Generalities of Quarter BPS Dyon Partition Function and Dyons of Torsion Two

Shamik Banerjee, Ashoke Sen and Yogesh K. Srivastava

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211019, INDIA

E-mail: bshamik, sen, yogesh@mri.ernet.in

Abstract

We propose a general set of constraints on the partition function of quarter BPS dyons in any $\mathcal{N} = 4$ supersymmetric string theory by drawing insight from known examples, and study the consequences of this proposal. The main ingredients of our analysis are duality symmetries, wall crossing formula and black hole entropy. We use our analysis to constrain the dyon partition function for two hitherto unknown cases – the partition function of dyons of torsion two (i.e. $\gcd(Q \wedge P)$=2) in heterotic string theory on T^6 and the partition function of dyons carrying untwisted sector electric charge in $\mathbb{Z}_2$ CHL model. With the help of these constraints we propose a candidate for the partition function of dyons of torsion two in heterotic string theory on T^6. This leads to a novel wall crossing formula for decay of quarter BPS dyons into half BPS dyons with non-primitive charge vectors. In an appropriate limit the proposed formula reproduces the known result for the spectrum of torsion two dyons in gauge theory.

Contents

1 Introduction and summary

The partition function of quarter BPS dyons is now known in a variety of $\mathcal{N} = 4$ supersymmetric string theories [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Generalization of these results to a class of $\mathcal{N} = 2$ supersymmetric string theories have also been proposed [19]. Our goal in this paper is to draw insight from these known results to postulate the general structure of dyon partition function for any class of quarter BPS dyons in any $\mathcal{N} = 4$ supersymmetric string theory.

The results of our analysis can be summarized as follows.

1. **Definition of the partition function:** Let (Q, P) denote the electric and magnetic charges carried by a dyon, and Q^2, P^2 and $Q \cdot P$ be the T-duality invariant quadratic

forms constructed from these charges.[1] In order to define the dyon partition function we first need to identify a suitable infinite subset $\mathcal{B}$ of dyons in the theory with the property that if we have two pairs of charges $(Q, P) \in \mathcal{B}$ and $(Q', P') \in \mathcal{B}$ with $Q^2 = Q'^2$, $P^2 = P'^2$ and $Q \cdot P = Q' \cdot P'$, then they must be related by a T-duality transformation. Furthermore given a pair of charge vectors $(Q, P) \in \mathcal{B}$, all other pairs of charge vectors related to it by T-duality should be elements of the set $\mathcal{B}$. We shall generate such a set $\mathcal{B}$ by beginning with a family $\mathcal{A}$ of charge vectors (Q, P) labelled by three integers such that Q^2, P^2 and $Q \cdot P$ are independent linear functions of these three integers, and then define $\mathcal{B}$ to be the set of all (Q, P) which are in the T-duality orbit of the set $\mathcal{A}$. We denote by $d(Q, P)$ the degeneracy, – or more precisely an index measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets – of quarter BPS dyons of charge (Q, P). Since $d(Q, P)$ should be invariant under a T-duality transformation, for $(Q, P) \in \mathcal{B}$ it should depend on (Q, P) only via the T-duality invariant combinations Q^2, P^2 and $Q \cdot P$:

$$d(Q, P) = f(Q^2, P^2, Q \cdot P)\,. \tag{1.1}$$

Note that for (1.1) to hold it is necessary to choose $\mathcal{B}$ in the way we have described. In particular if $\mathcal{B}$ had contained two elements with same Q^2, P^2 and $Q \cdot P$ but not related by a T-duality transformation, then $d(Q, P)$ can be different for these two elements and (1.1) will not hold. Even when $\mathcal{B}$ is chosen according to the prescription given above, eq.(1.1) cannot be strictly correct since $d(Q, P)$ could depend on the asymptotic moduli besides the charges and one can construct more general T-duality invariants using these moduli and the charges. Indeed, even though the index is not expected to change under a continuous change in the moduli, it could jump across the walls of marginal stability giving $d(Q, P)$ a dependence on the moduli. The reason that we can still write eq.(1.1) is that it is possible to label the different domains bounded by the walls of marginal stability by a set of discrete parameters $\vec{c}$ such that T-duality transformation does not change the parameters $\vec{c}$ [13, 18, 20]. Physically, if a domain is bounded by n walls of marginal stability, with the ith wall associated with the decay $(Q, P) \to (\alpha_i Q + \beta_i P, \gamma_i Q + \delta_i P) + ((1 - \alpha_i)Q - \beta_i P, -\gamma_i Q + (1 - \delta_i)P)$, then $\vec{c}$ is

[1] Irrespective of what description we are using, we shall denote by S-duality transformation the symmetry that acts on the complex scalar belonging to the gravity multiplet. In heterotic string compactification this would correspond to the axion-dilaton modulus. On the other hand T-duality will denote the symmetry that acts on the matter multiplet scalars. In the heterotic description these scalars arise from the components of the metric, anti-symmetric tensor fields and gauge fields along the compact directions.

the collection of the numbers $\{(\alpha_i, \beta_i, \gamma_i, \delta_i); 1 \le i \le n\}$. Due to T-duality invariance of $\vec{c}$, $d(Q, P)$ inside a given domain labelled by $\vec{c}$ will be invariant under a T-duality transformation on the charges only and will have the form (1.1). For different $\vec{c}$ the function f will be different, i.e. f has a hidden $\vec{c}$ dependence. We now define the dyon partition function associated with the set $\mathcal{B}$ to be[2]

$$\frac{1}{\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})} \equiv \sum_{Q^2, P^2, Q \cdot P} (-1)^{Q\cdot P+1} f(Q^2, P^2, Q \cdot P) e^{i\pi(\check{\sigma}Q^2 + \check{\rho}P^2 + 2\check{v}Q\cdot P)}, \tag{1.2}$$

where the sum runs over all the distinct triplets $(Q^2, P^2, Q \cdot P)$ which are present in the set $\mathcal{B}$. This relation can be inverted as

$$f(Q^2, P^2, Q \cdot P) \propto (-1)^{Q\cdot P+1} \int_{\mathcal{C}} d\check{\rho} d\check{\sigma} d\check{v}\, e^{-i\pi(\check{\sigma}Q^2 + \check{\rho}P^2 + 2\check{v}Q\cdot P)} \frac{1}{\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})}, \tag{1.3}$$

where $\mathcal{C}$ denotes an appropriate three dimensional subspace of the complex $(\check{\rho}, \check{\sigma}, \check{v})$ space. Along the 'contour' $\mathcal{C}$ the imaginary parts of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ are fixed at values where the sum in (1.2) converges, and the real parts of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ vary over an appropriate unit cell determined by the quantization laws of Q^2, P^2 and $Q \cdot P$ inside the set $\mathcal{B}$.

In all known cases the function f in different domains $\vec{c}$ is given by (1.3) with identical integrand, but the integration contour $\mathcal{C}$ depends on the choice of $\vec{c}$. Put another way, the same function $1/\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})$ admits different Fourier expansion in different regions in the complex $(\check{\rho}, \check{\sigma}, \check{v})$ space, since a Fourier expansion that is convergent in one region may not be convergent in another region. The coefficients of expansion in these different regions in the complex $(\check{\rho}, \check{\sigma}, \check{v})$ plane may then be regarded as the index $f(Q^2, P^2, Q \cdot P)$ in different domains in the asymptotic moduli space labelled by $\vec{c}$. We shall assume that *this result holds for all sets of dyons in all $\mathcal{N} = 4$ string theories.*

2. **Consequences of S-duality symmetry:** We now consider the effect of an S-duality transformation on the set $\mathcal{B}$. A generic S-duality transformation will take an element of $\mathcal{B}$ to outside $\mathcal{B}$, – we denote by H the subgroup of the S-duality group that leaves $\mathcal{B}$ invariant. This is the subgroup relevant for constraining the dyon partition function associated with the set $\mathcal{B}$. Since a generic element of H takes us from one domain bounded by walls of marginal stability to another such domain, it relates the function

[2]For $\mathcal{N} = 4$ supersymmetric $\mathbb{Z}_N$ orbifolds reviewed in [18] the function $\check{\Phi}$ is related to the function $\widetilde{\Phi}$ of [18] by the relation $\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) = \widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ with $(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = (\check{\sigma}/N, N\check{\rho}, \check{v})$.

f for one choice of $\vec{c}$ to the function f for another choice of $\vec{c}$. However since we have assumed that the dyon partition function $1/\check{\Phi}$ is independent of the domain label $\vec{c}$, we can use invariance under H to constrain the form of $\check{\Phi}$. In particular one finds that an S-duality symmetry of the form $(Q,P) \to (aQ+bP, cQ+dP)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$ gives the following constraint on $\check{\Phi}$:

$$\check{\Phi}(\check{\rho},\check{\sigma},\check{v}) = \check{\Phi}(d^2\check{\rho}+b^2\check{\sigma}+2bd\check{v}, c^2\check{\rho}+a^2\check{\sigma}+2ac\check{v}, cd\check{\rho}+ab\check{\sigma}+(ad+bc)\check{v}) . \tag{1.4}$$

Defining

$$\check{\Omega} = \begin{pmatrix} \check{\rho} & \check{v} \\ \check{v} & \check{\sigma} \end{pmatrix} , \tag{1.5}$$

we can express (1.4) as

$$\check{\Phi}((A\check{\Omega}+B)(C\check{\Omega}+D)^{-1}) = (\det(C\check{\Omega}+D))^k \, \check{\Phi}(\Omega) , \tag{1.6}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix} , \tag{1.7}$$

and k is as yet undermined since $\det(C\Omega+D)=1$.

