(2.7) and (2.9)-(2.11). In order that the state $(\vec{Q},\vec{P})$ is marginally stable against decay into $(ad\vec{Q}-ab\vec{P}, cd\vec{Q}-cb\vec{P})$ and $(-bc\vec{Q}+ab\vec{P}, -cd\vec{Q}+ad\vec{P})$, we need

$$m(\vec{Q},\vec{P}) = m(ad\vec{Q}-ab\vec{P}, cd\vec{Q}-cb\vec{P}) + m(-bc\vec{Q}+ab\vec{P}, -cd\vec{Q}+ad\vec{P})\,. \tag{3.15}$$

Using (3.14), (3.15) and some tedious algebra, we arrive at the condition

$$\left(a_\infty - \frac{ad+bc}{2cd}\right)^2 + \left(S_\infty + \frac{E}{2cd}\right)^2 = \frac{1}{4c^2d^2}(1+E^2)\,, \tag{3.16}$$

where

$$E \equiv \frac{cd(Q^T(M_\infty+L)Q) + ab(P^T(M_\infty+L)P) - (ad+bc)(P^T(M_\infty+L)Q)}{[(Q^T(M_\infty+L)Q)(P^T(M_\infty+L)P)-(P^T(M_\infty+L)Q)^2]^{1/2}}\,. \tag{3.17}$$

Note that E depends on M_∞, the constants a, b, c, d and the charges $\vec{Q}$, $\vec{P}$, but is independent of τ_∞. Thus for fixed P, Q and M_∞, the wall of marginal stability describes a circle in the (a_∞, S_∞) plane with radius

$$R = \sqrt{1+E^2}/2|cd|\,, \tag{3.18}$$

and center at

$$C = \left(\frac{ad+bc}{2cd}, -\frac{E}{2cd}\right)\,. \tag{3.19}$$

This circle intersects the real τ_∞ axis at

$$a/c \quad \text{and} \quad b/d\,. \tag{3.20}$$

The cases where either c or d vanish require special attention. First consider the case $c = 0$. In this case the condition $ad - bc = 1$ implies that $a = d = 1$. By taking the $c \to 0$

limit of (3.16), (3.17) we see that the wall of marginal stability becomes a straight line in the (a_∞, S_∞) plane for a fixed M_∞:

$$a_\infty - \frac{b(P^T(M_\infty+L)P) - (P^T(M_\infty+L)Q)}{\left[(Q^T(M_\infty+L)Q)(P^T(M_\infty+L)P) - (P^T(M_\infty+L)Q)^2\right]^{1/2}} S_\infty - b = 0\,. \tag{3.21}$$

On the other hand for $d=0$ we have $bc=-1$ and we can choose $b=-1$, $c=1$. In the $d\to 0$ limit of (3.16), (3.17) we get another straight line

$$a_\infty - \frac{a(P^T(M_\infty+L)P) - (P^T(M_\infty+L)Q)}{\left[(Q^T(M_\infty+L)Q)(P^T(M_\infty+L)P) - (P^T(M_\infty+L)Q)^2\right]^{1/2}} S_\infty - a = 0\,. \tag{3.22}$$

We now notice that this has exactly the same form as (3.21) with b replaced by a. Thus these do not give rise to new walls of marginal stability. In fact the $c=0$ and $d=0$ cases are related by the equivalence relation (3.5).

In order to get some insight into the geometric structure of the domain bounded by these marginal stability walls it will be useful to study the possible intersection points of these walls in the upper half τ_∞ plane. Let us consider a pair of such walls characterized by the matrices $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. A convenient procedure for studying their intersection is to convert one of them (say the first one) to a straight line by an SL(2, $\mathbb{Z}$) transformation. We define

$$\tau'_\infty \equiv a'_\infty + iS'_\infty = \frac{d_1\tau - b_1}{-c_1\tau + a_1}\,. \tag{3.23}$$

Then it is easy to see that in the (a'_∞, S'_∞) plane the two walls get mapped to the curves

$$a'_\infty + \frac{\widetilde{P}^T(M_\infty+L)\widetilde{Q}}{\left[(\widetilde{Q}^T(M_\infty+L)\widetilde{Q})(\widetilde{P}^T(M_\infty+L)\widetilde{P}) - (\widetilde{P}^T(M_\infty+L)\widetilde{Q})^2\right]^{1/2}} S'_\infty = 0\,, \tag{3.24}$$

and

$$\left(a'_\infty - \frac{\widetilde{a}\widetilde{d}+\widetilde{b}\widetilde{c}}{2\widetilde{c}\widetilde{d}}\right)^2 + \left(S'_\infty + \frac{\widetilde{E}}{2\widetilde{c}\widetilde{d}}\right)^2 = \frac{1}{4\widetilde{c}^2\widetilde{d}^2}(1+\widetilde{E}^2)\,, \tag{3.25}$$

where

$$\vec{\widetilde{Q}} = d_1\vec{Q} - b_1\vec{P}, \qquad \vec{\widetilde{P}} = -c_1\vec{Q} + a_1\vec{P}\,, \tag{3.26}$$

$$\begin{pmatrix} \widetilde{a} & \widetilde{b} \\ \widetilde{c} & \widetilde{d} \end{pmatrix} = \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\,, \tag{3.27}$$

$$\widetilde{E} \equiv \frac{\widetilde{c}\widetilde{d}(\widetilde{Q}^T(M_\infty+L)\widetilde{Q}) + \widetilde{a}\widetilde{b}(\widetilde{P}^T(M_\infty+L)\widetilde{P}) - (\widetilde{a}\widetilde{d}+\widetilde{b}\widetilde{c})(\widetilde{P}^T(M_\infty+L)\widetilde{Q})}{\Big[(\widetilde{Q}^T(M_\infty+L)\widetilde{Q})(\widetilde{P}^T(M_\infty+L)\widetilde{P}) - (\widetilde{P}^T(M_\infty+L)\widetilde{Q})^2\Big]^{1/2}} \,. \qquad (3.28)$$

If either $\widetilde{c}$ or $\widetilde{d}$ vanishes i.e. if either $a_1/c_1 = a_2/c_2$ or $a_1/c_1 = b_2/d_2$ then (3.25) reduces to a straight line of the form (3.21) or (3.22) with P, Q, b (or a) replaced by $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{b}$ (or $\widetilde{a}$) respectively, and one finds that the only point of intersection of this line with (3.24) in the upper half τ'_∞ plane is at $i\infty$. In the τ_∞ plane this corresponds to the point a_1/c_1. If neither $\widetilde{c}$ nor $\widetilde{d}$ vanishes, then by eliminating a'_∞ from (3.24) and (3.25) we get

$$\begin{aligned}
&(S'_\infty)^2 \, \frac{\widetilde{Q}^T(M_\infty+L)\widetilde{Q}\; \widetilde{P}^T(M_\infty+L)\widetilde{P}}{(\widetilde{Q}^T(M_\infty+L)\widetilde{Q})(\widetilde{P}^T(M_\infty+L)\widetilde{P}) - (\widetilde{P}^T(M_\infty+L)\widetilde{Q})^2} \\
&+S'_\infty \, \frac{\widetilde{Q}^T(M_\infty+L)\widetilde{Q} + (\widetilde{a}\widetilde{b}/\widetilde{c}\widetilde{d})\widetilde{P}^T(M_\infty+L)\widetilde{P}}{\Big[(\widetilde{Q}^T(M_\infty+L)\widetilde{Q})(\widetilde{P}^T(M_\infty+L)\widetilde{P}) - (\widetilde{P}^T(M_\infty+L)\widetilde{Q})^2\Big]^{1/2}} \\
&+\frac{(\widetilde{a}\widetilde{d}+\widetilde{b}\widetilde{c})^2 - 1}{4\widetilde{c}^2\widetilde{d}^2} = 0 \,.
\end{aligned} \qquad (3.29)$$

Using the conditions $\widetilde{a}\widetilde{d} - \widetilde{b}\widetilde{c} = 1$ and $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \in \mathbb{Z}$ we see that $\widetilde{a}\widetilde{d}$ and $\widetilde{b}\widetilde{c}$ have same signs and also that $(\widetilde{a}\widetilde{d}+\widetilde{b}\widetilde{c})^2 \geq 1$. As a result for $S'_\infty \geq 0$, each term in the left hand side of (3.29) is non-negative and the only possible solution is

$$\widetilde{b} = 0 \quad \text{or} \quad \widetilde{a} = 0 \,, \qquad S'_\infty = 0 \,. \qquad (3.30)$$

Going back to the original variables using (3.23), (3.27) we see that these correspond to the following two cases:

$$\frac{b_1}{d_1} = \frac{a_2}{c_2} \quad \text{or} \quad \frac{b_1}{d_1} = \frac{b_2}{d_2}, \qquad \tau_\infty = \frac{b_1}{d_1} \,. \qquad (3.31)$$

Collecting all the cases together we see that the pair of circles characterized by the matrices $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ never intersect in the interior of the upper half plane and intersect on the real axis if and only if the sets

$$\left\{ \frac{a_1}{c_1}, \frac{b_1}{d_1} \right\} \quad \text{and} \quad \left\{ \frac{a_2}{c_2}, \frac{b_2}{d_2} \right\} \qquad (3.32)$$

have an overlap.

