Duality Orbits, Dyon Spectrum and Gauge Theory Limit of Heterotic String Theory on T^6

Shamik Banerjee and Ashoke Sen

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211019, INDIA

E-mail: bshamik, sen@mri.ernet.in

Abstract

For heterotic string theory compactified on T^6, we derive the complete set of T-duality invariants which characterize a pair of charge vectors (Q, P) labelling the electric and magnetic charges of the dyon. Using this we can identify the complete set of dyons to which the previously derived degeneracy formula can be extended. By going near special points in the moduli space of the theory we derive the spectrum of quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric gauge theory with simply laced gauge groups. The results are in agreement with those derived from field theory analysis.

1

Contents

1 Introduction

We now have a good understanding of the exact spectrum of a class of quarter BPS dyons in a variety of $\mathcal{N} = 4$ supersymmetric string theories [1–18]. Explicit computation of the spectrum was carried out for a special class of charge vectors in a specific region of the moduli space. Using the various duality invariances of the theory we can extend the results to various other charge vectors in various other regions in the moduli space. However in order to do this we need to find out the duality orbits of the charge vectors for which the spectrum has been computed. This is one of the goals of this paper. Throughout this paper we shall focus on a particular $\mathcal{N} = 4$ supersymmetric string theory – heterotic string theory compactified on a six dimensional torus T^6.

A duality transformation typically acts on the charges as well as the moduli. Thus using duality invariance we can relate the degeneracy of a given state at one point of the moduli space to that of a different state, carrying different set of charges, at another point of the moduli space. For BPS states however the degeneracy – or more precisely an appropriate index measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets for a given set of charges – is invariant under changes in the moduli unless we cross a wall of marginal stability on which the state under consideration becomes marginally unstable. Thus for BPS states, instead of having to describe the spectrum as a function of the continuous moduli parameters we only need to specify it in different domains bounded by walls of marginal stability [13,17,18]. It turns out that a T-duality transformation takes a point inside one such

domain to another point inside the same domain in a sense described precisely in [13, 18]. Thus once we have calculated the spectrum in one domain for a given charge, T-duality symmetry can be used to find the spectrum in the same domain for all other charges related to the initial charge by a T-duality transformation. For this reason it is important to understand under what condition two different charges are related to each other by a T-duality transformation, ı.e. to classify the T-duality orbits. S-duality transformation, on the other hand, takes a point inside one domain to a point in another domain. Thus once we have calculated the spectrum in one domain, S-duality transformation allows us to calculate the spectrum in other domains.

Our results for the T-duality orbit of charges can be summarized as follows. Since heterotic string theory on T^6 has a gauge group of rank 28, a typical state is characterized by a 28 dimensional electric charge vector Q and a 28 dimensional magnetic charge vector P, each taking values on the Narain lattice Λ of signature (6,22). We shall take Q and P to be primitive vectors of the lattice; if not we can express them as integer multiples of primitive vectors and apply our analysis to these primitive vectors, treating the integer factors as additional T-duality invariants. Let Q_i and P_i denote the components of Q and P along some basis of primitive vectors of the lattice Λ and L_{ij} denote the natural metric of signature (6,22) under which the lattice is even and self-dual. Then the complete set of T-duality invariants are as follows. First of all we have the invariants of the continuous T-duality group:

$$Q^2 = Q^T L Q, \qquad P^2 = P^T L P, \qquad Q \cdot P = Q^T L P. \tag{1.1}$$

Next we have the combination [14, 19]

$$r(Q, P) = \text{g.c.d.}\{Q_i P_j - Q_j P_i, \quad 1 \le i, j \le 28\}. \tag{1.2}$$

Finally we have

$$u_1(Q, P) = \alpha \cdot P \bmod r(Q, P), \qquad \alpha \in \Lambda, \quad \alpha \cdot Q = 1. \tag{1.3}$$

$u_1(Q, P)$ can be shown to be independent of the choice of $\alpha \in \Lambda$. One finds first of all that each of the five combinations Q^2, P^2, $Q \cdot P$, $r(Q, P)$ and $u_1(Q, P)$ is invariant under T-duality transformation. Furhermore two pairs (Q, P) and (Q', P') having the same set of invariants can be transformed to each other by a T-duality transformation. Thus a necessary and sufficient condition for two pairs of charge vectors (Q, P) and (Q', P') to be related via a T-duality transformation is that all the five invariants are identical for the two pairs.

3

The computation of [9] of the spectrum of quarter BPS states in heterotic string theory on T^6 has been carried out for a special class of charge vectors for which $r(Q, P) = 1$, and Q^2, P^2 and $Q \cdot P$ are arbitrary. The invariant $u_1(Q, P)$ is trivially 0 for states with $r(Q, P) = 1$. Let us denote the calculated index by $f(Q^2, P^2, Q \cdot P)$. Then T-duality invariance tells us that for all states with $r(Q, P) = 1$ the index is given by the same function $f(Q^2, P^2, Q \cdot P)$ in the domain of the moduli space in which the original calculation was performed. Since S-duality maps states with $r(Q, P) = 1$ to states with $r(Q, P) = 1$, but maps the original domain to other domains, S-duality invariance allows us to extend the result to all states with $r(Q, P) = 1$ in all domains of the moduli space.

Since at special points in the moduli space of heterotic string theory on T^6 we can get $\mathcal{N} = 4$ supersymmetric gauge theories with simply laced gauge groups [20, 21] in the low energy limit, we can use the dyon spectrum of string theory to extract information about the dyon spectrum of $\mathcal{N} = 4$ supersymmetric gauge theories. For this we need to work near the point in the moduli space where we have enhanced gauge symmetry. Slightly away from this point we have the non-abelian part of the gauge symmetry spontaneously broken at a scale small compared to the string scale, and the spectrum of string theory contains quarter BPS dyons whose masses are of the order of the symmetry breaking scale. These dyons can be identified as dyons in the $\mathcal{N} = 4$ supersymmetric gauge theory. Thus the knowledge of the quarter BPS dyon spectrum in heterotic string theory on T^6 gives us information about the quarter BPS dyon spectrum in all $\mathcal{N} = 4$ supersymmetric gauge theories which can be obtained from the heterotic string theory on T^6. This method has been used in [22] to compute the spectrum of a class of quarter BPS states in $\mathcal{N} = 4$ supersymmetric SU(3) gauge theory.