Besides this symmetry, quantization of Q^2, P^2 and $Q\cdot P$ within the set $\mathcal{B}$ also gives rise to some translational symmetries of $\check{\Phi}$ of the form $\check{\Phi}(\check{\rho},\check{\sigma},\check{v}) = \check{\Phi}(\check{\rho}+a_1, \check{\sigma}+a_2, \check{v}+a_3)$ with a_1, a_2, a_3 taking values in an appropriate set. These can also be expressed as (1.6) with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_1 & a_3 \\ 0 & 1 & a_3 & a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.8}$$

3. **Wall crossing formula:** Given that the indices in different domains in the moduli space are given by different choices of the 3-dimensional integration contour in the $(\check{\rho},\check{\sigma},\check{v})$ space, the jump in the index as we cross a wall of marginal stability must be given by the residue of the integrand at the pole(s) encountered while deforming one contour to another. The walls across which the index jumps are the ones associated with decays

into a pair of half-BPS states.[3] We can label the decay products as [13]

$$(Q,P) \to (a_0 d_0 Q - a_0 b_0 P, c_0 d_0 Q - c_0 b_0 P) + (-b_0 c_0 Q + a_0 b_0 P, -c_0 d_0 Q + a_0 d_0 P)\,, \quad (1.9)$$

where a_0, b_0, c_0 and d_0 are normalized so that $a_0 d_0 - b_0 c_0 = 1$. In a generic situation a_0, b_0, c_0 and d_0 are not necessarily integers but are constrained by the fact that the final charges satisfy the charge quantization laws. In all known examples there is a specific correlation between a wall corresponding to a given decay and the location of the pole of the integrand that the contour crosses as we cross the wall in the moduli space. The location of the pole associated with the decay (1.9) is given by:

$$\check\rho c_0 d_0 + \check\sigma a_0 b_0 + \check v (a_0 d_0 + b_0 c_0) = 0\,. \quad (1.10)$$

We shall assume that *this formula continues to hold in all cases.* This then relates the jump in the index across a given wall of marginal stability to the residue of the partition function at a specific pole. An explicit choice of moduli dependent contour that satisfies this requirement can be found by generalizing the result of Cheng and Verlinde [17] to generic quarter BPS dyons in generic $\mathcal{N}=4$ supersymmetric string theories:

$$\begin{aligned}
\Im(\check\rho) &= \Lambda\left(\frac{|\tau|^2}{\tau_2} + \frac{Q_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}}\right), \\
\Im(\check\sigma) &= \Lambda\left(\frac{1}{\tau_2} + \frac{P_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}}\right), \\
\Im(\check v) &= -\Lambda\left(\frac{\tau_1}{\tau_2} + \frac{Q_R\cdot P_R}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}}\right), \quad (1.11)
\end{aligned}$$

where Λ is a large positive number, $\Im(z)$ denotes the imaginary part of z,

$$Q_R^2 = Q^T(M+L)Q, \quad P_R^2 = P^T(M+L)P, \quad Q_R\cdot P_R = Q^T(M+L)P\,, \quad (1.12)$$

[3]For a certain class of dyons kinematics allows decay into a pair of quarter BPS states or a half BPS and a quarter BPS states on a codimension 1 subspace of the moduli space. These correspond to decays of the form $(Q,P) \to (\alpha Q+\beta P, \gamma Q+\delta P) + ((1-\alpha)Q - \beta P, -\gamma Q + (1-\delta)P)$ with some of the $\alpha,\beta,\gamma,\delta$ fractional so that we can have $0 < (\alpha\delta - \beta\gamma) < 1$ and $0 \le ((1-\alpha)(1-\delta) - \beta\gamma < 1$ [21,22]. However a naive counting of the number of fermion zero modes on a half BPS - quarter BPS and quarter BPS - quarter BPS combination suggests that there are additional fermion zero modes besides the ones associated with the broken supersymmetry generators. This makes the index associated with such a configuration vanish. Although a rigorous analysis of this system is lacking at present, we shall proceed with the assumption that the result is valid so that such decays do not change the index. Otherwise the dyon partition function will have additional poles associated with the jump in the index across these additional walls of marginal stability. We wish to thank F. Denef for a discussion on this point.

$\tau \equiv \tau_1 + i\tau_2$ denotes the asymptotic value of the axion-dilaton moduli which belong to the gravity multiplet and M is the asymptotic value of the symmetric matrix valued moduli field of the matter multiplet satisfying $MLM^T = L$. The choice (1.11) of course is not unique since we can deform the contour without changing the result for the index as long as we do not cross a pole of the partition function.

Independent of the above analysis, the change in the index across a wall of marginal stability can be computed using the wall crossing formula [23, 24, 25, 26, 27, 13, 28]. This tells us that as we cross a wall of marginal stability associated with the decay $(Q, P) \to (Q_1, P_1) + (Q_2, P_2)$, the index jumps by an amount[4]

$$(-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}\, (Q_1 \cdot P_2 - Q_2 \cdot P_1)\, d_h(Q_1, P_1) d_h(Q_2, P_2) \tag{1.13}$$

up to a sign, where $d_h(Q, P)$ denotes the index of half-BPS states carrying charge (Q, P). For the decay described in (1.9) the relevant half-BPS indices are of the form $d_h(a_0M_0, c_0M_0)$ and $d_h(b_0N_0, d_0N_0)$ where $M_0 \equiv d_0Q - b_0P$ and $N_0 \equiv -c_0Q + a_0P$. T-duality invariance implies that – modulo some subtleties discussed below eqs.(4.11) – the dependence of $d_h(a_0M_0, c_0M_0)$ and $d_h(b_0N_0, d_0N_0)$ on M_0 and N_0 must come via the combinations M_0^2 and N_0^2 respectively. We now define

$$\phi_e(\tau; a_0, c_0) \equiv \sum_{M_0^2} e^{\pi i \tau M_0^2} d_h(a_0M_0, c_0M_0), \quad \phi_m(\tau; b_0, d_0) \equiv \sum_{N_0^2} e^{\pi i \tau N_0^2} d_h(b_0N_0, d_0N_0)\,, \tag{1.14}$$

where the sums are over the sets of (M_0^2, N_0^2) values which arise in the possible decays of the dyons in the set $\mathcal{B}$ via (1.9). Then (1.13) agrees with the residue of the partition function at the pole (1.10) if we assume that $\check\Phi$ has a double zero at (1.10) where it behaves as

$$\begin{aligned} \check\Phi(\check\rho, \check\sigma, \check v) \quad &\propto \quad \check v'^2\, \phi_e(\check\sigma'; a_0, c_0)\, \phi_m(\check\rho'; b_0, d_0)\,, \qquad \check v' \equiv \check\rho c_0 d_0 + \check\sigma a_0 b_0 + \check v(a_0d_0 + b_0c_0), \\ &\check\sigma' \equiv c_0^2\check\rho + a_0^2\check\sigma + 2a_0c_0\check v, \quad \check\rho' \equiv d_0^2\check\rho + b_0^2\check\sigma + 2b_0d_0\check v\,. \end{aligned} \tag{1.15}$$

Since for any given system the allowed values of (a_0, b_0, c_0, d_0) can be found from charge quantization laws, (1.15) gives us information about the locations of the zeroes on $\check\Phi$ and its behaviour at these zeroes in terms of the spectrum of half-BPS states in the theory.

[4]Eq.(1.13) holds only if the dyons (Q_1, P_1) and (Q_2, P_2) are primitive. As will be discussed later, this formula gets modified for non-primitive decay.