From this analysis we see that the set of marginal stability walls divides the upper half τ_∞ plane into many domains, with each domain bounded by a set of walls described by a set of matrices $\left\{\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\right\}$. Since the walls meet on the real axis or at $i\infty$, the regions have possible vertices on the real axis or at $i\infty$, but never in the interior of the upper half τ_∞ plane. We also note that while the shape of the walls in the τ_∞ plane depends on the charges and the moduli M_∞, their intersetion points are determined only in terms of the associated matrices $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$.

How do we determine the walls bordering a given domain? We shall illustrate this for the domain that includes the large S_∞ region and is bounded on the left (right) by the line (3.21) for $b = 0$ ($b = 1$). Since these lines intersect the real τ_∞ axis at 0 and 1 respectively, the domain is bounded from below by set of circle segments in the upper half plane described by (3.16), of which the first one begins at 0 and ends at some point P_1 on the positive real axis, the second one begins at P_1 and ends at some point P_2 to the right of P_1 etc. with the final segment ending at 1. The parameters a, b, c, d for each segment must be chosen such that given the starting point, it travels maximum possible distance to the right; any other segment that travels less distance will lie beneath the maximal segment and will not be of relevance for this computation. Thus in order to determine these segments we first need to study the following general question: given a rational number $p/q \geq 0$, what is the maximum distance a wall can travel given that it starts at p/q?

Let us first consider the case $p/q > 0$. We choose p, q to be relatively prime and $p > 0$, $q > 0$. Now eq.(3.20) shows that the circle (3.16) intersects the real axis at a/c and b/d. Thus either a/c or b/d should be equal to p/q. Without any loss of generality we can choose $a/c = p/q$ and hence $a = p$, $c = q$ up to an irrelevant overall sign. The constraint $ad - bc = 1$ now gives

$$\frac{b}{d} = \frac{p}{q} - \frac{1}{qd}\,. \tag{3.33}$$

We need to choose d so as to maximize the right hand side. Thus d must be chosen to be the maximum possible negative number subject to the requirement that $cd = qd$ is a multiple of N and $b = (pd - 1)/q$ is an integer.

In the special case where $p/q = 0$, we have $a/c = 0$ and hence $a = 0$. The condition $ad - bc = 1$ tells us that we have (up to an overall sign) $b = -1$, $c = 1$. The second intersection point $b/d = -1/d$ is maximized for maximum negative value of d subject to

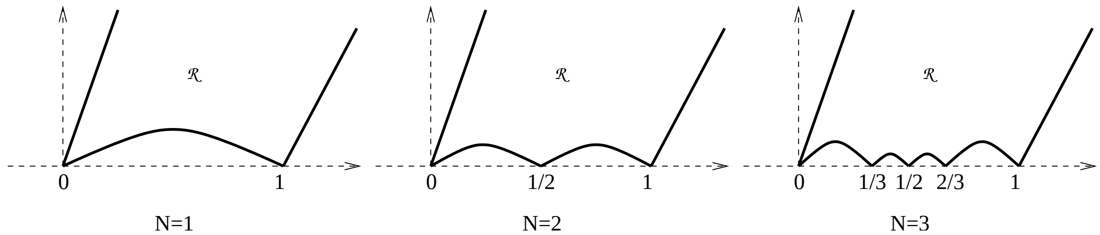

Figure 1: A schematic diagram representing the domain $\mathcal{R}$ in the upper half τ_∞ plane, bounded by the walls of marginal stability, for $\mathbb{Z}_1$, $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds. The shapes of the circles and the slopes of the straight lines bordering the domain depend on the charges and other asymptotic moduli, but the vertices are universal.

the requirement that cd is a multiple of N. This gives $d = -N$ and hence

$$\frac{b}{d} = \frac{1}{N}\,. \tag{3.34}$$

Following this rule we can now construct the circles which border the domain from below. For the purpose of illustration we shall carry out the first few steps. Using (3.34) we see that the first circle segment beginning at 0 ends at $1/N$. For $N = 1$ this completes the story since we have already reached the point 1. For $N \geq 2$ we need to proceed further. Taking $p/q = 1/N$ we see from (3.33) that $b = (d-1)/N$. Since $q = N$, the condition that qd is a multiple of N is trivially satisfied. Thus we need to choose d to be the maximum negative number for which $b = (d-1)/N$ is integer. This gives $d = -(N-1)$ and $b = -1$. Thus the circle ends at $b/d = 1/(N-1)$. For $N = 2$ this completes the story but for $N \geq 3$ we need to proceed further. At the next stage we begin with $p/q = 1/(N-1)$ and get $b = -(N-1)$, $d = -N(N-2)$. Thus $b/d = (N-1)/((N(N-2))$. This does not complete the story for any $N \geq 3$; *e.g.* for $N = 3$ this gives $b/d = 2/3$. By continuing this process one can show that for $N = 3$ we reach the point 1 at the next step via the wall corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$ but for higher N the story continues further. A schematic diagram representing these domains for $N = 1$, 2 and 3 have been shown in Fig. 1. As will be discussed in §6, for $N \geq 4$ the number of such walls is infinite.

This finishes our general analysis of marginal stability walls and domains bounded by them. Now we focus on one particular domain, – the one in which the degeneracy formula given in (2.14), (2.15) is valid. Since the calculation leading to (2.14), (2.15) was performed in a specific corner of the moduli space, – the weakly coupled type IIB string theory, – all we need to know is how this region is situated with respect to the various marginal stability walls described here. However to address this issue we need to determine the relation beween the moduli parameters in the first description, – as an orbifold of type IIB string theory, – and the moduli τ_∞, M_∞ appearing in the BPS mass formula (3.14). We have already stated that $\tau_\infty = a_\infty + iS_\infty$ denotes the asymptotic value of the complex structure of the torus $(S^1 \times \widetilde{S}^1)/\ \mathbb{Z}_N$. We have worked in a region of the moduli space where it is finite, i.e. S_∞ is neither too large nor too small. The relation between the other moduli fields in the first description and the matrix valued moduli field M can be found by following carefully the duality chain that takes the theory to its second description, and using the identification of M with the geometric quantities in this description as given in (2.3). Let us denote, in the first description of the theory, by g the ten dimensional coupling constant, by $(2\pi\sqrt{\alpha'})^4 V$ the volume of $\mathcal{M}$ measured in the string metric, and by $(2\pi\sqrt{\alpha'})^2 A$ the area of $S^1 \times \widetilde{S}^1/\ \mathbb{Z}_N$ measured in the string metric. Let us also set the anti-symmetric tensor field and all the Ramond-Ramond fields to zero. Then using the standard duality transformation rules one can show that the relevant 4×4 component part of the matrix M that couples to Q and P are given by

$$M_\infty = \begin{pmatrix} 1/V & & & \\ & g^2/(A^2V) & & \\ & & V & \\ & & & A^2V/g^2 \end{pmatrix}. \tag{3.35}$$

Using the expression for Q and P given in (2.12) one now gets

$$\begin{aligned} Q^T(M_\infty + L)Q &= \frac{g^2}{A^2V}\left(\frac{n}{N}\right)^2 + \frac{A^2V}{g^2} + 2\frac{n}{N}, \\ P^T(M_\infty + L)P &= \frac{(Q_1-\zeta)^2}{V} + \frac{g^2}{A^2V}J^2 + V + 2(Q_1-\zeta), \\ P^T(M_\infty + L)Q &= \frac{g^2}{A^2V}\frac{nJ}{N} + J\,. \end{aligned} \tag{3.36}$$

The computation of the degeneracy was done in a region where g is small but A and V are of order 1. Using (3.36) one can now see that in this region

$$Q^T(M_\infty + L)Q >> P^T(M_\infty + L)P,\ |P^T(M_\infty + L)Q|\,. \tag{3.37}$$

Even when we deform M_∞ away a little from its diagonal form (3.35), eq.(3.37) continues to hold.