Since the result for the quarter BPS dyon spectrum in heterotic string theory on T^6 is known only for the states with $r(Q, P) = 1$, we can use this information to compute the index of only a subset of dyons in $\mathcal{N} = 4$ super Yang-Mills theory with simply laced gauge groups. For this subset of states the result for the index can be stated in a simple manner, – we find that the index is non-zero only for those charges which can be embedded in the root lattice of an SU(3) subalgebra. Thus these states fall within the class of states analyzed in [22] and can be represented as arising from a 3-string junction with the three external strings ending on three parallel D3-branes [23]. This result for general $\mathcal{N} = 4$ supersymmetric gauge theories is in agreement with previous results obtained either by direct analysis in gauge theory [24, 25] or by the analysis of the spectrum of string network on a system of D3-branes [26].[1]

---

[1]Different aspects of dyon spectrum in $\mathcal{N} = 4$ supersymmetric gauge theories have been discussed in [27].

Some related issues have been addressed in [28].

2 T-duality orbits of dyon charges in heterotic string theory on T^6

We consider heterotic string theory compactified on T^6. In this case a general dyon is charac-terized by its electric and magnetic charge vectors (Q, P) where Q and P are 28 dimensional charge vectors taking values in the Narain lattice Λ [20]. We shall express Q and P as linear combinations of a primitive basis of lattice vectors so that the coefficients Q_i and P_i are inte-gers. There is a natural metric L of signature (6,22) on Λ under which the lattice is even and self-dual. The discrete T-duality transformations of the theory take the form

$$Q \to \Omega Q, \qquad P \to \Omega P, \tag{2.1}$$

where Ω is a 28×28 matrix that preserves the metric L and the Narain lattice Λ

$$\Omega^T L \Omega = L, \qquad \Omega \Lambda = \Lambda. \tag{2.2}$$

Since Ω must map an arbitrary integer valued vector to another integer valued vector, the elements of Ω must be integers.

We shall assume from the beginning that Q and P are primitive elements of the lattice.[2] Our goal is to find the T-duality invariants which characterize the pair of charge vectors (Q, P). First of all we have the continuous T-duality invariants

$$Q^2 = Q^T L Q, \qquad P^2 = P^T L P, \qquad Q \cdot P = Q^T L P. \tag{2.3}$$

Besides these we can introduce some additional invariants as follows. Consider the combination [14, 19]

$$r(Q, P) = \text{g.c.d.}\{Q_i P_j - Q_j P_i, \quad 1 \le i, j \le 28\}. \tag{2.4}$$

We shall first show that $r(Q, P)$ is independent of the choice of basis in which we expand Q and P. For this we note that the component form of Q and P in a different choice of basis will be related to the ones given above by multiplication by a matrix S with integer elements

---

[2]If this is not the case then the gcd a_1 of all the elements of Q and the gcd a_2 of all the elements of P will be separately invariant under discrete T-duality transformation. We can factor these out as $Q = a_1 \overline{Q}$, $P = a_2 \overline{P}$ with $a_1, a_2 \in \mathbb{Z}$, $\overline{Q}, \overline{P} \in \Lambda$, and then apply our analysis on the resulting primitive elements $\overline{Q}$ and $\overline{P}$.

and unit determinant so that the elements of S^{-1} are also integers. Thus in this new basis r will be given by

$$r(SQ, SP) = \gcd\{S_{ik}S_{jl}(Q_kP_l - Q_lP_k), \quad 1 \le i, j \le 28\}.$$ (2.5)

Since S_{ik} are integers, eq.(2.5) shows that $r(SQ, SP)$ must be divisible by $r(Q, P)$. Applying the S^{-1} transformation on (SQ, SP), and noting that S^{-1} also has integer elements, we can show that $r(Q, P)$ must be divisible by $r(SQ, SP)$. Thus we have

$$r(Q, P) = r(SQ, SP),$$ (2.6)

ı.e. $r(Q, P)$ is independent of the choice of basis used to describe the vectors (Q, P). As a special case where we restrict S to T-duality transformation matrices Ω, we find

$$r(Q, P) = r(\Omega Q, \Omega P).$$ (2.7)

Thus $r(Q, P)$ is invariant under a T-duality transformation.

Another set of T-duality invariants may be constructed as follows. Let $\alpha, \beta \in \Lambda$ satisfy

$$\alpha \cdot Q = 1, \qquad \beta \cdot P = 1.$$ (2.8)

Since Q and P are primitive and the lattice is self-dual one can always find such α, β. Then we define

$$u_1(Q, P) = \alpha \cdot P \bmod r(Q, P), \qquad u_2(Q, P) = \beta \cdot Q \bmod r(Q, P).$$ (2.9)

One can show that [19]

1. u_1 and u_2 are independent of the choice of α, β.

2. u_1 and u_2 are T-duality invariants.

3. u_2 is determined uniquely in terms of u_1.

The proof of these statements goes as follows. To prove that u_1 is independent of the choice of α we note that since Q is a primitive vector we can choose a basis of lattice vectors so that

the first element of the basis is Q itself. Then in this basis[3]

$$Q = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \qquad P = \begin{pmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ P_{28} \end{pmatrix}, \tag{2.10}$$

and we have

$$r(Q, P) = \gcd(P_2, \cdots P_{28}). \tag{2.11}$$

Now suppose α_1 and α_2 are two vectors which satisfy $Q \cdot \alpha_1 = Q \cdot \alpha_2 = 1$. Then $(\alpha_1 - \alpha_2) \cdot Q = 0$, and hence we have

$$(\alpha_1 - \alpha_2) \cdot P = (\alpha_1 - \alpha_2) \cdot (P - P_1 Q) = (\alpha_1 - \alpha_2) \cdot \begin{pmatrix} 0 \\ P_2 \\ P_3 \\ \cdot \\ \cdot \\ P_{28} \end{pmatrix}. \tag{2.12}$$

Eq.(2.11) shows that the right hand side of (2.12) is divisible by r. Thus $\alpha_1 \cdot P = \alpha_2 \cdot P$ modulo r. This shows that u_1 defined through (2.9) is independent of the choice of α. A similar analysis shows that u_2 defined in (2.9) is independent of the choice of β. From now on all equalities involving $u_1(Q, P)$ and $u_2(Q, P)$ will be understood to hold modulo $r(Q, P)$ although we shall not always mention it explicitly.