4. **Additional modular symmetries:** Often the partition functions associated with $d_h(Q,P)$ have modular properties, *e.g.* the function $\phi_m(\tau; a_0, c_0)$ could transform as a modular form under $\tau \to (\alpha\tau+\beta)/(\gamma\tau+\delta)$ and $\phi_e(\tau; b_0, d_0)$ could transform as a modular form under $\tau \to (p\tau+q)/(r\tau+s)$ with $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and $\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ belonging to certain subgroups of $SL(2, \mathbb{Z})$. Some of these may be accidental symmetries, but some could be consequences of exact symmetries of the full partition function $\check\Phi(\check\rho, \check\sigma, \check v)^{-1}$. Using (1.15) one finds that those which can be lifted to exact symmetries of $\check\Phi$ can be represented as symplectic transformations of the form (1.6) with $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ given by

$$\begin{pmatrix} d_0 & b_0 & 0 & 0\\ c_0 & a_0 & 0 & 0\\ 0 & 0 & a_0 & -c_0\\ 0 & 0 & -b_0 & d_0\end{pmatrix}^{-1} \begin{pmatrix}\alpha & 0 & \beta & 0\\ 0 & 1 & 0 & 0\\ \gamma & 0 & \delta & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0\\ c_0 & a_0 & 0 & 0\\ 0 & 0 & a_0 & -c_0\\ 0 & 0 & -b_0 & d_0\end{pmatrix} \tag{1.16}$$

and

$$\begin{pmatrix} d_0 & b_0 & 0 & 0\\ c_0 & a_0 & 0 & 0\\ 0 & 0 & a_0 & -c_0\\ 0 & 0 & -b_0 & d_0\end{pmatrix}^{-1} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & p & 0 & q\\ 0 & 0 & 1 & 0\\ 0 & r & 0 & s\end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0\\ c_0 & a_0 & 0 & 0\\ 0 & 0 & a_0 & -c_0\\ 0 & 0 & -b_0 & d_0\end{pmatrix} \tag{1.17}$$

respectively. These represent additional symmetries of $\check\Phi$ besides the ones associated with S-duality invariance and charge quantization laws. Furthermore the constant k appearing in (1.6) is given by the weight of ϕ_e and ϕ_m minus 2.

It is these additional symmetries which make the symmetry group of $\check\Phi$ a non-trivial subgroup of $Sp(2, \mathbb{Z})$. The S-duality transformations (1.7) and the translation symmetries (1.8) are both associated with $Sp(2, \mathbb{Z})$ matrices $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ with $C=0$. In contrast the trnsformations (1.16), (1.17) typically have $C \neq 0$.

Since we do not *a priori* know which part of the modular symmetries of ϕ_e and ϕ_m survive as symmetries of $\check\Phi$, this does not give a foolproof method for identifying symmetries of $\check\Phi$. However often by combining information from the behaviour of $\check\Phi$ around different zeroes one can make a clever guess.

5. **Black hole entropy:** Additional constraints may be found be requiring that in the limit of large charges the index reproduces correctly the black hole entropy.[5] In particular, by

[5]Here we are implicitly assuming that when the effect of interactions are taken into account, only index

requiring that we reproduce the black hole entropy $\pi\sqrt{Q^2P^2-(Q\cdot P)^2}$ that arises in the supergravity approximation one finds that $\check\Phi$ is required to have a zero at [1, 2, 6, 9, 18]

$$\check\rho\check\sigma-\check v^2+\check v=0\,. \tag{1.18}$$

In order to find the behaviour of $\check\Phi$ near this zero one needs to calculate the first non-leading correction to the black hole entropy and compare this with the first non-leading correction to the formula for the index. In general the former requires the knowledge of the complete set of four derivative terms in the effective action, but in all known examples one can reproduce the answer for the index just by taking into account the effect of the Gauss-Bonnet term in the action. If we assume that this continues to hold in general then by matching the first non-leading corrections on both sides one can relate the behaviour of $\check\Phi$ near (1.18) to the coefficient of the Gauss-Bonnet term. The result is

$$\check\Phi(\check\rho,\check\sigma,\check v)\propto(2v-\rho-\sigma)^k\left\{v^2\,g(\rho)\,g(\sigma)+\mathcal{O}(v^4)\right\}, \tag{1.19}$$

where

$$\rho=\frac{\check\rho\check\sigma-\check v^2}{\check\sigma},\quad \sigma=\frac{\check\rho\check\sigma-(\check v-1)^2}{\check\sigma},\quad v=\frac{\check\rho\check\sigma-\check v^2+\check v}{\check\sigma}, \tag{1.20}$$

and $g(\tau)$ is a modular form of weight $k+2$ of the S-duality group, related to the Gauss-Bonnet term

$$\int d^4x\,\sqrt{-\det g}\,\phi(a,S)\,\left\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2\right\}, \tag{1.21}$$

via the relation

$$\phi(a,S)=-\frac{1}{64\pi^2}\left((k+2)\ln S+\ln g(a+iS)+\ln g(-a+iS)\right)+\text{constant}\,. \tag{1.22}$$

Here $\tau=a+iS$ is the axion-dilaton modulus.

In §6 we apply the considerations described above to several examples. These include known examples involving unit torsion dyons in heteroric string theory on T^6 and CHL orbifolds and also some unknown cases like dyons of torsion 2 in heterotic string theory on T^6 (i.e. dyons for which gcd($Q\wedge P$)=2 [14]) and dyons carrying untwisted sector charges in $\mathbb{Z}_2$ CHL orbifold [29, 30]. In the latter cases we determine the constraints imposed by the S-duality

worth of states remain as BPS states so that the black hole entropy can be compared to the logarithm of the index.

invariance and wall crossing formulæ and also try to use the known modular properties of half-BPS states to guess the symmetry group of the quarter BPS dyon partition function.

In §7 we propose a formula for the dyon partitions function of torsion two dyons in heterotic string theory on T^6. The formula for the partition function when Q and P are both primitive but $(Q \pm P)$ are twice primitive vectors is

$$\begin{aligned}\frac{1}{\check{\Phi}(\check{\rho},\check{\sigma},\check{v})} &= \frac{1}{8}\Bigg[\frac{1}{\Phi_{10}(\check{\rho},\check{\sigma},\check{v})}+\frac{1}{\Phi_{10}(\check{\rho}+\frac{1}{2},\check{\sigma}+\frac{1}{2},\check{v})}+\frac{1}{\Phi_{10}(\check{\rho}+\frac{1}{4},\check{\sigma}+\frac{1}{4},\check{v}+\frac{1}{4})}\\ &+\frac{1}{\Phi_{10}(\check{\rho}+\frac{3}{4},\check{\sigma}+\frac{3}{4},\check{v}+\frac{1}{4})}+\frac{1}{\Phi_{10}(\check{\rho}+\frac{1}{2},\check{\sigma}+\frac{1}{2},\check{v}+\frac{1}{2})}\\ &+\frac{1}{\Phi_{10}(\check{\rho},\check{\sigma},\check{v}+\frac{1}{2})}+\frac{1}{\Phi_{10}(\check{\rho}+\frac{3}{4},\check{\sigma}+\frac{3}{4},\check{v}+\frac{3}{4})}+\frac{1}{\Phi_{10}(\check{\rho}+\frac{1}{4},\check{\sigma}+\frac{1}{4},\check{v}+\frac{3}{4})}\Bigg]\\ &+\frac{2}{\Phi_{10}(\check{\rho}+\check{\sigma}+2\check{v},\check{\rho}+\check{\sigma}-2\check{v},\check{\sigma}-\check{\rho})}\end{aligned} \tag{1.23}$$

where Φ_{10} is the weight 10 Igusa cusp form of $Sp(2, \mathbb{Z})$ describing the inverse partition function of torsion one dyons. The sum of the first eight terms on the right hand side of (1.23) coincides with the partition function of unit torsion dyons subject to the constraints that $Q^2+P^2 \pm 2Q\cdot P$ are multiples of 8; the last term is a new addition. We show that (1.23) satisfies all the required consistency conditions. First of all it has the required S-duality invariance. It also satisfies the wall crossing formulæ at all the walls of marginal stability at which the original dyon decays into a pair of primitive dyons. It satisfies the constraint (1.19) coming from the requirement that the statistical entropy and the black hole entropy agrees up to the first non-leading order in inverse powers of charges. Furthermore by taking an appropriate limit of this formula we can reproduce the known results for torsion two dyons in gauge theories [31, 32, 33, 34, 35].