We now study the implication of (3.37) on the location of the walls of marginal stability in the τ_∞ plane. First consider the case where $cd \neq 0$. In this case we see from (3.17), (3.37) that

$$\frac{E}{cd} \simeq \sqrt{\frac{Q^T(M_\infty + L)Q}{P^T(M_\infty + L)P}} >> 1\,. \tag{3.38}$$

Hence for the circle in the τ_∞ plane described by (3.16) we have

$$|S_\infty| < \frac{\sqrt{1+E^2}}{|2cd|} - \frac{E}{2cd} << 1\,. \tag{3.39}$$

In other words in the (a_∞, S_∞) plane the uppermost point on the circle (3.16) lies little above the $S_\infty = 0$ axis and its center lies deep down in the lower half plane. Since $S_\infty \sim 1$ in the region of the moduli space in which we have worked, we see that this region lies *above* all the circles described by (3.16) in the τ_∞ plane.

Next we consider the case $c = 0$, – as discussed earlier the $d = 0$ case is equivalent to this. In this case eq.(3.36) shows that the coefficient of S_∞ in (3.21) is small for finite b and small g. Thus for fixed M_∞ these lines are almost vertical in the (a_∞, S_∞) plane and are given by

$$a_\infty \simeq b\,. \tag{3.40}$$

The $b = 0$ line corresponds to the wall of marginal stability described at the end of §2, and as we described in §2, the degeneracy formula actually jumps across this wall.[7] Since our degeneracy formula has been derived in the small a_∞ region, we see that the region of validity of our formula is bounded by the $b = -1$ line on the left, $b = 1$ line on the right and a set of circle segments below.

To summarize, the region of the moduli space in which we have carried out our analysis consists of two domains. One of them, lying between the $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ line and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ line extends to the large S_∞ region, and is bounded from below by a set of circles. For later reference we shall call this the right domain $\mathcal{R}$ and denote by $\mathcal{B}_R$ the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ labelling the boundaries of this domain. The other domain,

[7] In the analysis of [1] the jump occured exactly across the line $a_\infty = 0$. This can be traced to the fact that there the analysis was carried out using weak coupling approximation and did not take into account the backreaction due to switching on J. Indeed for $J = 0$ we have $P^T(M_\infty + L)Q = 0$, and (3.21) for $b = 0$ reduces to the vertical line $a_\infty = 0$.

lying between the $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ line and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ line also extends to the large S_∞ region and is bounded from below by a set of circles. We shall call this the left domain $\mathcal{L}$, and denote by $\mathcal{B}_L$ the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ labelling the boundaries of this domain. We shall argue in §6 that the set $\mathcal{B}_L$ is obtained simply by multiplying the elements of $\mathcal{B}_R$ by the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ from the left.

One important issue that we would like to address is: how does the degeneracy formula change as we move across a wall of marginal stability? We already know that as we move across the wall (3.21) for $b = 0$ the degeneracy formula continues to be given by (2.14) except for a change in the location of the integration contour from $\mathcal{C}$ to $\widehat{\mathcal{C}}$. We shall argue in §4 that this is a general phenomenon; as we move across any line of marginal stability the degeneracy formula will be given by the same expression except for a change in the integration contour.

4 Duality Transformation of the Degeneracy Formula

As noted in §2, the degeneracy formula (2.14), (2.15) has been written in terms of T-duality invariant combinations Q^2, P^2 and $Q \cdot P$ although we have derived the formula only for a special class of charge vectors. In this section we shall discuss what information about the degeneracy formula can be extracted using the T- and S-duality symmetries of the theory.

We begin by studying the consequences of the T-duality symmetries of the theory. It follows from (2.5), (2.6) that if a T-duality transformation takes a charge vector $(\vec{Q}, \vec{P})$ to $(\vec{Q}', \vec{P}')$ then

$$Q'^2 = Q^2, \quad P'^2 = P^2, \quad Q' \cdot P' = Q \cdot P \,. \tag{4.1}$$

However there may be pairs of charge vectors with the same Q^2, P^2 and $Q \cdot P$ which are not related by a T-duality transformation. Clearly T-duality invariance of the theory cannot give us any relation between the degeneracies associated with such a pair of charge vectors. In what follows we shall focus on charge vectors $(\vec{Q}', \vec{P}')$ which are in the same T-duality orbit of a charge vector $(\vec{Q}, \vec{P})$ for which we have derived (2.14).

We have denoted by $\mathcal{R}$ the right domain of the region of the moduli space described in §3 in which the original formula for $d(\vec{Q}, \vec{P})$ is valid. It is bounded by a set of marginal stability walls labelled by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{B}_R$. Let $\mathcal{R}'$ denote the image of $\mathcal{R}$ under the T-duality map. In this case we expect $d(\vec{Q}', \vec{P}')$ in the region $\mathcal{R}'$ to be equal to $d(\vec{Q}, \vec{P})$ given in (2.14):

$$\begin{aligned} d(\vec{Q}', \vec{P}') &= \frac{1}{N} \int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i(N\widetilde{\rho}Q^2 + \widetilde{\sigma}P^2/N + 2\widetilde{v}Q\cdot P)} \frac{1}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})}, \\ &= \frac{1}{N} \int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i(N\widetilde{\rho}Q'^2 + \widetilde{\sigma}P'^2 2/N + 2\widetilde{v}Q'\cdot P')} \frac{1}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})}, \end{aligned} \tag{4.2}$$

where $\mathcal{C}$ has been defined in (2.15). In going from the first to the second line of (4.2) we have used (4.1).

Let us now determine the region $\mathcal{R}'$. Since under a T-duality transformation $M \to \Omega M \Omega^T$, and since $\mathcal{R}'$ is the image of $\mathcal{R}$ under this map, $\mathcal{R}'$ is bounded by walls of marginal stability described in (3.16), (3.17) with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{B}_R$ and M_∞ in (3.17) replaced by $\Omega^{-1} M_\infty (\Omega^T)^{-1}$. Using (2.6) we see that this effectively replaces $(\vec{Q}, \vec{P})$ by $(\vec{Q}', \vec{P}')$ in (3.17). Thus $\mathcal{R}'$ is the region of the upper half plane bounded by the circles:

$$\left(a_\infty - \frac{ad+bc}{2cd}\right)^2 + \left(S_\infty + \frac{E'}{2cd}\right)^2 = \frac{1}{4c^2d^2}(1 + E'^2), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{B}_R, \tag{4.3}$$

where

$$E' \equiv \frac{cd(Q'^T(M_\infty + L)Q') + ab(P'^T(M_\infty + L)P') - (ad+bc)(P'^T(M_\infty + L)Q')}{[(Q'^T(M_\infty + L)Q')(P'^T(M_\infty + L)P') - (P'^T(M_\infty + L)Q')^2]^{1/2}}. \tag{4.4}$$

(4.2)-(4.4) are valid for any charge vector $(\vec{Q}', \vec{P}')$ which can be related to the charge vectors given in (2.12) via a T-duality transformation.

Next we shall analyze the consequences of S-duality symmetry. An S-duality transformation changes the vector $(\vec{Q}, \vec{P})$ to another vector $(\vec{Q}'', \vec{P}'')$ and τ to τ'' via the formulæ (2.9), (2.10). Thus if $\mathcal{R}''$ denotes the image of the region $\mathcal{R}$ under the map (2.10), then S-duality invariance implies that inside $\mathcal{R}''$ the degeneracy $d(\vec{Q}'', \vec{P}'')$ is given by the same expression (2.14) for $d(\vec{Q}, \vec{P})$:

$$d(\vec{Q}'', \vec{P}'') = \frac{1}{N} \int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i(N\widetilde{\rho}Q^2 + \widetilde{\sigma}P^2/N + 2\widetilde{v}Q\cdot P)} \frac{1}{\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})}. \tag{4.5}$$