T-duality invariance of u_1 follows from the fact that if $\alpha \cdot Q = 1$ then $\Omega\alpha \cdot \Omega Q = 1$. Thus

$$u_1(\Omega Q, \Omega P) = \Omega\alpha \cdot \Omega P \bmod r(\Omega Q, \Omega P) = \alpha \cdot P \bmod r(Q, P) = u_1(Q, P). \tag{2.13}$$

A similar analysis shows the T-duality invariance of u_2.

To show that u_2 is determined in terms of u_1 and vice versa we first note that for the choice of (Q, P) given in (2.10), we have

$$u_1(Q, P) = \alpha \cdot P = \alpha \cdot (P - P_1 Q) + P_1 \alpha \cdot Q = P_1 \bmod r(Q, P) \tag{2.14}$$

since $P - P_1 Q$ is divisible by r due to eqs.(2.10), (2.11), and $\alpha \cdot Q = 1$. On the other hand we have

$$1 = \beta \cdot P = \{\beta \cdot (P - P_1 Q) + P_1 \beta \cdot Q\} = u_1(Q, P) u_2(Q, P) \bmod r(Q, P), \tag{2.15}$$

---

[3]Note that in this basis the metric takes a complicated form, *e.g.* the 11 component of the metric must be equal to Q^2. However all components of the metric are still integers since the inner product between two arbitrary integer valued vectors – representing a pair of elements of the lattice – must be integer.

since $(P - P_1 Q) = 0$ modulo r, $P_1 = u_1$, and $\beta \cdot Q = u_2$. Thus we have

$$u_1(Q, P)\, u_2(Q, P) = 1 \bmod r(Q, P)\,. \tag{2.16}$$

This shows that neither u_1 nor u_2 shares a common factor with r. We shall now show that (2.16) also determines u_2 uniquely in terms of u_1. To prove this assume the contrary, that there exists another number v_2 satisfying $u_1 v_2 = 1 \bmod r(Q, P)$. Then we have

$$u_1(Q, P)\, (u_2(Q, P) - v_2(Q, P)) = 0 \bmod r(Q, P)\,. \tag{2.17}$$

Since u_1 has no common factor with r, this shows that $v_2 = u_2$ modulo r. Hence u_2 is determined in terms of u_1 modulo r.

Thus we have so far identified five separate T-duality invariants characterizing the pair of vectors (Q, P): Q^2, P^2, $Q \cdot P$, $r(Q, P)$ and $u_1(Q, P)$. We shall now show that these are sufficient to characterize a T-duality orbit, ı.e. given any two pairs (Q, P) and (Q', P') with the same set of invariants they are related by a T-duality transformation. We begin by defining[4]

$$\widehat{P} = Q^2 P - Q \cdot P\, Q\,, \tag{2.18}$$

and

$$\widetilde{P} = \frac{1}{K}\, \widehat{P}\,, \qquad K \equiv \gcd\{\widehat{P}_1, \cdots \widehat{P}_{28}\}\,. \tag{2.19}$$

By construction $\widetilde{P}$ is a primitive vector of the lattice satisfying

$$Q \cdot \widetilde{P} = 0\,. \tag{2.20}$$

We shall now use the result of [19] that the T-duality orbit of a pair of primitive vectors $(Q, \widetilde{P})$ satisfying $Q \cdot \widetilde{P} = 0$ is characterized completely by the invariants Q^2, $\widetilde{P}^2$, $r(Q, \widetilde{P})$ and $u_1(Q, \widetilde{P})$. A proof of this statement has been reviewed in appendix A. Given this, we shall show that the five invariants Q^2, P^2, $Q \cdot P$, $r(Q, P)$ and $u_1(Q, P)$ completely characterize the duality orbits of an arbitrary pair of charge vectors (Q, P). The steps involved in the proof are as follows:

1. We shall first show that the quantities $\widetilde{P}^2$, $r(Q, \widetilde{P})$, $u_1(Q, \widetilde{P})$ and the constant K appearing in (2.19) are determined completely in terms of Q^2, P^2, $Q \cdot P$, $r(Q, P)$ and $u_1(Q, P)$

---

[4]This procedure breaks down for $Q^2 = 0$, but as long as $P^2 \neq 0$ we can carry out our analysis by reversing the roles of Q and P. If both Q^2 and P^2 vanish then our analysis does not apply. However a different proof given in §3 applies to this case as well.

via the relaions

$$K = r(Q, P) \gcd \left\{ (u_1(Q, P)Q^2 - Q \cdot P)/r(Q, P), Q^2 \right\},$$

$$r(Q, \widetilde{P}) = Q^2 r(Q, P)/K, \quad u_1(Q, \widetilde{P}) = \frac{1}{K} \left(u_1(Q, P)Q^2 - Q \cdot P \right) \mod r(Q, \widetilde{P}),$$

$$\widetilde{P}^2 = \frac{1}{K^2} Q^2 (Q^2 P^2 - (Q \cdot P)^2). \tag{2.21}$$

The last equation follows trivially from the definition of $\widetilde{P}$. To prove the other relations we again use the form of (Q, P) given in (2.10). We have

$$\widehat{P} = Q^2 P - Q \cdot P Q = Q^2 (P - P_1 Q) - Q \cdot (P - P_1 Q) Q = r(Q, P) \{ Q^2 \gamma - Q \cdot \gamma Q \}, \tag{2.22}$$

where

$$\gamma = \frac{1}{r(Q, P)} (P - P_1 Q) = \frac{1}{r(Q, P)} \begin{pmatrix} 0 \\ P_2 \\ \cdot \\ \cdot \\ P_{28} \end{pmatrix}. \tag{2.23}$$

γ has integer elements due to (2.11). The same equation tells us that

$$\gcd(\gamma_2, \cdots \gamma_{28}) = 1. \tag{2.24}$$

Expressing (2.22) as

$$\widehat{P} = r(Q, P) \begin{pmatrix} -Q \cdot \gamma \\ Q^2 \gamma_2 \\ \cdot \\ \cdot \\ Q^2 \gamma_{28} \end{pmatrix}, \tag{2.25}$$

and using (2.24) we see that K defined in (2.19) is given by

$$K = r(Q, P) \gcd(-Q \cdot \gamma, Q^2). \tag{2.26}$$

Using (2.23) and that $P_1 = u_1(Q, P)$ modulo $r(Q, P)$ we may express (2.26) as

$$K = r(Q, P) \gcd \left\{ (u_1(Q, P)Q^2 - Q \cdot P)/r(Q, P), Q^2 \right\}. \tag{2.27}$$

This establishes the first equation in (2.21). Note that a shift in u_1 by $r(Q, P)$ does not change the value of K. Thus K given in (2.27) is independent of which particular representative we use for $u_1(Q, P)$.