In the case of torsion two dyons with Q, P both primitive, the vectors $Q \pm P$ are not primitive, but $(Q \pm P)/2$ are primitive vectors [20]. As a result for the decay into

$$(Q_1, P_1) = (Q - P, 0), \qquad (Q_2, P_2) = (P, P)\,, \tag{1.24}$$

the charge vector (Q_1, P_1) is not primitive. Computing the jump in the index from (1.23) we find that in this case the change in the index across this wall of marginal stability is given by

$$\Delta d(Q,P) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}(Q_1\cdot P_2 - Q_2\cdot P_1)\left\{d_h\left(Q_1,P_1\right)+d_h\left(\frac{1}{2}Q_1,\frac{1}{2}P_1\right)\right\}d_h\left(Q_2,P_2\right)\,. \tag{1.25}$$

This differs from the formula (1.13). A similar modification of the wall crossing formula for decays into non-primitive states in $\mathcal{N}=2$ supersymmetric string theories has been suggested in [28].

There are two more classes of dyons of torsion two, – one where Q is primitive and P is twice a primitive vector and the other where P is primitive and Q is twice a primitive vector. The partition functions for these dyons can be recovered from the one given above by S-duality transformations $(Q,P)\to(Q,P-Q)$ and $(Q,P)\to(Q-P,P)$ respectively [20]. This amount to making replacements $(\check\rho,\check\sigma,\check v)\to(\check\rho,\check\sigma+\check\rho+2\check v,\check v+\check\rho)$ and $(\check\rho,\check\sigma,\check v)\to(\check\rho+\check\sigma+2\check v,\check\sigma,\check v+\check\sigma)$ respectively in eq.(1.23).

Our analysis can also be used to predict the form of the partition function of dyons of higher torsion. These results will be presented in a forthcoming publication [36].

Although we have presented most of our analysis as a way of extracting information about the partition function of quarter BPS states from known spectrum of half-BPS states, we could also use it in the reverse direction. In the final section §8 we provide some examples in the context of $\mathbb{Z}_N$ orbifold models where the knowledge of the quarter BPS partition function can be used to compute the spectrum of a certain class of half-BPS states.

2 The dyon partition function

Let us consider a particular $\mathcal{N}=4$ supersymmetric string theory in four dimensions with a total of r U(1) gauge fields including the six graviphotons. The electric and magnetic charges in this theory are represented by r dimensional vectors Q and P, and there is a T-duality invariant metric L of signature $(6,r-6)$ that can be used to define the inner product of the charges. Let us consider an (infinite) set $\mathcal{B}$ of dyon charge vectors (Q,P) with the property that if two different members of the set have the same values of $Q^2\equiv Q^TLQ$, $P^2\equiv P^TLP$ and $Q\cdot P\equiv Q^TLP$ then there must exist a T-duality transformation that relates the two members. In other words if there are T-duality invariants other than Q^2, P^2 and $Q\cdot P$ then for all members of the set $\mathcal{B}$ with a given set of values of $(Q^2,P^2,Q\cdot P)$ these other T-duality invariants must have the same values. We shall generate such a set $\mathcal{B}$ by beginning with a family $\mathcal{A}$ of charge vectors (Q,P) labelled by three integers such that the triplet $(Q^2,P^2,Q\cdot P)$ are independent linear functions of these three integers, and then define $\mathcal{B}$ to be the set of all (Q,P) which are in the T-duality orbit of the set $\mathcal{A}$. Such a set $\mathcal{B}$ automatically satisfies the restriction mentioned above since given two elements of $\mathcal{B}$ with the same values of $(Q^2,P^2,Q\cdot P)$, each will be related by a T-duality transformation to the unique element of $\mathcal{A}$ with these values of $(Q^2, P^2, Q\cdot P)$. An example of such a set $\mathcal{A}$ can be found in eqs.(6.1.3), (6.1.4).

Our object of interest is the index $d(Q,P)$, measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets of quarter BPS dyons carrying charges $(Q,P)\in\mathcal{B}$. Typically the index, besides depending on (Q,P), also depends of the domain in which the asymptotic moduli lie. These domains are bounded by walls of marginal stability associated with decays of the form $(Q,P)\to(\alpha Q+\beta P, \gamma Q+\delta P)+((1-\alpha)Q-\beta P, -\gamma Q+(1-\delta)P)$ for appropriate values of $(\alpha,\beta,\gamma,\delta)$ associated with the quantization conditions [22, 21, 37]. For fixed values of the other moduli these walls describe circles or straight lines in the axion-dilaton moduli space labelled by the complex parameter τ [13,22]. We denote by $\vec{c}$ the collection of $(\alpha_i,\beta_i,\gamma_i,\delta_i)$ bordering a particular domain in the moduli space; inside any such domain the index remains unchanged. It has been shown in [13] that the parameters $\vec{c}$ labelling a domain remain invariant under a simultaneous T-duality transformation on the charges and the moduli. Since $d(Q,P)$ must be invariant under simultaneous T-duality transformation on the charges and the moduli, we can conclude that for a given $\vec{c}$ the index $d(Q,P)$ for $(Q,P)\in\mathcal{B}$ will be a function only of the T-duality invariants $(Q^2,P^2,Q\cdot P)$. We shall express this as $f(Q^2,P^2,Q\cdot P,\vec{c})$.

Let us now introduce the partition function

$$\frac{1}{\check{\Phi}(\check{\rho},\check{\sigma},\check{v})}\equiv\sum_{Q^2,P^2,Q\cdot P}(-1)^{Q\cdot P+1}\, f(Q^2,P^2,Q\cdot P;\vec{c}_0)e^{i\pi(\check{\sigma}Q^2+\check{\rho}P^2+2\check{v}Q\cdot P)}\,. \tag{2.1}$$

where $\vec{c}_0$ denotes some specific domain in the moduli space bounded by a set of walls of marginal stability. The sum runs over allowed values of Q^2, P^2 and $Q\cdot P$ for the dyons belonging to the set $\mathcal{B}$. The factor of $(-1)^{Q\cdot P+1}$ has been included for convenience. $\check{\Phi}$ so defined is expected to be a periodic function of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$, with the periods depending on the quantization condition on P^2, Q^2 and $Q\cdot P$. Let the periods be T_1, T_2 and T_3 respectively – these represent inverses of the quanta of $P^2/2$, $Q^2/2$ and $Q\cdot P$ belonging to the set $\mathcal{B}$. The sum given in (2.1) is typically not convergent for real values of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$. However often it may be made convergent by treating $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ as complex variables and working in appropriate domain in the complex plane. We shall assume that this can be done. We may now invert (2.1) as

$$\begin{aligned} f(Q^2,P^2,Q\cdot P;\vec{c}_0) &= \frac{(-1)^{Q\cdot P+1}}{T_1T_2T_3}\int_{iM_1-T_1/2}^{iM_1+T_1/2}d\check{\rho}\int_{iM_2-T_2/2}^{iM_2+T_2/2}d\check{\sigma}\int_{iM_3-T_3/2}^{iM_3+T_3/2}d\check{v} \\ &\qquad e^{-i\pi(\check{\sigma}Q^2+\check{\rho}P^2+2\check{v}Q\cdot P)}\,\frac{1}{\check{\Phi}(\check{\rho},\check{\sigma},\check{v})}\,, \end{aligned} \tag{2.2}$$

provided the imaginary parts M_1, M_2 and M_3 of $\check\rho$, $\check\sigma$ and $\check v$ are fixed in a region where the original sum (2.1) is convergent.

During the above discussion we have implicitly assumed that the quantization laws of Q^2, P^2 and $Q\cdot P$ are uncorrelated so that $\check\Phi(\check\rho,\check\sigma,\check v)$ is separately invariant under $\check\rho\to\check\rho+T_1$, $\check\sigma\to\check\sigma+T_2$ and $\check v\to\check v+T_3$. In general we can have more complicated periods which involve simultaneous shifts of $\check\rho$, $\check\sigma$ and $\check v$. In this case the integration in (2.2) needs to be carried out over an appropriate unit cell in the $(\Re(\check\rho),\Re(\check\sigma),\Re(\check v))$ space and the factor of $T_1T_2T_3$ in the denominator will be replaced by the volume of the unit cell.