We would like to express the right hand side of (4.5) in terms of the vectors $\vec{Q}''$ and $\vec{P}''$. For this we define

$$\begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix} = \begin{pmatrix} \delta & \gamma/N \\ \beta N & \alpha \end{pmatrix} \in \Gamma_1(N). \tag{4.6}$$

and

$$\begin{pmatrix} \widetilde{\rho}'' \\ \widetilde{\sigma}'' \\ \widetilde{v}'' \end{pmatrix} \equiv \begin{pmatrix} \widetilde{\rho}_1'' + i\widetilde{\rho}_2'' \\ \widetilde{\sigma}_1'' + i\widetilde{\sigma}_2'' \\ \widetilde{v}_1'' + i\widetilde{v}_2'' \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}^2 & \tilde{\beta}^2 & -2\tilde{\alpha}\tilde{\beta} \\ \tilde{\gamma}^2 & \tilde{\delta}^2 & -2\tilde{\gamma}\tilde{\delta} \\ -\tilde{\alpha}\tilde{\gamma} & -\tilde{\beta}\tilde{\delta} & (\tilde{\alpha}\tilde{\delta} + \tilde{\beta}\tilde{\gamma}) \end{pmatrix} \begin{pmatrix} \widetilde{\rho} \\ \widetilde{\sigma} \\ \widetilde{v} \end{pmatrix} . \tag{4.7}$$

Using (2.9), (4.6), (4.7) one can easily verify that

$$e^{-\pi i(N\widetilde{\rho}Q^2 + \widetilde{\sigma}P^2/N + 2\widetilde{v}Q\cdot P)} = e^{-\pi i(N\widetilde{\rho}''Q''^2 + \widetilde{\sigma}''P''^2/N + 2\widetilde{v}''Q''\cdot P'')} , \tag{4.8}$$

and

$$d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v} = d\widetilde{\rho}''\, d\widetilde{\sigma}''\, d\widetilde{v}'' . \tag{4.9}$$

Furthermore, with the help of eq.(2.27) one can show that [3]

$$\widetilde{\Phi}(\widetilde{\rho}'', \widetilde{\sigma}'', \widetilde{v}'') = \widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) . \tag{4.10}$$

If $\mathcal{C}''$ denotes the image of $\mathcal{C}$ under the map (4.7) then eqs.(4.8)-(4.10) allow us to express (4.5) as

$$d(\vec{Q}'', \vec{P}'') = \frac{1}{N} \int_{\mathcal{C}''} d\widetilde{\rho}''\, d\widetilde{\sigma}''\, d\widetilde{v}''\, e^{-\pi i(N\widetilde{\rho}''Q''^2 + \widetilde{\sigma}''P''^2/N + 2\widetilde{v}Q''\cdot P'')} \frac{1}{\widetilde{\Phi}(\widetilde{\rho}'', \widetilde{\sigma}'', \widetilde{v}'')} . \tag{4.11}$$

To find the location of $\mathcal{C}''$ we note that under the map (4.7) the real parts of $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$ mix among themselves and the imaginary parts of $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$ mix among themselves. The initial contour $\mathcal{C}$ corresponded to a unit cell of the cubic lattice in the ($\widetilde{\rho}_1$,$\widetilde{\sigma}_1$,$\widetilde{v}_1$) space spanned by the basis vectors $(1, 0, 0)$, $(0, N, 0)$ and $(0, 0, 1)$. The unimodular map (4.7) transforms this into a different unit cell of the same lattice. We can now use the shift symmetries

$$\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = \widetilde{\Phi}(\widetilde{\rho}+1, \widetilde{\sigma}, \widetilde{v}) = \widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}+N, \widetilde{v}) = \widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}+1) , \tag{4.12}$$

which are manifest from (2.16), to bring the integration region back to the original unit cell. Thus $\mathcal{C}''$ and $\mathcal{C}$ differ only in the values of the imaginary parts of $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$. Using (2.15), (4.7) we see that for the contour $\mathcal{C}''$,

$$\begin{aligned} \widetilde{\rho}_2'' &= \tilde{\alpha}^2\, M_1 + \tilde{\beta}^2\, M_2 + 2\tilde{\alpha}\tilde{\beta}\, M_3 , \\ \widetilde{\sigma}_2'' &= \tilde{\gamma}^2\, M_1 + \tilde{\delta}^2\, M_2 + 2\tilde{\gamma}\tilde{\delta}\, M_3 , \\ v_2'' &= -\tilde{\alpha}\tilde{\gamma}\, M_1 - \tilde{\beta}\tilde{\delta}\, M_2 - (\tilde{\alpha}\tilde{\delta} + \tilde{\beta}\tilde{\gamma})\, M_3 . \end{aligned} \tag{4.13}$$

Thus $\mathcal{C}''$ is not identical to $\mathcal{C}$, – a fact first noticed in [13]. We could try to deform $\mathcal{C}''$ back to $\mathcal{C}$, but in that process we might pick up contribution from the residues at the poles of $\widetilde{\Phi}(\widetilde{\rho}'', \widetilde{\sigma}'', \widetilde{v}'')$. Thus we see that the degeneracy formula (4.11) for $d(\vec{Q}'', \vec{P}'')$ is not obtained by simply replacing $(\vec{Q}, \vec{P})$ by $(\vec{Q}'', \vec{P}'')$ in the expression for $d(\vec{Q}, \vec{P})$. The integration contour $\mathcal{C}$ also gets deformed to a new contour $\mathcal{C}''$.

Let us now analyze the region $\mathcal{R}''$ of the asymptotic moduli space in which (4.11) is valid. This is obtained by taking the image of the region $\mathcal{R}$ under the transformation (2.10). To determine this region we need to first study the images of the curves described in (3.16) in the $a_\infty - S_\infty$ plane. A straightforward analysis shows that the image of (3.16) is described by the curve

$$\left(a_\infty - \frac{a''d'' + b''c''}{2c''d''}\right)^2 + \left(S_\infty + \frac{E''}{2c''d''}\right)^2 = \frac{1}{4c''^2 d''^2}(1 + E''^2)\,, \tag{4.14}$$

where

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \tag{4.15}$$

and

$$E'' \equiv \frac{c''d''(Q''^T(M_\infty + L)Q'') + a''b''(P''^T(M_\infty + L)P'') - (a''d'' + b''c'')(P''^T(M_\infty + L)Q'')}{[(Q''^T(M_\infty + L)Q'')(P''^T(M_\infty + L)P'') - (P''^T(M_\infty + L)Q'')^2]^{1/2}}\,. \tag{4.16}$$

It can be easily seen that $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$ satisfy the relations (3.1), (3.6), and the equivalence relations (3.4), (3.5) translate to identical equivalence relations on $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$. Thus the collection of all the matrices $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$ describes same set $\mathcal{A}$ as the collection of all the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Now recall that the original region $\mathcal{R}$ was bounded by a set of marginal stability walls $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\} \in \mathcal{B}_R$. Thus the region $\mathcal{R}''$ is bounded by the collection of walls described by (4.14), (4.16) with $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mathcal{B}_R$.

At this stage it will be instructive to compare the expressions for the degeneracies $d(\vec{Q}, \vec{P})$ and $d(\vec{Q}'', \vec{P}'')$. There are two key differences. First of all although both are represented by similar looking integrals, the contours of integration are different in the two cases. Second the region of validity $\mathcal{R}''$ of the expression for $d(\vec{Q}, \vec{P})$ is not obtained by simple replacement of $(\vec{Q}, \vec{P}) \to (\vec{Q}'', \vec{P}'')$ in the expression for $d(\vec{Q}, \vec{P})$; the original region of validity of our formula was bounded by a set of marginal stability walls corresponding to a set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{B}_R$, whereas the new region of validity of the formula is bounded by a set of walls corresponding to the matrices $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mathcal{B}_R$.

Typically the new charge vectors $(\vec{Q}'', \vec{P}'')$ are not of the type (2.12), but in some cases it is possible to obtain the charge vectors $(\vec{Q}'', \vec{P}'')$ by T-duality transformation of another charge vector $(\vec{\widetilde{Q}}, \vec{\widetilde{P}})$ of the form given in (2.12) satisfying

$$\widetilde{Q}^2 = Q''^2, \qquad \widetilde{P}^2 = P''^2, \qquad \widetilde{Q} \cdot \widetilde{P} = Q'' \cdot P''\,. \tag{4.17}$$

Then for $d(\vec{\widetilde{Q}}, \vec{\widetilde{P}})$ our original formula for degeneracy holds. Using the results given in (4.2)-(4.4) we can now conclude that in the region $\widetilde{\mathcal{R}}''$ bounded by the walls (4.16) with

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in \mathcal{B}_R\,, \tag{4.18}$$

the degeneracy $d(\vec{Q}'', \vec{P}'')$ is given by (4.11) with the contour $\mathcal{C}''$ replaced by $\mathcal{C}$.