9

To derive an expression for $r(Q, \widetilde{P})$ we note from the form of Q given in (2.10), the form of $\widehat{P}$ given in (2.25), and (2.24) that

$$r(Q, \widehat{P}) = \gcd\{Q_i \widehat{P}_j - Q_j \widehat{P}_i, \ 1 \le i, j \le 28\} = r(Q, P) \, Q^2 \,. \tag{2.28}$$

Since $\widetilde{P} = \widehat{P}/K$ we have

$$r(Q, \widetilde{P}) = Q^2 \, r(Q, P)/K \,. \tag{2.29}$$

This establishes the second equation in (2.21). Finally to calculate $u_1(Q, \widetilde{P})$ we pick the vector α for which $\alpha \cdot Q = 1$, and express $u_1(Q, \widetilde{P})$ as

$$u_1(Q, \widetilde{P}) = \alpha \cdot \widetilde{P} = \frac{1}{K} \left(Q^2 \alpha \cdot P - Q \cdot P \, \alpha \cdot Q \right) = \frac{1}{K} \left(Q^2 u_1(Q, P) - Q \cdot P \right) \,. \tag{2.30}$$

This establishes the third equation in (2.21). Note that under a shift of $u_1(Q, P)$ by $r(Q, P)$, the expression for $u_1(Q, \widetilde{P})$ given above shifts by $r(Q, \widetilde{P})$. Thus $u_1(Q, \widetilde{P})$ given above is determined unambiguously modulo $r(Q, \widetilde{P})$.

2. Now suppose we have two pairs (Q, P) and (Q', P') with the same set of invariants:

$$Q^2 = Q'^2, \quad P^2 = P'^2, \quad Q \cdot P = Q' \cdot P', \quad r(Q, P) = r(Q', P'), \quad u_1(Q, P) = u_1(Q', P') \,. \tag{2.31}$$

Let us define $\widehat{P}'$, K' and $\widetilde{P}'$ as in (2.18), (2.19) with (Q, P) replaced by (Q', P') so that $Q' \cdot \widetilde{P}' = 0$. Then by eq.(2.21), its analog with (Q, P) replaced by (Q', P'), and eq.(2.31), we have

$$Q^2 = Q'^2, \quad K' = K, \quad \widetilde{P}^2 = \widetilde{P}'^2, \quad r(Q, \widetilde{P}) = r(Q', \widetilde{P}'), \quad u_1(Q, \widetilde{P}) = u_1(Q', \widetilde{P}') \,. \tag{2.32}$$

Thus by the result of [19], reviewed in appendix A, $(Q, \widetilde{P})$ and $(Q', \widetilde{P}')$ must be related to each other by a T-duality transformation Ω:

$$Q' = \Omega Q, \qquad \widetilde{P}' = \Omega \widetilde{P} \,. \tag{2.33}$$

It follows from this that

$$\widehat{P}' = \Omega \widehat{P}, \quad \longrightarrow \quad P' = \Omega P \,. \tag{2.34}$$

Thus (Q, P) and (Q', P') are related by the duality transformation Ω.

This establishes that the T-duality orbits of pairs of charge vectors (Q, P) are completely characterized by the invariants Q^2, P^2, $Q \cdot P$, $r(Q, P)$ and $u_1(Q, P)$. Two pairs of charge vectors, having the same values of all the invariants, can be related to each other by a T-duality transformation.

3   An Alternative Proof

In this section we shall give a different proof of the results of the previous section.

We shall begin by giving a physical interpretation of the discrete T-duality invariants $r(Q, P)$ and $u_1(Q, P)$. Let E denote the two dimensional vector space spanned by the vectors Q and P, and $\Lambda' = E \cap \Lambda$ denote the two dimensional lattice containing the points of the Narain lattice in E. Let (e_1, e_2) denote a pair of primitive basis elements of the lattice Λ'. Since Q is a primitive vector, we can always choose $e_1 = Q$. Then we claim that in this basis

$$Q = e_1, \qquad P = u_1(Q, P)\, e_1 + r(Q, P)\, e_2\,. \tag{3.1}$$

The proof goes as follows. First of all since (e_1, e_2) form a primitive basis of Λ', by a standard result [29] one can show that (e_1, e_2) can be chosen as the first two elements of a primitive basis of the full lattice Λ. In such a basis $Q_1 = 1$, $P_1 = u_1$, $P_2 = r$ and all the other componets of Q and P vanish. Thus we have gcd $\{Q_i P_j - Q_j P_i\} = r$ as required. Furthermore, it is clear from (3.1) that if $\alpha \cdot Q = 1$ then $\alpha \cdot P = u_1$ modulo r as required by the definition of u_1. Finally, we see that a different choice of e_2 that preserves the primitivity of the basis (e_1, e_2) is related to the original choice by $e_2 \to e_2 + s\, e_1$ for some integer s. Under such a transformation u_1 defined through (3.1) is shifted by a multiple of r. Thus u_1 defined through (3.1) is unambiguous modulo r as required. We shall choose e_2 such that u_1 appearing in (3.1) lies between 0 and $r - 1$.

Eq.(3.1) provides a physical interpretation of u_1 and r in terms of the components of Q and P along a primitive basis of the Narain lattice in the plane spanned by Q and P. As a consequence of (3.1) we have

$$e_1^2 = Q^2, \qquad e_2^2 = \left\{P^2 + u_1(Q, P)^2 Q^2 - 2 u_1(Q, P) Q \cdot P\right\} / r(Q, P)^2,$$
$$e_1 \cdot e_2 = \left\{Q \cdot P - u_1(Q, P) Q^2\right\} / r(Q, P)\,. \tag{3.2}$$

Now take a different pair of charges (Q', P') with the same invariants, e.g. satisfying (2.31), and define (e_1', e_2') as in (3.1) with (Q, P) replaced by (Q', P'). Then as a consequence of (2.31) and (3.2) we have

$$e_1^2 = (e_1')^2, \qquad e_2^2 = (e_2')^2, \qquad e_1 \cdot e_2 = e_1' \cdot e_2'\,. \tag{3.3}$$

Thus the lattices generated by (e_1, e_2) and (e_1', e_2') can be regarded as different primitive embeddings into Λ of an abstract even lattice of rank two with a given metric. We now use the result of [30–32] that an even lattice of signature (m, n) has a unique primitive embedding in an even

11

self-dual lattice Λ of signature (p, q) up to a T-duality transformation if $m+n \leq \min(p, q) - 1$. Setting $m+n = 2$ and $(p, q) = (6, 22)$ we see that the required condition is satisfied and hence (e_1, e_2) must be related to (e_1', e_2') by a T-duality transformation:

$$e_1' = \Omega e_1, \qquad e_2' = \Omega e_2 . \tag{3.4}$$

Eq.(3.1) and its analog with $(Q, P) \longrightarrow (Q', P')$, $(e_1, e_2) \longrightarrow (e_1', e_2')$ then tells us that

$$Q' = \Omega Q, \qquad P' = \Omega P . \tag{3.5}$$

This is the desired result.