3 Consequences of S-duality symmetry

Let us now assume that the theory has S-duality symmetries of the form

$$Q\to Q''=aQ+bP, \qquad P\to P''=cQ+dP\,, \tag{3.1}$$

for appropriate choice of (a,b,c,d). Under this transformation

$$Q''^2=a^2Q^2+b^2P^2+2abQ\cdot P, \quad P''^2=c^2Q^2+d^2P^2+2cdQ\cdot P, \quad Q''\cdot P''=acQ^2+bdP^2+(ad+bc)Q\cdot P\,. \tag{3.2}$$

A generic S-duality transformation acting on an arbitrary element of $\mathcal{B}$ will give rise to (Q'',P'') outside the set $\mathcal{B}$ for which the index formula is given by the function f. We shall restrict ourselves to a subset of S-duality transformations which takes an element of the set $\mathcal{B}$ to another element of the set $\mathcal{B}$. For such transformations, the S-duality invariance of the theory tells us that

$$f(Q''^2,P''^2,Q''\cdot P'',\vec c_0'')=f(Q^2,P^2,Q\cdot P,\vec c_0)\,, \tag{3.3}$$

where $\vec c_0''$ denotes the collection $\{(\alpha_i'',\beta_i'',\gamma_i'',\delta_i'')\}$ of domain walls related to the set $\{(\alpha_i,\beta_i,\gamma_i,\delta_i)\}$ associated with $\vec c_0$ by the relation [13, 20]

$$\begin{pmatrix}\alpha_i'' & \beta_i''\\ \gamma_i'' & \delta_i''\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha_i & \beta_i\\ \gamma_i & \delta_i\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}. \tag{3.4}$$

Physically the domain corresponding to $\vec c_0''$ represents the image of the one corresponding to $\vec c_0$ under simultaneous S-duality transformation on the charges and the moduli. Making a change of variables

$$\check\rho=d^2\check\rho''+b^2\check\sigma''+2bd\check v'', \quad \check\sigma=c^2\check\rho''+a^2\check\sigma''+2ac\check v'', \quad \check v=cd\check\rho''+ab\check\sigma''+(ad+bc)\check v''\,, \tag{3.5}$$

in (2.2) and using the fact that

$$(-1)^{Q\cdot P} = (-1)^{Q''\cdot P''}, \quad \check\sigma Q^2 + \check\rho P^2 + 2\check v Q\cdot P = \check\sigma'' Q''^2 + \check\rho'' P''^2 + 2\check v'' Q''\cdot P'',$$
$$d\check\rho\wedge d\check\sigma\wedge d\check v = d\check\rho''\wedge d\check\sigma''\wedge d\check v''\,, \tag{3.6}$$

under an S-duality transformation, we can express (3.3) as

$$f(Q''^2, P''^2, Q''\cdot P'', \vec c_0'') = \frac{(-1)^{Q''\cdot P''+1}}{T_1T_2T_3}\int_{\mathcal C} d\check\rho'' d\check\sigma'' d\check v''\, e^{-i\pi(\check\sigma'' Q''^2+\check\rho'' P''^2+2\check v'' Q''\cdot P'')}\,\frac{1}{\check\Phi(\check\rho,\check\sigma,\check v)}\,, \tag{3.7}$$

where $\mathcal C$ is the image of the original region of integration (2.2) in the complex $(\check\rho'', \check\sigma'', \check v'')$ plane:

$$\Im(\check\rho'') = a^2M_1 + b^2M_2 - 2abM_3, \quad \Im(\check\sigma'') = c^2M_1 + d^2M_2 - 2cdM_3,$$
$$\Im(\check v'') = -acM_1 - bdM_2 + (ad+bc)M_3\,. \tag{3.8}$$

We would like to get some constraint on the function $\check\Phi$ by comparing (2.2) with (3.7). For this we note that we can replace (Q,P) by (Q'',P'') and $(\check\rho,\check\sigma,\check v)$ by $(\check\rho'',\check\sigma'',\check v'')$ everywhere in (2.2) since they are dummy variables. This gives

$$f(Q''^2, P''^2, Q''\cdot P''; \vec c_0) = \frac{(-1)^{Q''\cdot P''+1}}{T_1T_2T_3}\int_{iM_1-T_1/2}^{iM_1+T_1/2} d\check\rho'' \int_{iM_2-T_2/2}^{iM_2+T_2/2} d\check\sigma'' \int_{iM_3-T_3/2}^{iM_3+T_3/2} d\check v'' \; e^{-i\pi(\check\sigma'' Q''^2+\check\rho'' P''^2+2\check v'' Q''\cdot P'')}\frac{1}{\check\Phi(\check\rho'',\check\sigma'',\check v'')}\,. \tag{3.9}$$

Since in general $\vec c_0$ and $\vec c_0''$ describe different domains, we cannot compare (3.7) and (3.9) to constrain the form of $\check\Phi$ without any further input.[6] However the dyon spectrum in a variety of $\mathcal N = 4$ supersymmetric string theories displays the feature that the spectrum in two different domains $\vec c_0''$ and $\vec c_0$ are both given as integrals with the same integrand, but for $\vec c_0''$ the integration over $(\check\rho'',\check\sigma'',\check v'')$ is carried out over a different subspace than the one given in (3.9). In particular if $\vec c_0''$ is related to $\vec c_0$ by an S-duality transformation then this subspace is given by the integration region $\mathcal C$ given in (3.8). We shall assume that this feature continues to hold in the general situation. In that case the effect of replacing $\vec c_0$ by $\vec c_0''$ in eq.(3.9) is to replace the integration contour by $\mathcal C$ on the right hand side:

$$f(Q''^2, P''^2, Q''\cdot P''; \vec c_0'') = \frac{(-1)^{Q''\cdot P''+1}}{T_1T_2T_3}\int_{\mathcal C} d\check\rho'' d\check\sigma'' d\check v''\, e^{-i\pi(\check\sigma'' Q''^2+\check\rho'' P''^2+2\check v'' Q''\cdot P'')}\,\frac{1}{\check\Phi(\check\rho'',\check\sigma'',\check v'')}\,. \tag{3.10}$$

[6]The only exceptions are those S-duality transformations which leave the domain $\vec c_0$ unchanged [13].

Comparing (3.10) and (3.7) we get

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) = \check{\Phi}(\check{\rho}'', \check{\sigma}'', \check{v}'') \,. \tag{3.11}$$

For future reference we shall rewrite the transformation laws (3.5) in a suggestive form. We define

$$\check{\Omega} = \begin{pmatrix} \check{\rho} & \check{v} \\ \check{v} & \check{\sigma} \end{pmatrix} \,. \tag{3.12}$$

Then the transformations (3.5) may be written as

$$\check{\Omega} = (A\check{\Omega}'' + B)(C\check{\Omega}'' + D)^{-1} \,, \tag{3.13}$$

where A, B, C and D are 2×2 matrices, given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix} \,. \tag{3.14}$$

Eq.(3.11) now gives (after replacing the dummy variable $\check{\Omega}''$ by $\check{\Omega}$ on both sides),

$$\check{\Phi}((A\check{\Omega} + B)(C\check{\Omega} + D)^{-1}) = \det(C\check{\Omega} + D)^k \check{\Phi}(\check{\Omega}) \,, \tag{3.15}$$

for A, B, C, D given in (3.14). Here k is an arbitrary number. Since $\det(C\Omega + D) = 1$, we cannot yet ascertain the value of k.

To this we can also append the translational symmetries of $\check{\Phi}$:

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) = \check{\Phi}(\check{\rho} + a_1, \check{\sigma} + a_2, \check{v} + a_3) \,, \tag{3.16}$$

where a_i's are integer multiples of the T_i's. It is convenient, although not necessary, to work with appropriately rescaled Q and/or P so that the T_i's and hence the a_i's are integers. This symmetry can also be rewritten as (3.15) with the choice

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_1 & a_3 \\ 0 & 1 & a_3 & a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,. \tag{3.17}$$

Again since $\det(C\check{\Omega} + D) = 1$ the choice of k is arbitrary.

The alert reader would have noticed that although we have expressed the consequences of S-duality invariance and charge quantization conditions as symmetries of the function $\check{\Phi}$ under a symplectic transformation, the symplectic transformations arising this way are trivial, – for all the transformations arising this way the matrix C vanishes and hence the transformations act linearly on the variables $\check\rho$, $\check\sigma$ and $\check v$. In order to show that the function $\check\Phi$ has non-trivial modular properties we need to find symmetries of $\check\Phi$ which have non-vanishing C. This will also determine the weight of $\check\Phi$ under the modular transformation. To get a hint about any possible additional symmetries of $\check\Phi$ we need to make use of the wall crossing formula for the dyon spectrum of $\mathcal{N}=4$ supersymmetric string theories. This will be the subject of discussion in §4.