Thus we now have expressions for $d(\vec{Q}'', \vec{P}'')$ in two different domains, – $\widetilde{\mathcal{R}}''$ and $\mathcal{R}''$. In both regions the degeneracy is given by an integral. The integrand in both cases are same, but in one case the integration contour is $\mathcal{C}$ while in the other case it is $\mathcal{C}''$. This shows that as we cross the walls of marginal stability to move from the region $\widetilde{\mathcal{R}}''$ to $\mathcal{R}''$ the expression for $d(\vec{Q}'', \vec{P}'')$ changes by a modification in the location of the contour of integration.

Even though the result was derived under the assumption that $(\vec{Q}'', \vec{P}'')$ can be related to a charge vector of the type (2.12) by T-duality, it is natural to assume that this phenomenon is more general. In particular, a natural postulate will be that *as we move from the domain $\mathcal{R}$ corresponding to the set $\mathcal{B}_R$ to another domain corresponding to the set $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mathcal{B}_R$, with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1(N)$, the expression for $d(\vec{Q}, \vec{P})$ gets modified by a replacement of the integration contour $\mathcal{C}$ to the contour $\mathcal{C}''$ corresponding to*

$$\begin{aligned}
\widetilde{\rho}_2 &= \tilde{\alpha}^2\, M_1 + \tilde{\beta}^2\, M_2 + 2\tilde{\alpha}\tilde{\beta}\, M_3\,, \\
\widetilde{\sigma}_2 &= \tilde{\gamma}^2\, M_1 + \tilde{\delta}^2\, M_2 + 2\tilde{\gamma}\tilde{\delta}\, M_3\,, \\
v_2 &= -\tilde{\alpha}\tilde{\gamma}\, M_1 - \tilde{\beta}\tilde{\delta}\, M_2 - (\tilde{\alpha}\tilde{\delta} + \tilde{\beta}\tilde{\gamma})\, M_3 \\
0 \le \widetilde{\rho}_1 \le 1, &\quad 0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1\,,
\end{aligned} \tag{4.19}$$

M_1, M_2 and M_3 being large but fixed positive numbers with $M_3 << M_1, M_2$ and $\widetilde{\alpha} = \delta$, $\widetilde{\beta} = \gamma/N$, $\widetilde{\gamma} = N\beta$, $\widetilde{\delta} = \alpha$.

Can every domain be related to $\mathcal{R}$ or $\mathcal{L}$ via S-duality transformation? If this is so then we can use duality invariance together with the information about the contours $\mathcal{C}$ and $\widehat{\mathcal{C}}$ appropriate for $\mathcal{R}$ and $\mathcal{L}$, – as given in (2.15) and (2.29), – to find out the location of the integration contour in every other domain. For $N = 1, 2, 3$ we can answer this question in the affirmative as follows. In these cases the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ labelling any wall can be represented by an element of $\Gamma_1(N)$ and hence can be related to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ via a duality transformation of the form (4.15). Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ describes the boundary between the domains $\mathcal{R}$ and $\mathcal{L}$, the two domains bordering the wall $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must be related to the domains $\mathcal{R}$ and $\mathcal{L}$ by the same duality transformation. Since this can be done for every wall of marginal stability, we see that every domain must be related to either $\mathcal{R}$ or $\mathcal{L}$ via a duality transformation.

For $N \geq 4$ not all the walls can be related to the wall $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ via duality transformation. Nevertheless it may still be possible to relate all the domains to $\mathcal{R}$ and $\mathcal{L}$ via duality transfomation, with the walls not related to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ being related to some other wall of $\mathcal{R}$ or $\mathcal{L}$. We shall not attempt to settle this issue here.

Given two contours on two sides of a marginal stability wall, if we try to deform one to the other then typically we shall encounter poles of the integrand, and the two results will differ by the sum of the residues at these poles. To get a physical insight into how much this change is as we move across a given wall, let us consider the wall corresponding to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is the wall separating the regions $\mathcal{R}$ and $\mathcal{L}$, and across this wall the original dyon becomes unstable against decay into states with charges $(\vec{Q}, 0)$ and $(0, \vec{P})$. Since across this wall the contour changes from constant negative $\widetilde{v}_2$ to constant positive $\widetilde{v}_2$, we pick up the residue at the pole $\widetilde{v} = 0$. Now near $\widetilde{v} = 0$ the function $\widetilde{\Phi}$ has the behaviour [2, 3, 5, 6]

$$\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \simeq -4\pi^2 \widetilde{v}^2 \, f(N\widetilde{\rho}) g(\widetilde{\sigma}/N) + \mathcal{O}(\widetilde{v}^4) \, , \tag{4.20}$$

where $f(\tau)$ and $g(\tau)$ are two functions which have the interpretation of inverse of partition functions associated with electric and magnetic half BPS states of the theory. Performing the integral over $\widetilde{v}$ in (2.14) around the $\widetilde{v}=0$ point now gives

$$-(Q\cdot P)\left\{\int_0^1 d\widetilde{\rho} e^{-i\pi N\widetilde{\rho}Q^2}\left(f(N\widetilde{\rho})\right)^{-1}\right\}\left\{\frac{1}{N}\int_0^N d\widetilde{\sigma} e^{-i\pi\widetilde{\sigma}P^2/N}\left(g(\widetilde{\sigma}/N)\right)^{-1}\right\}. \tag{4.21}$$

This formula can be given a simple physical interpretation. The second and the third factors represent the degeneracies of the electric and magnetic half BPS states into which the original dyon decays on the marginal stability wall. The $Q\cdot P$ factor on the other hand is associated with the supersymmetric quantum mechanics describing the relative motion of the electric and the magnetic system [1]. In particular it represents the number of states whose binding energy vanishes as we reach the marginal stability wall [22, 23]. Thus this is the number of states which disappear from the spectrum as we cross the wall.

5 The Large Charge Limit

In this section we shall argue that even though the complete spectrum changes discontinuously when the asymptotic value of the axion field changes sign, the large charge expansion is not affected by this change. Our starting point is the result derived in [1–3] that the poles in the integrand in the expression for $d(\vec{Q},\vec{P})$ come from the second order zeroes of $\widetilde{\Phi}$ at[8]

$$\begin{aligned}
& n_2(\widetilde{\sigma}\widetilde{\rho}-\widetilde{v}^2)+j\widetilde{v}+n_1\widetilde{\sigma}-m_1\widetilde{\rho}+m_2=0\,,\\
& m_1n_1+m_2n_2+\frac{j^2}{4}=\frac{1}{4}\,,\\
& m_1\in N\,\mathbb{Z},\quad n_1,m_2,n_2\in\,\mathbb{Z},\quad j\in 2\,\mathbb{Z}+1\,.
\end{aligned} \tag{5.1}$$

For large charges the leading contribution to the degeneracy comes from the pole at $n_2=1$ [1–5, 24]. Contribution from the poles with $n_2\neq 1$ are exponentially suppressed compared to the leading contribution.

Consider now the contours $\mathcal{C}$ and $\mathcal{C}''$ given in (2.15) and (4.19) respectively. Both contours have the same range of integration over the real parts of $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$

$$0\le\widetilde{\rho}_1<1,\qquad 0\le\widetilde{\sigma}_1<N,\qquad 0\le\widetilde{v}<1\,. \tag{5.2}$$

[8]For $\mathcal{M}=K3$ this result can be found in appendix E of [1]. For the general case the set of all the zeroes and poles of $\widetilde{\Phi}$ were listed in [3], but we did not attempt to separate out the zeroes from the poles. A careful analysis shows that the only zeroes come from the set (5.1).

However they differ in the values of $\widetilde{\rho}_2$, $\widetilde{\sigma}_2$ and $\widetilde{v}_2$. We have for

$$\begin{aligned}
\mathcal{C} \quad &: \quad \widetilde{\rho}_2 = M_1, \quad \widetilde{\sigma}_2 = M_2, \quad \widetilde{v}_2 = -M_3\,, \\
\mathcal{C}'' \quad &: \quad \widetilde{\rho}_2 = \tilde{\alpha}^2\, M_1 + \tilde{\beta}^2\, M_2 + 2\tilde{\alpha}\tilde{\beta}\, M_3, \quad \widetilde{\sigma}_2 = \tilde{\gamma}^2\, M_1 + \tilde{\delta}^2\, M_2 + 2\tilde{\gamma}\tilde{\delta}\, M_3, \\
& \quad\ \ \widetilde{v}_2 = -\tilde{\alpha}\tilde{\gamma}\, M_1 - \tilde{\beta}\tilde{\delta}\, M_2 - (\tilde{\alpha}\tilde{\delta} + \tilde{\beta}\tilde{\gamma})\, M_3\,,
\end{aligned} \tag{5.3}$$

where M_1, M_2 and M_3 are fixed numbers with $M_1, M_2 >> 1$ and $|M_3| << M_1, M_2$. Our goal will be to show that we can deform the contour $\mathcal{C}$ to $\mathcal{C}''$ without hitting any of the poles given in (5.1) except those with $n_2 = 0$. Since for large charges the contribution to the degeneracy comes from the poles at $n_2 = 1$ up to exponentially suppressed terms, this will show that the change in the degeneracy due to the change in the contour of integration is exponentially suppressed compared to the leading contribution.