One interesting question is: for a given set of values of Q^2, P^2, $Q \cdot P$ and r, what is the maximum number of possible orbits? This is given by the maximum number of allowed values of u_1. Since u_1 and r cannot share a common factor, the number is bounded from above by the number of positive integers below $(r - 1)$ with no common factor with r. This in turn is given by

$$r \times \prod_{\text{primes } p,\, p|r} \left(1 - \frac{1}{p}\right) . \tag{3.6}$$

4 Predictions for gauge theory

At special points in the moduli space heterotic string theory on T^6 has enhanced gauge symmetry. As we move away from this point the gauge symmetry gets spontaneously broken, with the moduli fields describing deformations away from the enhanced symmetry point playing the role of the Higgs field. When the deformation parameter is small the scale of gauge symmetry breaking is small compared to the string scale and the theory contains massive states with mass of the order of the gauge symmetry breaking scale and small compared to the string scale. These states can be identified as the states of the spontaneously broken gauge theory. Thus if we know the spectrum of the string theory, we can determine the spectrum of spontaneously broken gauge theory. In particular the known spectrum of quarter BPS dyons in string theory should give us information about the spectrum of quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

The dyon charges in a gauge theory of rank n are labelled by a pair of n-dimensional vectors (q, p) in the root lattice of the gauge algebra. If we choose a set of n simple roots as the basis of the root lattice then the components q_a and p_a will label the coefficients of the simple roots

in an expansion of the charge vectors in this basis. When the root lattice is embedded in the Narain lattice the vectors (q, p) correpond to a pair of vectors (Q, P) in the Narain lattice, and the metric L on the Narain lattice, restricted to the root lattice, gives the negative of the Cartan metric. Denoting by $\circ$ the inner product with respect to the Cartan metric, we have

$$q^2 \equiv q \circ q = -Q^2, \qquad p^2 \equiv p \circ p = -P^2, \qquad q \circ p = -Q \cdot P. \qquad (4.1)$$

Since the Cartan metric is positive definite, we must have $q^2, p^2 \geq 0$, $|q \circ p| \leq (q^2 + p^2)/2$. Furthermore quarter BPS dyons require q and p to be both non-zero and non-parallel. Hence none of the above inequalities can be saturated. This translates to the following conditions on Q, P:

$$Q^2 < 0, \qquad P^2 < 0 \qquad |Q \cdot P| < \left(|Q^2| + |P^2|\right)/2. \qquad (4.2)$$

Finally, since the string theory dyon spectrum is known only for charges (Q, P) with $r(Q, P) = 1$ we need to know what this condition translates to on the vectors (q, p). This is done most easily if the Narain lattice admits a primitive embedding of the root lattice, ı.e. if we can choose the n simple roots of the root lattice as the first n basis elements of the full 28 dimensional Narain lattice. In that case we can easily identify (q, p) in the root lattice as a pair of charge vectors (Q, P) in the Narain lattice where the first n components of Q (P) are equal to the components of q (p) and the rest of the components of Q, P vanish. Thus we have

$$r(Q, P) = \gcd\{q_i p_j - q_j p_i\} \equiv r_{gauge}(q, p). \qquad (4.3)$$

The condition $r(Q, P) = 1$ then translates to $r_{gauge}(q, p) = 1$.

Let us now investigate under what condition the Narain lattice does not admit a primitive embedding of the root lattice. Let F be the n-dimensional vector space spanned by the root lattice, and let $\Lambda' = F \cap \Lambda$. Then by a standard result [29] one finds that the root lattice has a primitive embedding in the Narain lattice if Λ' does not contain any element other than the ones in the root lattice. So we need to examine under what condition Λ' can contain elements other than the ones in the root lattice. Now clearly the elements of Λ' must belong to the weight lattice of the algebra. Furthermore, since Narain lattice is even, any element of Λ' will be even. Thus we can classify all possible extra elements of Λ' by examining the possible even elements of the weight lattice outside the root lattice. For many algebras we have no such element, and hence in those cases the embedding of the root lattice in the Narain lattice is necessarily primitive. Exceptions among the rank ≤ 22 algebras are $so(16)$, $so(32)$, $su(8)$, $su(9)$, $su(16)$ and $su(18)$; for each of these the weight lattice has even elements other than

those in the root lattice [33].[5] Hence in these cases Λ' could contain elements other than the ones in the root lattice, preventing the root lattice from having a primitive embedding in the Narain lattice. But since Λ' would have a primitive embedding in the Narain lattice, if we choose a basis for Λ', and define q_i, p_i as the componets of q and p expanded in this basis, then (4.3) continues to reproduce the value of $r(Q, P)$.