4 Constraints from wall crossing

As has already been discussed, the index associated with the quarter BPS dyon spectrum in $\mathcal{N}=4$ supersymmetric string theories can undergo discontinuous jumps across walls of marginal stability. *A priori* the formula for the dyon spectrum in different domains labelled by the vector $\vec c$ could be completely different. However the study of dyon spectrum in a variety of $\mathcal{N}=4$ supersymmetric string theories shows that in different domains the index continues to be given by an expression similar to (2.2), the only difference being that the choice of the 3 real dimensional subspace (contour) over which we carry out the integration in the complex $(\check\rho,\check\sigma,\check v)$ plane is different in different domains. As a result the difference between the indices in two different domains is given by the sum of residues of the integrand at the poles we encounter while deforming the contour associated with one domain to the contour associated with another domain. As a special example of this we can consider the decay $(Q,P)\to(Q,0)+(0,P)$. In all known examples change in the index across this wall of marginal stability is accounted for by the residue of a double pole of the integrand at $\check v=0$, i.e. as we cross this particular wall of marginal stability in the moduli space, the integration contour crosses the pole at $\check v=0$. Since the change in the index as we cross a given wall can be found using the wall crossing formula [23, 24, 25, 26, 27, 13, 28], this provides information on the residue of the integrand at the $\check v=0$ pole.

There are many other possible decays of a quarter BPS state into a pair of half BPS states. All such decays may be parametrized as [13]

$$(Q,P)\to(a_0d_0Q-a_0b_0P,c_0d_0Q-c_0b_0P)+(-b_0c_0Q+a_0b_0P,-c_0d_0Q+a_0d_0P)\,,\quad a_0d_0-b_0c_0=1\,. \tag{4.1}$$

a_0, b_0, c_0, d_0 are not necessarily all integers, but must be such that the charges carried by the decay products belong to the charge lattice. One can try to use the wall crossing formulæ associated with these decays to further constrain the form of $\check{\Phi}$. For unit torsion states in heterotic string theory on T^6, a_0, b_0, c_0 and d_0 are integers and the decay given in (4.1) is related to the decay $(Q,P) \to (Q,0) + (0,P)$ via an S-duality transformation $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. Thus the change in the index across the wall is controlled by the residue of the partition function at a new pole that is related to the $\check{v}=0$ pole by the S-duality transformation (3.5). This gives the location of the pole to be at

$$\check{\rho} c_0 d_0 + \check{\sigma} a_0 b_0 + \check{v}(a_0 d_0 + b_0 c_0) = 0\,. \tag{4.2}$$

As long as $\check{\Phi}$ is manifestly S-duality invariant, i.e. satisfies (3.14), (3.15), the residues at these poles will automatically satisfy the wall crossing formula. Thus they do not provide any new information. However in a generic situation new walls may appear, labelled by fractional values of a_0, b_0, c_0, d_0. Also the S-duality group is smaller. As a result not all the walls can be related to each other by S-duality transformation. It is tempting to speculate that the jump across any wall of marginal stability associated with the decay (4.1) is described by the residue of the partition function at the pole at (4.2). We shall proceed with this assumption – this will be one of our key postulates.[7]

Before we proceed we shall show that this postulate is internally consistent, i.e. it is possible to choose $\mathcal{C}$ at different points in the moduli space consistent with this postulate. For this we generalize the contour prescription of [17], assuming that it holds for all $\mathcal{N}=4$ supersymmetric string theories. Let $\tau = \tau_1 + i\tau_2$ be the axion-dilaton moduli, M be the usual $r\times r$ symmetric matrix valued moduli satisfying $MLM^T = L$, and

$$Q_R^2 \equiv Q^T(M+L)Q, \quad P_R^2 \equiv P^T(M+L)P, \quad Q_R\cdot P_R \equiv Q^T(M+L)P\,. \tag{4.3}$$

Then at the point (τ, M) in the space of asymptotic moduli we choose $\mathcal{C}$ to be

$$\begin{aligned} \Im(\check{\rho}) &= \Lambda\left(\frac{|\tau|^2}{\tau_2} + \frac{Q_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}}\right), \\ \Im(\check{\sigma}) &= \Lambda\left(\frac{1}{\tau_2} + \frac{P_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}}\right), \end{aligned}$$

[7] Of course the translation symmetries (3.16) allow us to shift a pole at (4.2) to other equivalent locations. Our postulate asserts that the contribution comes from poles which can be brought to (4.2) using the translation symmetries (3.16). In that case we can choose the unit cell over which we carry out the integration in (2.2) in such a way that only the pole at (4.2) contributes to the jump in the index across the wall at (4.1). A possible exception to this will be discussed in the paragraphs above eq.(4.12) where we address some subtle issues.

$$\Im(\check{v}) \quad = \quad -\Lambda \left(\frac{\tau_1}{\tau_2} + \frac{Q_R \cdot P_R}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} \right) , \tag{4.4}$$

where Λ is a large positive number. Then on $\mathcal{C}$

$$\begin{aligned} & \Im(c_0 d_0 \check{\rho} + a_0 b_0 \check{\sigma} + (a_0 d_0 + b_0 c_0)\check{v}) \\ = \; & \frac{c_0 d_0}{\tau_2} \Lambda \left\{ \left(\tau_2 + \frac{E}{2c_0 d_0} \right)^2 + \left(\tau_1 - \frac{a_0 d_0 + b_0 c_0}{2c_0 d_0} \right)^2 - \left(1 + \frac{E^2}{4c_0^2 d_0^2} \right) \right\} , \end{aligned} \tag{4.5}$$

where

$$E = \frac{c_0 d_0 Q_R^2 + a_0 b_0 P_R^2 - (a_0 d_0 + b_0 c_0) Q_R \cdot P_R}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} . \tag{4.6}$$

As shown in [13], the right hand side of (4.5) vanishes on the wall of marginal stability associated with the decay given in (4.1). Thus it follows from (4.5) that as we cross this wall of marginal stability, the contour (4.4) crosses the pole at (4.2) in accordance with our postulate.

This postulate allows us to identify the possible poles of the partition function besides those related to the $\check{v} = 0$ pole by the S-duality transformation (3.5), – they occur at (4.2) for those values of a_0, b_0, c_0 and d_0 for which the decay (4.1) is consistent with the charge quantization laws. One can also get information about the residues at these poles since they are given by the jumps in the index. This jump can be expressed using the wall crossing formula [23, 24, 25, 26, 27, 13, 28] that tells us that as we cross a wall of marginal stability associated with the decay $(Q, P) \to (Q_1, P_1) + (Q_2, P_2)$ the index jumps by an amount

$$(-1)^{Q_1 \cdot P_2 - Q_2 \cdot P_1 + 1} \left(Q_1 \cdot P_2 - Q_2 \cdot P_1 \right) d_h(Q_1, P_1) d_h(Q_2, P_2) \tag{4.7}$$

up to a sign, where $d_h(Q, P)$ denotes the index measuring the number of bosonic minus the number of fermionic half BPS supermultiplets carrying charges (Q, P). Thus this relates the residues at the poles of the integrand to the indices of half BPS states.

We shall now study the consequence of (4.7) on the residue at the pole (4.2). First let us consider the special case associated with the decay $(Q, P) \to (Q, 0) + (0, P)$. In this case the jump in the index is given by

$$(-1)^{Q \cdot P + 1} \, Q \cdot P \, d_e(Q) \, d_m(P) \, , \tag{4.8}$$

where $d_e(Q) = d_h(Q, 0)$ is the index of purely electrically charged states and $d_m(P) = d_h(0, P)$ is the index of purely magnetically charged state. This jump is to be accounted for by the residue of a pole of the integrand at $\check{v} = 0$. The result (4.8) is reproduced if near $\check{v} = 0$, $\check{\Phi}$ behaves as

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})^{-1} \propto \{\phi_m(\check{\rho})^{-1}\, \phi_e(\check{\sigma})^{-1} \check{v}^{-2} + \mathcal{O}(\check{v}^0)\}\,, \tag{4.9}$$

where $1/\phi_m(\check{\rho})$ and $1/\phi_e(\check{\sigma})$ denote respectively the partition functions of purely magnetic and purely electric states:

$$d_m(P) = \frac{1}{T_1} \int_{iM_1 - T_1/2}^{iM_1 + T_1/2} d\check{\rho}\, e^{-i\pi P^2 \check{\rho}} \frac{1}{\phi_m(\check{\rho})}\,, \qquad d_e(Q) = \frac{1}{T_2} \int_{iM_2 - T_2/2}^{iM_2 + T_2/2} d\check{\sigma}\, e^{-i\pi Q^2 \check{\sigma}} \frac{1}{\phi_e(\check{\sigma})}\,. \tag{4.10}$$

Substituting (4.10) into the integrand in (2.2) and picking up the residue from the pole at $\check{v} = 0$ we get the change in the index to be

$$(-1)^{Q\cdot P+1}\, Q\cdot P\, d_e(Q)\, d_m(P)\,, \tag{4.11}$$

in agreement with (4.8), provided we choose the constant of proportionality in (4.9) appropriately. Note that the $Q \cdot P$ factor comes from the $\check{v}$ derivative of the exponential factor in (2.2) arising due to the double pole of $\check{\Phi}^{-1}$ at $\check{v} = 0$.