To proceed we shall choose a specific path along which we deform the contour. We take this to be along the straight line joining the points (M_1, M_2, M_3) and $(\tilde{\alpha}^2\, M_1 + \tilde{\beta}^2\, M_2 + 2\tilde{\alpha}\tilde{\beta}\, M_3, \tilde{\gamma}^2\, M_1 + \tilde{\delta}^2\, M_2 + 2\tilde{\gamma}\tilde{\delta}\, M_3, -\tilde{\alpha}\tilde{\gamma}\, M_1 - \tilde{\beta}\tilde{\delta}\, M_2 - (\tilde{\alpha}\tilde{\delta} + \tilde{\beta}\tilde{\gamma})\, M_3)$ in the $(\widetilde{\rho}_2, \widetilde{\sigma}_2, \widetilde{v}_2)$ space. Points on this line can be parametrized by a real number λ lying between 0 and 1, with

$$\begin{aligned}
\widetilde{\rho}_2 &= M_1 + \lambda\, \{(\tilde{\alpha}^2 - 1)\, M_1 + \tilde{\beta}^2\, M_2 + 2\tilde{\alpha}\tilde{\beta}\, M_3\} \\
\widetilde{\sigma}_2 &= M_2 + \lambda\, \{\tilde{\gamma}^2\, M_1 + (\tilde{\delta}^2 - 1)\, M_2 + 2\tilde{\gamma}\tilde{\delta}\, M_3\} \\
\widetilde{v}_2 &= -M_3 + \lambda\, \{-\tilde{\alpha}\tilde{\gamma}\, M_1 - \tilde{\beta}\tilde{\delta}\, M_2 - (\tilde{\alpha}\tilde{\delta} + \tilde{\beta}\tilde{\gamma} - 1)\, M_3\}\,.
\end{aligned} \tag{5.4}$$

For each λ we take the real parts of $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$ to be in the range (5.2).

We want to show that (5.1) and (5.4) have no simultaneous solutions except for $n_2 = 0$. For this we write down separately the real and imaginary parts of eq.(5.1), as well as the constraint on $\vec{m}$, $\vec{n}$, j:

$$\widetilde{\sigma}_2\widetilde{\rho}_2 - \widetilde{v}_2^2 - \widetilde{\sigma}_1\widetilde{\rho}_1 + \widetilde{v}_1^2 - \frac{j}{n_2}\widetilde{v}_1 - \frac{n_1}{n_2}\widetilde{\sigma}_1 + \frac{m_1}{n_2}\widetilde{\rho}_1 - \frac{m_2}{n_2} = 0\,, \tag{5.5}$$

$$n_2(\widetilde{\sigma}_1\widetilde{\rho}_2 + \widetilde{\sigma}_2\widetilde{\rho}_1 - 2\widetilde{v}_1\widetilde{v}_2) + j\widetilde{v}_2 + n_1\widetilde{\sigma}_2 - m_1\widetilde{\rho}_2 = 0\,, \tag{5.6}$$

$$m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,. \tag{5.7}$$

We now eliminate m_1 and m_2 using (5.6), (5.7) to write (5.5) as

$$(\widetilde{\sigma}_2\widetilde{\rho}_2 - \widetilde{v}_2^2) + \left[\begin{pmatrix} j & n_1 \end{pmatrix} A \begin{pmatrix} j \\ n_1 \end{pmatrix} + aj + bn_1 + c\right] = 0\,, \tag{5.8}$$

where

$$A = \frac{1}{4n_2^2}\begin{pmatrix} 1 & 2\widetilde{v}_2/\widetilde{\rho}_2 \\ 2\widetilde{v}_2/\widetilde{\rho}_2 & 4\widetilde{\sigma}_2/\widetilde{\rho}_2 \end{pmatrix}, \tag{5.9}$$

$$a = \frac{1}{n_2}\left(\frac{\widetilde{v}_2}{\widetilde{\rho}_2}\widetilde{\rho}_1 - \widetilde{v}_1\right), \quad b = 2\frac{\widetilde{\sigma}_2}{n_2\widetilde{\rho}_2}\left(\widetilde{\rho}_1 - \frac{\widetilde{v}_2}{\widetilde{\sigma}_2}\widetilde{v}_1\right), \quad c = v_1^2 + \widetilde{\rho}_1\left(\frac{\widetilde{\sigma}_2}{\widetilde{\rho}_2}\widetilde{\rho}_1 - 2\frac{\widetilde{v}_2}{\widetilde{\rho}_2}\widetilde{v}_1\right) - \frac{1}{4n_2^2}. \tag{5.10}$$

Now for our contour $\widetilde{\rho}_1$, $\widetilde{\sigma}_1$ and $\widetilde{v}_1$ are always finite. For M_1 and M_2 large and of the same order with $|M_3| << M_1, M_2$, the quantities $\widetilde{\rho}_2$ and $\widetilde{\sigma}_2$ given in (5.4) always remain large and positive, and the ratios $\widetilde{\sigma}_2/\widetilde{\rho}_2$, $\widetilde{v}_2/\widetilde{\rho}_2$, $\widetilde{v}_2/\widetilde{\sigma}_2$ etc. are bounded from above by finite numbers. Thus the quantities a, b and c remain finite. On the other hand in this limit we have

$$\widetilde{\rho}_2\widetilde{\sigma}_2 - \widetilde{v}_2^2 \simeq \lambda(1-\lambda)\left(\tilde{\gamma}^2 M_1^2 + \tilde{\beta}^2 M_2^2\right) + \left(\lambda^2 + (1-\lambda)^2 + \lambda(1-\lambda)(\tilde{\alpha}^2 + \tilde{\delta}^2)\right) M_1 M_2 \,. \tag{5.11}$$

In the range $0 \le \lambda \le 1$ each term in this expression is non-negative, and (5.11) remains large and positive, – of order M_1M_2, – in the limit of large M_1, M_2. This also shows that the matrix A defined in (5.9) is nondegenerate in this limit and in fact has finite positive eigenvalues. As a result the term in the square bracket in (5.8) is bounded from below by a finite number

$$c - \frac{1}{4}\begin{pmatrix} a & b \end{pmatrix} A^{-1}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{5.12}$$

and can never cancel the first term in (5.8) for any value of j and n_1 for sufficiently large M_1 and M_2. Hence (5.8) cannot be satisfied. This shows that it is impossible to find a similtaneous solution to the eqs.(5.1) and (5.4) for $n_2 \neq 0$, and hence the contour $\mathcal{C}$ can be deformed to $\mathcal{C}''$ along the path (5.4) without encountering the poles of the integrand given in (5.1) for $n_2 \neq 0$.

This establishes that in the limit of large charges the degeneracy remains the same in different domains in the moduli space up to nonperturbative terms. This result is consistent with the fact that for a black hole of charge $(\vec{Q}, \vec{P})$ in this theory we have a stable supersymmetric attractor for $P^2 > 0$, $Q^2 > 0$, $P^2Q^2 > (Q \cdot P)^2$. Thus the near horizon geometry of these black holes is always given by this attractor point and is independent of the asymptotic moduli even if this requires the attractor flow to cross one or more walls of marginal stability.

6 Test of S-duality Invariance

In §4 we used S-duality invariance to determine the locations of the integration contour in the degeneracy formula in different domains in the moduli space. However this did not provide a test of S-duality. In this section we shall describe some tests of S-duality that one could perform.

1. If there is an S-duality transformation that leaves the set $\mathcal{B}_R$ invariant, then under such a transformation the contour $\mathcal{C}$ either should not transform, or should transform to another contour that is continuously deformable to $\mathcal{C}$ without passing through any poles.