With this understanding we can now study the implications of the known dyon spectrum in $\mathcal{N} = 4$ supersymmetric string theory. As is well known, for dyons with $r(Q, P) = 1$ the dyon spectrum in different parts of the moduli space can be different. The situation is best described in the axion-dilaton moduli space at fixed values of the other moduli [13]. In particular in the upper half plane labelled by the axion-dilaton field[6] $\tau = a + iS$ the spectrum jumps across walls of marginal stability, which are circles or straight lines passing through rational points on the real axis [13, 16, 17]. These curves do not intersect in the interior of the upper half plane and divide up the upper half plane into different domains, each with three vertices lying either at rational points on the real axis or at ∞. Inside a given domain the index $d(Q, P)$ that counts the number of bosonic supermultiplets minus the number of fermionic supermultiplets remains constant, but as we move from one domain to another the index changes. We shall first consider the domain bounded by a straight line passing through 0, a straight line passing through 1 and a circle passing through 0 and 1, – the domain called $\mathcal{R}$ in [13, 18]. This has vertices at 0, 1 and ∞. In this domain the only non-zero values of $d(Q, P)$ for $Q^2 < 0$, $P^2 < 0$ are obtained at $Q^2 = P^2 = -2$. For $Q^2 = P^2 = -2$ the result for $d(Q, P)$ is [9, 13, 22]

$$d(Q, P) = \begin{cases} 0 & \text{for } Q \cdot P \geq 0 \\ j(-1)^{j-1} & \text{for } Q \cdot P = -j, \, j > 0 \end{cases}. \tag{4.4}$$

The condition (4.2) on $Q \cdot P$ now shows that for (Q, P) describing the elements of the root lattice, non-vanishing index exists only for $Q^2 = P^2 = -2$, $Q \cdot P = -1$. Translated to a condition on the charge vectors in the gauge theory this gives[7]

$$d_{gauge}(q, p) = \begin{cases} 1 & \text{for } q^2 = p^2 = 2, \, q \circ p = 1, \, r_{gauge}(q, p) = 1 \\ 0 & \text{for other } (q, p) \text{ with } r_{gauge}(q, p) = 1 \end{cases}. \tag{4.5}$$

---

[5]Both for $so(16)$ and $su(9)$, inclusion of the extra even elements of the weight lattice makes the lattice $F \cap \Lambda$ into the root lattice of e_8. Thus for such embeddings we are actually counting the dyon spectrum of an E_8 gauge theory rather than $SO(16)$ or $SU(9)$ gauge theory. On the other hand for $su(8)$ the extra even elements of the weight lattice makes $F \cap \Lambda$ into the root lattice of e_7. Thus in this case we get an E_7 gauge theory.

[6]From the point of view of the gauge theory the axion-dilaton moduli correspond to the theta parameter and the inverse square of the coupling constant.

[7]It has been shown in appendix B that for states with $Q^2 = P^2 = -2$, $Q \cdot P = \pm 1$ the condition $r(Q, P) = 1$ is satisfied automatically. Thus we do not need to state this as a separate condition.

This condition in turn implies that q and $-p$ can be regarded as the simple roots of an $su(3)$ subalgebra of the full gauge algebra, with the Cartan metric of su(3) being equal to the restriction of the Cartan metric of the full algebra. Thus we learn that in the domain $\mathcal{R}$ the only dyons with $r_{gauge}(q, p) = 1$ and non-vanishing index are the ones which can be regarded as $SU(3)$ dyons for some level one $su(3)$ subalgebra of the gauge algebra, with q and $-p$ identified with the simple roots α and β of the $su(3)$ algebra.

The index in other domains can be found using the S-duality invariance of the theory. An S-duality transformation of the form $\tau \to (a\tau + b)/(c\tau + d)$ maps the domain $\mathcal{R}$ to another domain with vertices

$$\frac{a}{c}, \quad \frac{b}{d}, \quad \frac{a+b}{c+d}.$$ (4.6)

Under the same S-duality transformation the charge vector $(q, p) = (\alpha, -\beta)$ gets mapped to

$$(q, p) = (a\alpha - b\beta, c\alpha - d\beta).$$ (4.7)

It can be easily seen that $r(Q, P)$ remains invariant under an $SL(2, \mathbb{Z})$ transformation:

$$r(Q, P) = r(aQ + bP, cQ + dP), \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1.$$ (4.8)

Thus we conclude that in the domain (4.6), the index of gauge theory dyons with $r_{gauge}(q, p) = 1$ is given by

$$d_{gauge}(q, p) = \begin{cases} 1 & \text{for } (q, p) = (a\alpha - b\beta, c\alpha - d\beta), \\ 0 & \text{otherwise} \end{cases},$$ (4.9)

with (α, β) labelling the simple roots of some level one su(3) subalgebra of the full gauge algebra.

This general result agrees with the known results for quarter BPS dyons in gauge theories [23–26, 34].[8] In particular in the representation of $SU(N)$ dyons as string network with ends on a set of parallel D3-branes, this is a reflection of the fact that networks with three external strings ending on three D3-barnes are the only quarter BPS configurations at a generic point in the moduli space [26].

---

[8]In codimension ≥ 1 subspaces of the moduli space the dyon spectrum, computed in some approximation, has a rich structure [24–26, 34]. However the index associated with these dyons vanish and these results are not in contradiction with the spectrum of string theory.

A T-duality orbits of pair $(Q, \widetilde{P})$ with $Q \cdot \widetilde{P} = 0$

In this appendix we shall review the proof, given in [19], of the fact that for a pair of primitive lattice vectors $(Q, \widetilde{P})$ with $Q \cdot \widetilde{P} = 0$, the invariants Q^2, $\widetilde{P}^2$, $r(Q, \widetilde{P})$ and $u_1(Q, \widetilde{P})$ completely characterize the T-duality orbit. We shall choose a basis in which the metric L given by the direct sum of six σ_1's and two $-L_{E_8}$'s where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and L_{E_8} is the Cartan metric of E_8. Using the known result [35] that in this lattice any pair of primitive vectors of the same norm can be related by a T-duality transformation, we can choose the vector Q to be

$$Q = \begin{pmatrix} -1 \\ n \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \qquad n = -Q^2/2. \tag{A.1}$$

After this T-duality transformation the new $\widetilde{P}$ satisfying $Q \cdot \widetilde{P} = 0$ has the form $\begin{pmatrix} k \\ kn \\ \vec{p} \end{pmatrix}$ for some 26 dimensional vector $\vec{p}$ and some integer k. In general $\vec{p}$ is not a primitive vector; however if l is the gcd of the elements of $\vec{p}$ then $\vec{p}/l$ is a primitive vector. We can now use the result of [35] on the vector $\vec{p}/l$, belonging to the Narain lattice of signature (5,21), to bring $\widetilde{P}$ into the form[9]

$$\widetilde{P}' = \begin{pmatrix} k \\ kn \\ l \\ lm \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \qquad m = \frac{\vec{p}^2}{2\, l^2} \in \mathbb{Z}, \tag{A.2}$$

via a T-duality transformation acting on the last 26 elements that does not affect the form of Q. Furthermore since $\widetilde{P}$ is a primitive vector so is $\widetilde{P}'$, and we have

$$\gcd(k, l) = 1. \tag{A.3}$$

It follows from (A.3) and the definitions of $r(Q, P)$ and $u_1(Q, P)$ given in (2.4), (2.9) that

$$r(Q, \widetilde{P}) = r(Q, \widetilde{P}') = \gcd(l, 2kn) = \gcd(l, 2n), \qquad u_1(Q, \widetilde{P}) = u_1(Q, \widetilde{P}') = -k. \tag{A.4}$$

---

[9]The result of [35] is valid on a Lorenzian lattice of signature $(k, k + 16)$ if $k \geq 2$.