In writing (4.9), (4.10) we have implicitly assumed that the allowed values of Q^2 and P^2 inside the set $\mathcal{B}$ are independent of each other, i.e. the possible values that Q^2 can take for a given P^2 is independent of P^2 and vice versa. If this is not so then instead of having a single product the right hand side of (4.9) will contain a sum of products. For example if in the set $\mathcal{B}$, $Q^2/2$ and $P^2/2$ are correlated so that $Q^2/2$ is odd (even) when $P^2/2$ is odd (even) then the coeffcient of $\check{v}^{-2}$ in the expression for $\check{\Phi}^{-1}$ will contain two terms, – the product of the partition function with odd $Q^2/2$ electric states with that of odd $P^2/2$ magnetic states and the product of the partition function of even $Q^2/2$ electric states with that of even $P^2/2$ magnetic states.

There is one more assumption that has gone into writing (4.9), (4.10). We have assumed that given two pairs of charge vectors (Q, P) and $(\widehat{Q}, \widehat{P})$ in $\mathcal{B}$, if $Q^2 = \widehat{Q}^2$ then Q and $\widehat{Q}$ are related by a T-duality transformation. Otherwise $d_e(Q)$ will not be a function of Q^2 and one cannot define an electric partition function via eq.(4.10). A similar restriction applies to the magnetic charges as well. Now since the set $\mathcal{B}$ has been chosen such that if the triplets $(Q^2, P^2, Q \cdot P)$ are identical for two charge vectors then they must be related by T-duality transformation, if two different Q's with same Q^2 are not related by T-duality then they must come from triplets with different values of P^2 and/or $Q \cdot P$. In other words the different T-duality orbits for a given Q^2 must be correlated with P^2 and/or $Q \cdot P$. If the correlation is with P^2 then we follow the procedure described in the previous paragraph, *e.g.* if one set of

Q's arise from even $P^2/2$ and another set of Q's arise from odd $P^2/2$, we define two separate electric partition function for these two different sets of Q's and identify the coefficient of $\check{v}^{-2}$ in the partition function $\check{\Phi}^{-1}$ as a sum of terms. If on the other hand the correlation is with $Q \cdot P$ then the procedure is more complicated. We first project onto different $Q \cdot P$ sectors by adding to $\check{\Phi}^{-1}$ other terms obtained by appropriate shifts of $\check{v}$, so that the subset of states which contribute to the new partition function now has a unique Q for a given Q^2 up to T-duality transformations. The singularities of this new partition functions near $\check{v} = 0$ will now be described by equation of the type (4.9), (4.10). For example if one set of Q's come from odd $Q \cdot P$ and the second set of Q's come from even $Q \cdot P$, then we can consider the quarter BPS partition functions $\frac{1}{2}\{\check{\Phi}^{-1}(\check{\rho}, \check{\sigma}, \check{v}) \pm \check{\Phi}^{-1}(\check{\rho}, \check{\sigma}, \check{v} + \frac{1}{2})\}$. These pick up even $Q \cdot P$ and odd $Q \cdot P$ states respectively, and hence the contribution to these partition functions will come from charge vectors (Q, P) with the property that for a given Q^2, there will be a unique Q up to a T-duality transformation. Thus the behaviour of these combinations will now be controlled by equations of the type given in (4.9), (4.10). Conversely, for the original set $\mathcal{B}$ the jump in the index associated with the decay $(Q, P) \to (Q, 0) + (0, P)$ is now controlled by the zeroes of $\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})$ at $\check{v} = 0$ and also at $\check{v} = 1/2$. Similar considerations apply when the same P^2 in the set $\mathcal{B}$ comes from more than one P's which are not related by T-duality.

Often both the subtleties mentioned above can be avoided by a judicious choice of the set $\mathcal{B}$. In fact in all the explicit examples we shall study in §6, we shall be able to avoid these subtleties.

We now return to the general case associated with the decay described in (4.1). Since here

$$(Q_1, P_1) = (a_0 d_0 Q - a_0 b_0 P, c_0 d_0 Q - c_0 b_0 P), \qquad (Q_2, P_2) = (-b_0 c_0 Q + a_0 b_0 P, -c_0 d_0 Q + a_0 d_0 P)\,, \tag{4.12}$$

we have

$$Q_1 \cdot P_2 - Q_2 \cdot P_1 = -Q^2 c_0 d_0 - P^2 a_0 b_0 + Q \cdot P (a_0 d_0 + b_0 c_0)\,. \tag{4.13}$$

Let us now make a change of variables

$$\check{\rho}' = d_0^2 \check{\rho} + b_0^2 \check{\sigma} + 2 b_0 d_0 \check{v}, \qquad \check{\sigma}' = c_0^2 \check{\rho} + a_0^2 \check{\sigma} + 2 a_0 c_0 \check{v}, \qquad \check{v}' = c_0 d_0 \check{\rho} + a_0 b_0 \check{\sigma} + (a_0 d_0 + b_0 c_0) \check{v}\,, \tag{4.14}$$

and define

$$Q' = d_0 Q - b_0 P, \qquad P' = -c_0 Q + a_0 P\,. \tag{4.15}$$

Under this change of variables

$$d\check{\rho} \wedge d\check{\sigma} \wedge d\check{v} = d\check{\rho}' \wedge d\check{\sigma}' \wedge d\check{v}'\,, \tag{4.16}$$

$$Q_1 \cdot P_2 - Q_2 \cdot P_1 = Q' \cdot P' \, , \tag{4.17}$$

$$(Q_1, P_1) = (a_0 Q', c_0 Q'), \qquad (Q_2, P_2) = (b_0 P', d_0 P') \, , \tag{4.18}$$

and

$$\frac{1}{2}\check\rho P^2 + \frac{1}{2}\check\sigma Q^2 + \check v Q \cdot P = \frac{1}{2}\check\rho' P'^2 + \frac{1}{2}\check\sigma' Q'^2 + \check v' Q' \cdot P' \, . \tag{4.19}$$

Thus the jump in the index given in (4.7) can be expressed as

$$(-1)^{Q'\cdot P'+1}\, Q' \cdot P' \, d_h(a_0 Q', c_0 Q') d_h(b_0 P', d_0 P') \, . \tag{4.20}$$

Furthermore in these variables the pole at (4.2) is at $\check v' = 0$. Thus we can identify (4.20) with the residue of the integrand from $\check v' = 0$. Using (4.16), (4.19) the latter may be expressed as

$$(-1)^{Q\cdot P+1} \int d\check\rho' d\check\sigma' d\check v' e^{i\pi(\check\rho' P'^2 + \check\sigma' Q'^2 + 2\check v' Q'\cdot P')} \, \frac{1}{\check\Phi(\check\rho, \check\sigma, \check v)} \, , \tag{4.21}$$

where the integration contour is around $\check v' = 0$. We now note that this result can be reproduced if we assume that near the pole (4.2) the partition function behaves as[8]

$$\check\Phi(\check\rho, \check\sigma, \check v)^{-1} \propto \{\phi_e(\check\sigma'; a_0, c_0)^{-1} \phi_m(\check\rho'; b_0, d_0)^{-1} \check v'^{-2} + \mathcal{O}(\check v'^0)\} \, , \tag{4.22}$$

where $1/\phi_{e,m}(\tau; k, l)$ denote the partition functions of half BPS dyons in the set $\mathcal{B}$ such that

$$\begin{aligned} d_h(a_0 Q', c_0 Q') &= \frac{1}{T} \int_{iM-T/2}^{iM+T/2} d\tau \, e^{-i\pi Q'^2 \tau} \frac{1}{\phi_e(\tau; a_0, c_0)}, \\ d_h(b_0 P', d_0 P') &= \frac{1}{T'} \int_{iM-T'/2}^{iM+T'/2} d\tau \, e^{-i\pi P'^2 \tau} \frac{1}{\phi_m(\tau; b_0, d_0)} \, . \end{aligned} \tag{4.23}$$

The integration over τ run parallel to the real axis over unit period with the imaginary part fixed at some large positive value M. Substituting (4.22) into (4.21) and picking up the residue from the pole at $\check v' = 0$ we get the change in the index to be

$$(-1)^{Q'\cdot P'+1}\, Q' \cdot P' \, d_h(a_0 Q', c_0 Q') d_h(b_0 P', d_0 P') \, , \tag{4.24}$$

in agreement with (4.20).