2. Analysis of [1] has shown that inside the left domain $\mathcal{L}$ corresponding to the set of matrices $\mathcal{B}_L$, the degeneracy is given by performing integration over the contour $\widehat{\mathcal{C}}$ described in (2.29). Thus if there is an S-duality transformation that maps the set $\mathcal{B}_R$ to the set $\mathcal{B}_L$ then such a transformation must map the contour $\mathcal{C}$ to the contour $\widehat{\mathcal{C}}$ or another contour deformable to $\widehat{\mathcal{C}}$ without passing through any pole.

In fact for all values of N there is an S-duality transformation that maps $\mathcal{B}_R$ to $\mathcal{B}_L$. It is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} . \tag{6.1}$$

To see this we note that it maps the $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}_R$ to $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}_L$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}_R$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}_L$. Since two domains sharing two common boundaries must be identical, the action of (6.1) must map $\mathcal{B}_R$ to $\mathcal{B}_L$. Using this transformation we can convert all tests of the first type into tests of the second type; all we need to do is to left multiply the duality transformation preserving $\mathcal{B}_R$ by (6.1) to construct a duality transformation that maps $\mathcal{B}_R$ to $\mathcal{B}_L$.

We shall now try to verify that the transformation (6.1) maps $\mathcal{C}$ to $\widehat{\mathcal{C}}$ or a contour deformable to $\widehat{\mathcal{C}}$ without passing through any poles. Using (4.6), (4.19) we see that this transformation maps the contour $\mathcal{C}$ given in (2.15) to

$$\widetilde{\rho}_2 = M_1, \qquad \widetilde{\sigma}_2 = N^2 M_1 + M_2 - 2NM_3, \qquad \widetilde{v}_2 = NM_1 - M_3 \, ,$$
$$0 \le \widetilde{\rho}_1 \le 1, \quad 0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1 \, . \tag{6.2}$$

This is different from $\widehat{\mathcal{C}}$ given in (2.29) for which $(\widetilde{\rho}_2, \widetilde{\sigma}_2, \widetilde{v}_2) = (M_1, M_2, M_3)$. Thus we need to verify that we can deform the contour (6.2) to $\widehat{\mathcal{C}}$ without encountering any pole. From the analysis in §5 we already know that the poles at (5.1) for $n_2 \neq 0$ are not encountered; thus we need to look for poles with $n_2 = 0$. They occur at

$$(j\widetilde{v} + n_1\widetilde{\sigma} - m_1\widetilde{\rho} + m_2) = 0\,, \tag{6.3}$$

with

$$m_1 \in N\,\mathbb{Z}, \quad n_1, m_2 \in \mathbb{Z}, \quad j \in 2\,\mathbb{Z}+1, \quad m_1 n_1 + \frac{j^2}{4} = \frac{1}{4}\,. \tag{6.4}$$

Taking the imaginary part of eq.(6.3) we get

$$j\widetilde{v}_2 + n_1\widetilde{\sigma}_2 - m_1\widetilde{\rho}_2 = 0\,. \tag{6.5}$$

For fixed j, n_1, m_1 this describes a plane in the $(\widetilde{\rho}_2, \widetilde{\sigma}_2, \widetilde{v}_2)$ space. Our job is to show that points in the $(\widetilde{\rho}_2, \widetilde{\sigma}_2, \widetilde{v}_2)$ space given in (2.29) and (6.2) lie on the same side of this plane so that we can deform them to each other without going through this plane. For this we need to show that the left-hand side of (6.5), evaluated at (2.29) and (6.2) have the same sign:

$$(jM_3 + n_1M_2 - m_1M_1)\left(j(NM_1 - M_3) + n_1(N^2M_1 + M_2 - 2NM_3) - m_1M_1\right) > 0\,. \tag{6.6}$$

We can simplify the analysis by setting $M_1 = M_2 = M$. In this case the left hand side of (6.6) has the form

$$\begin{aligned}
&(n_1 - m_1)\{Nj + (N^2+1)n_1 - m_1\}M^2 \\
&+\{(Nj + (N^2+1)n_1 - m_1)j - (n_1 - m_1)(j + 2Nn_1)\}\, M\, M_3 \\
&-j(j + 2Nn_1)M_3^2\,.
\end{aligned} \tag{6.7}$$

Using (6.4) the coefficient of the M^2 term can be brought to the form

$$\left(\frac{Nj}{2} + n_1 - m_1\right)^2 + N^2 n_1^2 - \frac{N^2}{4}\,. \tag{6.8}$$

If $n_1 \neq 0$ then this is strictly positive. If $n_1 = 0$ then from (6.4) we have $j = \pm 1$. We shall choose $j = 1$ by using the freedom of changing the signs of n_1, m_1, n_2, m_2 and j simultaneously without changing the location of the pole. Since $m_1 \in N\,\mathbb{Z}$, in this case (6.8) is strictly positive if $m_1 \neq 0, N$, and vanishes for $m_1 = 0, N$.

Now as long as (6.8) is non-vanishing and positive, we can make the first term of (6.7) dominate over others by taking M to be arbitrarily large, and hence (6.7) is positive as required. Thus we only need to worry about is the case $n_1 = 0$, $j = 1$, $m_1 = 0, N$ when the order M^2 term vanishes. In both cases (6.7) takes the form:

$$N\, M\, M_3 - M_3^2 \,. \tag{6.9}$$

Since M is large and positive, and $M >> M_3 > 0$, this is strictly positive. This shows that (6.6) holds for all j, m_1, n_1 and hence we do not encounter any pole while deforming the contour (6.2) to (2.29).

For the special case of $N = 1$ one can consider another map that takes us from the set $\mathcal{B}_R$ to $\mathcal{B}_L$. It is

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{6.10}$$

Under this map the vertices 0, 1 and $i\infty$ of the domain $\mathcal{R}$ get mapped to the points $i\infty$, -1 and 0 respectively. The latter set is precisely the vertices of the domain $\mathcal{L}$.

Using (4.6), (4.19) we see that this transformation maps the original contour to [13]

$$\begin{gathered} \widetilde{\rho}_2 = M_2, \quad \widetilde{\sigma}_2 = M_1, \quad \widetilde{v}_2 = M_3, \\ 0 \le \widetilde{\rho}_1 \le 1, \quad 0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1 \,. \end{gathered} \tag{6.11}$$

This exactly coincides with (2.29) up to an exchange of M_1 and M_2. Since exchange of M_1 and M_2 does not change the integral (we could take $M_1 = M_2$) we see that this constraint of S-duality is satisfied trivially.

Returning to the case of general N, one can identify the following additional $\Gamma_1(N)$ transformation that maps the set $\mathcal{B}_R$ to $\mathcal{B}_L$:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix} . \tag{6.12}$$

This maps the contour $\mathcal{C}$ to

$$\begin{gathered} \widetilde{\rho}_2 = M_1 + M_2 - 2M_3, \qquad \widetilde{\sigma}_2 = M_2, \qquad \widetilde{v}_2 = M_2 - M_3 \,, \\ 0 \le \widetilde{\rho}_1 \le 1, \quad 0 \le \widetilde{\sigma}_1 \le N, \quad 0 \le \widetilde{v}_1 \le 1 \,. \end{gathered} \tag{6.13}$$

Consistency with duality invariance requires that we must be able to deform this contour to $\widehat{\mathcal{C}}$ without encountering any pole. One can proceed to analyze this exactly in the same manner as we did for (6.2) and arrive at the condition that in order to be able to deform this contour to $\widehat{\mathcal{C}}$ given in (2.29) we need the following quantity to be positive (analog of eq.(6.7)):

$$\begin{aligned}&(n_1 - m_1)(n_1 - 2m_1 + j)M^2 \\ &+\{j(n_1 - 2m_1 + j) + (n_1 - m_1)(2m_1 - j)\}MM_3 \\ &+j(2m_1 - j)M_3^2\,.\end{aligned} \tag{6.14}$$

Using (6.4) the coefficient of the M^2 term can be brought to the form

$$\left(\frac{j}{2} + n_1 - m_1\right)^2 + m_1^2 - \frac{1}{4}\,. \tag{6.15}$$

If $m_1 \neq 0$ it is strictly positive. For $m_1 = 0$ from (6.4) we have $j = 1$. Since $n_1 \in \mathbb{Z}$, in this case (6.15) is strictly positive if $n_1 \neq 0, -1$, and vanishes for $n_1 = 0, -1$. As long as (6.15) is non-vanishing and positive, we can make the first term of (6.14) dominate over others by taking M to be arbitrarily large, and hence (6.14) is positive as required. Thus we only need to worry about is the case $m_1 = 0$, $j = 1$, $n_1 = 0, -1$ when the order M^2 term vanishes. In both cases (6.14) takes the form:

$$M\,M_3 - M_3^2\,. \tag{6.16}$$

Since M is large and positive, and $M >> M_3 > 0$, this is strictly positive. This shows that (6.14) is positive for all j, m_1, n_1 and hence we do not encounter any pole while deforming the contour (6.13) to (2.29).