In arriving at (A.4) we have chosen $\alpha = \begin{pmatrix} 0 \\ -1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$.

We now consider the T-duality transformation generated by

$$
\Omega' = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & \\ 0 & 1 & 0 & 0 & -n & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & \\ n & 1 & 0 & 0 & -n & 1 & \\ 0 & 0 & 0 & 0 & 1 & 0 & \\ & & & & & & I_{22} \end{pmatrix} .
\tag{A.5}
$$

This leaves the charge vector Q invariant, but transforms $\widetilde{P}'$ to

$$
\widetilde{P}'' = \begin{pmatrix} k \\ kn \\ l \\ lm \\ 2kn \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \equiv \begin{pmatrix} k \\ kn \\ \vec{p}'' \end{pmatrix} .
\tag{A.6}
$$

We now regard the vector $\vec{p}''$ as an element of Narain lattice of signature $(5, 21)$. The gcd of all the elements of $\vec{p}''$ is given by

$$
\gcd(l, 2kn) = \gcd(l, 2n) = r(Q, \widetilde{P}) ,
\tag{A.7}
$$

using (A.3) and (A.4). Thus $\vec{p}''$ is $r(Q, \widetilde{P})$ times a primitive lattice vector. Hence we can again use the result of [35] to show that by a T-duality transformation acting on the last 26 elements of the charge vector, $\vec{p}''$ can be brought into the form

$$
\begin{pmatrix} r(Q, \widetilde{P}) \\ r(Q, \widetilde{P}) a \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} , \qquad a = \frac{\vec{p}''^2}{2r(Q, P)^2} .
\tag{A.8}
$$

17

This does not change the form of Q. Thus at this stage we have brought $(Q, \widetilde{P})$ to the form

$$
Q = \begin{pmatrix} -1 \\ n \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \quad \widetilde{P}''' = \begin{pmatrix} k \\ kn \\ r(Q, \widetilde{P}) \\ r(Q, \widetilde{P})\, a \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}. \tag{A.9}
$$

Finally we apply another T-duality transformation generated by the matrix

$$
\Omega'' = \begin{pmatrix} 1 & 0 & -q & 0 & \\ 0 & 1 & -nq & 0 & \\ nq & q & -nq^2 & 1 & \\ 0 & 0 & 1 & 0 & \\ & & & & I_{24} \end{pmatrix}, \tag{A.10}
$$

with q is an integer to be specified below. This leaves Q unchanged but brings $\widetilde{P}'''$ to the form

$$
\widetilde{P}_{std} = \begin{pmatrix} k - qr(Q, \widetilde{P}) \\ nk - nqr(Q, \widetilde{P}) \\ 2knq - nq^2 r(Q, \widetilde{P}) + ar(Q, \widetilde{P}) \\ r(Q, \widetilde{P}) \\ 0 \\ . \\ . \\ 0 \end{pmatrix}. \tag{A.11}
$$

We choose q such that $k - q\, r(Q, \widetilde{P})$ is an integer between 0 and $r(Q, \widetilde{P}) - 1$. By eq.(A.4) this is a representative of $-u_1(Q, \widetilde{P})$ in the range $[0, r(Q, \widetilde{P}) - 1]$. Hence it is determined uniquely by $u_1(Q, \widetilde{P})$. Let us call this integer $d(Q, \widetilde{P})$. We can then express (A.11) as

$$
\widetilde{P}_{std} = \begin{pmatrix} d(Q, \widetilde{P}) \\ nd(Q, \widetilde{P}) \\ b \\ r(Q, \widetilde{P}) \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \tag{A.12}
$$

18

where b is a constant. It is determined by equating $(\widetilde{P}_{std})^2$ to $\widetilde{P}^2$:

$$2\,n\,d(Q,\widetilde{P})^2 + 2\,b\,r(Q,\widetilde{P}) = \widetilde{P}^2\,. \tag{A.13}$$

Since $n = -Q^2/2$, this determines the form of Q and $\widetilde{P}_{std}$ completely in terms of the invariants Q^2, $\widetilde{P}^2$, $r(Q,\widetilde{P})$ and $d(Q,\widetilde{P})$. Thus any two pairs of charge vectors $(Q_1,\widetilde{P}_1)$ and $(Q_2,\widetilde{P}_2)$ having same values of these invariants and satisfying $Q_1\cdot\widetilde{P}_1 = Q_2\cdot\widetilde{P}_2 = 0$ can be related to each other by a T-duality transformation, since each pair can be brought by a T-duality transformation to the standard form $(Q,\widetilde{P}_{std})$ given in (A.1), (A.12). This is the desired result.

B   Analysis of $r(Q,P) = 1$ condition

In this appendix we shall derive a condition on Q^2, P^2 and $Q\cdot P$ which is sufficient but not necessary to gurantee that $r(Q,P) = 1$.

As usual, we shall assume that Q and P are primitive vectors of the lattice. In this case we can represent Q and P as in (3.1). This gives

$$Q^2 P^2 - (Q\cdot P)^2 = r(Q,P)^2\left(e_1^2 e_2^2 - (e_1\cdot e_2)^2\right)\,. \tag{B.1}$$

Thus in order for $r(Q,P)$ to be different from 1, $Q^2 P^2 - (Q\cdot P)^2$ must have a factor that is square of an integer. Conversely, if $Q^2 P^2 - (Q\cdot P)^2$ is square free we can conclude that $r(Q,P) = 1$. In particular, for $Q^2 = P^2 = -2$ and $Q\cdot P = \pm 1$ we have $Q^2 P^2 - (Q\cdot P)^2 = 3$. Since this is square free we must have $r(Q,P) = 1$.

So far we have taken Q and P to be arbitrary vectors in the lattice. However if Q and P are to be identified as the elements of the root lattice of a gauge algebra then the induced metric on the vector space E spanned by Q and P is euclidean. In this case we can do slightly better by noting that since the lattice is even, e_1^2 and e_2^2 must be even, while $e_1\cdot e_2$ is an integer. Thus $e_1^2 e_2^2$ is a multiple of 4, while $(e_1\cdot e_2)^2$ has the form $4s$ or $4s+1$ for some integer s. This implies that for positive $e_1^2 e_2^2 - (e_1\cdot e_2)^2$ – which is the case since the induced metric in E is euclidean – we must have $e_1^2 e_2^2 - (e_1\cdot e_2)^2 \geq 3$. Thus in order for $r(Q,P)$ to be different from 1, the combination $Q^2 P^2 - (Q\cdot P)^2$ must have the form kl^2 with $k\geq 3$, $l\geq 2$.

While the above analysis tells us under what condition $r(Q,P) = 1$, it does not tell us that if $Q^2 P^2 - (Q\cdot P)^2$ has the form kl^2 with $k\geq 3$, $l\geq 2$ then $r(Q,P)$ is necessarily larger than one. Thus $Q^2 P^2 - (Q\cdot P)^2$ being square free is sufficient but not necessary for $r(Q,P)$ to be 1.

References

[1] R. Dijkgraaf, E. P. Verlinde and H. L. Verlinde, "Counting dyons in N = 4 string theory," Nucl. Phys. B **484**, 543 (1997) [arXiv:hep-th/9607026].

[2] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, "Asymptotic degeneracy of dyonic N = 4 string states and black hole entropy," JHEP **0412**, 075 (2004) [arXiv:hep-th/0412287].

[3] D. Shih, A. Strominger and X. Yin, "Recounting dyons in N = 4 string theory," arXiv:hep-th/0505094.

[4] D. Gaiotto, "Re-recounting dyons in N = 4 string theory," arXiv:hep-th/0506249.

[5] D. Shih and X. Yin, "Exact black hole degeneracies and the topological string," JHEP **0604**, 034 (2006) [arXiv:hep-th/0508174].

[6] D. P. Jatkar and A. Sen, "Dyon spectrum in CHL models," JHEP **0604**, 018 (2006) [arXiv:hep-th/0510147].

[7] J. R. David, D. P. Jatkar and A. Sen, "Product representation of dyon partition function in CHL models," JHEP **0606**, 064 (2006) [arXiv:hep-th/0602254].

[8] A. Dabholkar and S. Nampuri, "Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift," arXiv:hep-th/0603066.

[9] J. R. David and A. Sen, "CHL dyons and statistical entropy function from D1-D5 system," JHEP **0611**, 072 (2006) [arXiv:hep-th/0605210].

[10] J. R. David, D. P. Jatkar and A. Sen, "Dyon spectrum in N = 4 supersymmetric type II string theories," arXiv:hep-th/0607155.

[11] J. R. David, D. P. Jatkar and A. Sen, "Dyon spectrum in generic N = 4 supersymmetric Z(N) orbifolds," arXiv:hep-th/0609109.

[12] A. Dabholkar and D. Gaiotto, "Spectrum of CHL dyons from genus-two partition function," arXiv:hep-th/0612011.

[13] A. Sen, "Walls of marginal stability and dyon spectrum in N = 4 supersymmetric string theories," arXiv:hep-th/0702141.

[14] A. Dabholkar, D. Gaiotto and S. Nampuri, "Comments on the spectrum of CHL dyons," arXiv:hep-th/0702150.

[15] N. Banerjee, D. P. Jatkar and A. Sen, "Adding charges to N = 4 dyons," arXiv:0705.1433 [hep-th].

[16] A. Sen, "Two Centered Black Holes and N=4 Dyon Spectrum," arXiv:0705.3874 [hep-th].

[17] M. C. N. Cheng and E. Verlinde, "Dying Dyons Don't Count," arXiv:0706.2363 [hep-th].

[18] A. Sen, "Black Hole Entropy Function, Attractors and Precision Counting of Microstates," arXiv:0708.1270 [hep-th].

[19] N. Askitas, Embeddings of 2-spheres in 4-manifolds, Manuscripta Math. **89** 35 (1996).

[20] K. S. Narain, "New Heterotic String Theories In Uncompactified Dimensions < 10," Phys. Lett. B **169**, 41 (1986).

[21] K. S. Narain, M. H. Sarmadi and E. Witten, "A Note on Toroidal Compactification of Heterotic String Theory," Nucl. Phys. B **279**, 369 (1987).

[22] A. Sen, "Three String Junction and N=4 Dyon Spectrum," arXiv:0708.3715 [hep-th].

[23] O. Bergman, "Three-pronged strings and 1/4 BPS states in N=4 super-Yang-Mills theory," Nucl. Phys. B **525**, 104 (1998) [arXiv:hep-th/9712211].

[24] M. Stern and P. Yi, "Counting Yang-Mills dyons with index theorems," Phys. Rev. D **62**, 125006 (2000) [arXiv:hep-th/0005275].

[25] E. J. Weinberg and P. Yi, "Magnetic monopole dynamics, supersymmetry, and duality," Phys. Rept. **438**, 65 (2007) [arXiv:hep-th/0609055].

[26] O. Bergman and B. Kol, "String webs and 1/4 BPS monopoles," Nucl. Phys. B **536**, 149 (1998) [arXiv:hep-th/9804160].

[27] A. Dabholkar, K. Narayan and S. Nampuri, to appear.

[28] S. Nampuri, P. K. Tripathy and S. P. Trivedi, "Duality Symmetry and the Cardy Limit," arXiv:0711.4671 [hep-th].

[29] J. Martinet, Perfect Lattices in Euclidean Spaces, Springer (2000).

[30] D. G. James, On Wit's theorem for Unimodular Quadratic Forms, Pacific J. Math. **26** 303 (1968).

[31] E. Looijenga and C. Peters, Torelli theorems for K?ahler K3 surfaces, Comp. Math. **42** 145 (1980/81).

[32] V.V. Nikulin, Integral Symmetric Bilinear Forms and Some of their Applications, Math. Izvestija **14** 103 (1980).

[33] J. H. Conway and N. J. A. Sloane, Sphere Packings, Lattices and Groups, Springer Verlag (1993).

[34] D. Bak, K. M. Lee and P. Yi, "Quantum 1/4 BPS dyons," Phys. Rev. D **61**, 045003 (2000) [arXiv:hep-th/9907090].

[35] C.T.C. Wall, On the Orthogonal Groups of Unimodular Quadratic Forms, Math. Annalen **147** 328 (1962).