If we consider the $\Gamma_1(N)$ element

$$g_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix} = \begin{pmatrix} 1-N & 1 \\ -N & 1 \end{pmatrix}, \tag{6.17}$$

then it clearly leaves the set $\mathcal{B}_R$ unchanged since it is given by a map from $\mathcal{B}_R$ to $\mathcal{B}_L$ followed by the inverse of a map from $\mathcal{B}_L$ to $\mathcal{B}_R$. Our results establish that under this transformation the contour $\mathcal{C}$ is mapped to another contour that is continuously deformable to $\mathcal{C}$. This is turn establishes that any power of g_0 will also have the same property. We can now follow this up with the transformation (6.1) to construct a set of duality transformations that maps the set $\mathcal{B}_R$ to $\mathcal{B}_L$ and maps the contour $\mathcal{C}$ to another contour deformable to $\widehat{\mathcal{C}}$, thereby providing a test of the corresponding duality transformtion. For $N = 1$ this includes in particular the element (6.10) considered earlier.

Can every element of $\Gamma_1(N)$ that preserves the set $\mathcal{B}_R$ be expressed as a positive or negative power of g_0? If so then our test of S-duality invariance of the degeneracy formula would be complete. To address this issue note that a $\Gamma_1(N)$ transformation that maps $\mathcal{B}_R$ to $\mathcal{B}_R$ must take adjacent walls to adjacent walls. Furthermore the map must be orientation preserving. Thus the action of any such transformation on $\mathcal{B}_R$ must preserve the cyclic ordering of the walls and vertices. The map g_0 indeed has this property. It moves the walls and vertices by two steps clockwise, taking 0 to 1, $i\infty$ to $1-\frac{1}{N}$, 1 to $1-\frac{1}{N-1}$ etc. g_0^{-1} causes a shift by two steps in the anti-clockwise direction. Powers of g_0 will move the walls clockwise or anti-clockwise by even number of steps. Are there elements of $\Gamma_1(N)$ which shift the walls by odd number of steps? If so then by combining it with appropriate positive or negative powers of g_0 we can generate a transformation that shifts every wall of $\mathcal{B}_R$ by one step in the clockwise direction. Such a move will map 0 to $i\infty$, $i\infty$ to 1 and 1 to $1-\frac{1}{N}$. The unique SL(2,R) map that implements this is the matrix $\begin{pmatrix} \sqrt{N} & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$. This is clearly not an element of $\Gamma_1(N)$ for any N other than $N=1$.

This would seem to indicate that all elements of $\Gamma_1(N)$ preserving $\mathcal{B}_R$ are generated by g_0. However there is an additional subtlety arising out of the fact that for $N>4$ the element g_0 has a pair of fixed points on the real line corresponding to[9]

$$\tau_\infty = \frac{1}{2}\left(1 \pm \sqrt{1-\frac{4}{N}}\right) . \tag{6.18}$$

These provide accumulation points of the vertices of $\mathcal{R}$; indeed if we begin with any vertex of $\mathcal{R}$ and apply g_0 or g_0^{-1} transformation successively to generate other vertices they accumulate at the points given in (6.18). As a result there are infinite number of walls bordering $\mathcal{R}$, and a wall or a vertex that is situated within the range

$$\frac{1}{2}\left(1 - \sqrt{1-\frac{4}{N}}\right) < \tau_\infty < \frac{1}{2}\left(1 + \sqrt{1-\frac{4}{N}}\right) , \tag{6.19}$$

can never be related to a wall or a vertex lying outside this range by a g_0 transformation. This opens up the possibility that there may be additional elements of $\Gamma_1(N)$ which map the walls and vertices outside the range (6.19) to walls and vertices inside this range and vice versa, preserving the cyclic ordering.

[9]For $N=4$ there is a single fixed point and hence it does not divide the vertices into two sets. Thus every pair of vertices separated by even number of steps can still be related by a g_0 transformation.

To examine this issue we need to first identify some vertices lying within the range (6.19). For this we shall first prove that the point 1/2 must be a vertex. If it is not a vertex then there must be a wall that goes over 1/2; by symmetry of the problem the vertices at the two ends of this wall must be situated symmetrically about 1/2. Let us take them to be

$$\frac{1}{2} \pm \frac{p}{q} = \frac{q \pm 2p}{2q}\,, \tag{6.20}$$

with p, q relatively prime. This wall will correspond to a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a/c = (q-2p)/2q$ and $b/d = (q+2p)/2q$. Now we see that if q is odd then $(q \pm 2p)$ are odd and hence both c and d must be divisible by 2. This of course is incompatible with the relation $ad - bc = 1$. If on the other hand q is even (say $2m$) then p must be odd and we can express (6.20) as

$$\frac{m \pm p}{2m}\,. \tag{6.21}$$

Thus we have $a/c = (m-p)/2m$ and $b/d = (m+p)/2m$. m must be odd so that $(m \pm p)$ are even; otherwise we again run into the problem of both c and d being even. Now since c and d cannot have a common factor, between $m+p$ and $m-p$ they should be able to cancel all the factors in $2m$. This in particular will mean that their product $(m^2 - p^2)$ must be divisible by m. Thus p^2 should be divisible by m. This is in contradiction with the assumption that p and q (and hence p and m) are relatively prime except for $m = 1$. The latter corresponds to the wall connecting 0 and 1 and is not relevant for us.

This shows that 1/2 must be a vertex (or accumulation point of vertices) of $\mathcal{R}$. We can now begin with $\frac{1}{2}$ and identify the vertices to the left and right of this using the analysis described below (3.33). For odd N this gives three points

$$\frac{1}{2} - \frac{1}{2N}, \quad \frac{1}{2}, \quad \frac{1}{2} + \frac{1}{2N}\,, \tag{6.22}$$

in ascending order. On the other hand for $N = 2M$ with M odd we get the three points to be

$$\frac{1}{2} - \frac{1}{2M}, \quad \frac{1}{2}, \quad \frac{1}{2} + \frac{1}{2M}\,. \tag{6.23}$$

If N is a multiple of 4 then one can show that there is no wall ending at 1/2 and hence 1/2 must be an accumulation point. We shall not deal with this case here. We can now ask how g_0 acts on these vertices. For $N = 5$ and $N = 6$ one can check that it moves the left-most vertex in the set (6.22) or (6.23) to the right-most vertex. By continuity we can conclude that in the range (6.19), g_0 moves every vertex and wall by two steps in the *anti-clockwise* direction. There is no contradiction with the fact that outside the range (6.19) g_0 moves the points in the clockwise direction since the boundaries of this region are fixed points of g_0. Thus every wall in the range (6.19) can be related by g_0 action to one of the two walls connecting the vertices given in (6.22) or (6.23) for $N = 5, 6$. For $N \geq 7$ there are more than two vertices between a given vertex and its g_0 image and the situation is more complicated.

Restricting ourselves to the cases $N = 5, 6$ we now ask the following question: is there a $\Gamma_1(N)$ transformation that maps the vertices lying inside the range (6.19) to vertices outside the range (6.19)? We focus on the possible action of this map on the three vertices given in (6.22) or (6.23). If such a transformation exists then by left multiplying it with powers of g_0 we can always bring the three final points to 1, $i\infty$, 0 or $i\infty$, 0, $1/N$. Examining the two cases separately we find that there is no such $\Gamma_1(N)$ transformation mapping (6.22) or (6.23) to these points (although for odd N there is a $\Gamma_0(N)$ transformation $\begin{pmatrix} 2 & -1 \\ N & -(N-1)/2 \end{pmatrix}$ which takes the three points given in (6.22) to $i\infty$, 0 and $1/N$).

This finally establishes that for $N \leq 6$, all the elements of $\Gamma_1(N)$ which preserve the set $\mathcal{B}_R$ are obtained by taking positive or negative powers of g_0. Since we have checked that these transformations take the contour $\mathcal{C}$ to another contour deformable to $\mathcal{C}$ without encountering any poles, this completes our test of S-duality invariance for $N \leq 6$.

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