To summarize, (4.2) gives us the locations of the zeroes of $\check\Phi$, whereas eq.(4.9) and more generally (4.22) give us information about the behaviour of $\check\Phi$ near this zero. We shall now

[8]This formula suffers from the same type of subtleties described below eq.(4.11) with (Q, P) replaced by (Q', P') and $(\check\rho, \check\sigma, \check v)$ replaced by $(\check\rho', \check\sigma', \check v')$.

show that these results suggest additional symmetries of $\check{\Phi}$ of the type described in (3.15). Typically in any theory the partition functions of half BPS states have modular properties. Let us for definiteness consider the decay $(Q,P) \to (Q,0) + (0,P)$. In this case the functions $\phi_m(\check{\rho})$ and $\phi_e(\check{\sigma})$ transform as modular forms of a subgroup of $SL(2, \mathbb{Z})$ since they arise from quantization of a fundamental string or a dual magnetic string. These relations take the form

$$\phi_m((\alpha\check{\rho}+\beta)(\gamma\check{\rho}+\delta)^{-1}) = (\gamma\check{\rho}+\delta)^{k+2}\phi_m(\check{\rho}), \qquad \phi_e((p\check{\sigma}+q)(r\check{\sigma}+s)^{-1}) = (r\check{\sigma}+s)^{k+2}\phi_e(\check{\sigma}), \tag{4.25}$$

where k is an integer specific to the theory under study, and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ belong to appropriate subgroups of $SL(2, \mathbb{Z})$. Given that ϕ_m and ϕ_e have these symmetries, we conclude from (4.9) that near $\check{v} = 0$, $\check{\Phi}$ also has some additional symmetries. Even though there is no guarantee that these will be symmetries of the full quarter BPS partition function, one could hope that some part of these do lift to symmetries of the partition function and hence of $\check{\Phi}$. Those which do can be represented by symplectic transformations of the type (3.15) with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & s \end{pmatrix}. \tag{4.26}$$

The first transformation generates

$$\check{\rho} \to \frac{\alpha\check{\rho}+\beta}{\gamma\check{\rho}+\delta}, \quad \check{\sigma} \to \check{\sigma} - \frac{\gamma\check{v}^2}{\gamma\check{\rho}+\delta}, \quad \check{v} \to \frac{\check{v}}{\gamma\check{\rho}+\delta}, \tag{4.27}$$

while the second transformation generates

$$\check{\rho} \to \check{\rho} - \frac{r\check{v}^2}{r\check{\sigma}+s}, \quad \check{\sigma} \to \frac{p\check{\sigma}+q}{r\check{\sigma}+s}, \quad \check{v} \to \frac{\check{v}}{r\check{\sigma}+s}. \tag{4.28}$$

Both transformations leave the $\check{v} = 0$ surface invariant. Furthermore applying these transformations on (3.15) and using (4.9) near $\check{v} = 0$ we generate the transformation laws (4.25).

The symplectic transformations given in (4.26), if present, give us the additional symmetries required to have $\check{\Phi}$ transform as a modular form under a non-trivial subgroup of $Sp(2, \mathbb{Z})$. We can use this to determine the subgroup of $Sp(2, \mathbb{Z})$ under which we expect $\check{\Phi}$ to transform as a modular form and also the weight k of the modular form. However since we do not know *a priori* which part of the symmetry groups of ϕ_e and ϕ_m will lift to the symmetries of $\check{\Phi}$, this is not a fool proof method. Nevertheless these can serve as guidelines for making an educated guess.

The behaviour of $\check{\Phi}$ near the other zeroes given in (4.2) could provide us with additional information. If the zero of $\check{\Phi}$ at (4.2) is related to the one at $\check{v}=0$ by an S-duality transformation then this information is not new. Since S-duality transformation acts by multiplying the matrix $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ associated with a wall from the left [13], this means that if $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ itself is an S-duality transformation then we do not get a new information. To this we must also add the information that multiplying $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ from the right by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for any λ or by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ does not change the wall [13]. However in many cases even after imposing these equivalence relations one finds inequivalent walls.[9] In such cases the associated zero of $\check{\Phi}$ cannot be related to the zero at $\check{v}=0$ by an S-duality transformation, and we get new information.[10] Let $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ be the matrix associated with such a decay. If the corresponding partition functions $\phi_m(\tau; b_0, d_0)$ and $\phi_e(\tau; a_0, c_0)$ have modular groups containing matrices of the form $\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$ and $\begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix}$ respectively, then they may be regarded as symplectic transformations generated by the matrices

$$\begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 & 0 & \beta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_1 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix} \tag{4.29}$$

and

$$\begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p_1 & 0 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & r_1 & 0 & s_1 \end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix} \tag{4.30}$$

respectively, acting on the original variables $(\check{\rho}, \check{\sigma}, \check{v})$. Again we could hope that a part of this symmetry is a symmetry of $\check{\Phi}$.

We shall illustrate these by several examples in §6.

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[9]For example in $\mathbb{Z}_6$ CHL model with S-duality group $\Gamma_1(6)$ the wall corresponding to the matrix $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ is not equivalent to the wall corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We shall discuss this example in some detail in §8.

[10]Typically the number of such additional zeroes is a finite number, providing us with a finite set of additional information.

5 Black hole entropy

Another set of constraints may be derived by requiring that the formula for the index of quarter BPS states match the entropy of the black hole carrying the same charges in the limit when the charges are large. The consequences of this constraint have been analyzed in detail in the past [1, 2, 6, 9] and reviewed in [18]. Hence our discussion will be limited to a review of the salient features.

In the approximation where we keep the supergravity part of the action containing only the two derivative terms, the black hole entropy is given by

$$\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}\,. \tag{5.1}$$

In all known cases this result is reproduced by the asymptotic behaviour of (2.2) for large charges. Furthermore the leading asymptotic behaviour comes from the residue of the partition function at the pole at [1, 2, 6, 9, 18]

$$\check\rho\check\sigma - \check v^2 + \check v = 0\,, \tag{5.2}$$

up to translations of $\check\rho$, $\check\sigma$ and $\check v$ by their periods. We shall assume that this result continues to hold in the general case. Thus $\check\Phi(\check\rho,\check\sigma,\check v)$ must have a zero at (5.2). In order to find the behaviour of $\check\Phi$ near this zero one needs to know the first non-leading correction to the leading formula (5.1) for the black hole entropy. *A priori* these corrections depend on the complete set of four derivative terms in the quantum effective action of the theory and are difficult to calculate. However in all known examples one finds that the entropy calculated just by including the Gauss-Bonnet term in the effective action reproduces correctly the first non-leading correction to the statistical entropy. If we assume that this result continues to hold for a general theory then we can use this to determine the behaviour of $\check\Phi$ near (5.2) in terms of the coefficient of the Gauss-Bonnet term in the effective action.

Since this procedure has been extensively studied in [1,2,6,9] and reviewed in [18], we shall only quote the result. Typically the Gauss Bonnet term in the Lagrangian has the form

$$\int d^4x\,\sqrt{-\det g}\,\phi(a,S)\,\left\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right\}\,, \tag{5.3}$$

where $\tau = a + iS$ is the axion-dilaton modulus and the function $\phi(a,S)$ has the form

$$\phi(a,S) = -\frac{1}{64\pi^2}\,\left((k+2)\ln S + \ln g(a+iS) + \ln g(-a+iS)\right) + \text{constant}\,. \tag{5.4}$$

Here k is the same integer that appeared in (3.15) and $g(\tau)$ transforms as a modular form of weight $k+2$ under the S-duality group. In a given theory $g(\tau)$ can be calculated in string perturbation theory [38,39]. To the first non-leading order in the inverse power of charges, the effect of this term is to change the black hole entropy to [18]

$$S_{BH} = \pi\sqrt{Q^2P^2-(Q\cdot P)^2} + 64\,\pi^2\,\phi\left(\frac{Q\cdot P}{P^2}, \frac{\sqrt{Q^2P^2-(Q\cdot P)^2}}{P^2}\right) + \cdots \tag{5.5}$$

The analysis of [1,2,6,9,18] shows that this behaviour can be reproduced if we assume that near the zero at (5.2)

$$\check{\Phi}(\check{\rho},\check{\sigma},\check{v}) \propto (2v-\rho-\sigma)^k\left\{v^2\,g(\rho)\,g(\sigma)+\mathcal{O}(v^4)\right\}, \tag{5.6}$$

where

$$\rho = \frac{\check{\rho}\check{\sigma}-\check{v}^2}{\check{\sigma}}, \quad \sigma = \frac{\check{\rho}\check{\sigma}-(\check{v}-1)^2}{\check{\sigma}}, \quad v = \frac{\check{\rho}\check{\sigma}-\check{v}^2+\check{v}}{\check{\sigma}}. \tag{5.7}$$

If we assume that eq.(5.6) holds in general, then it gives us information about the behaviour of $\check{\Phi}(\check{\rho},\check{\sigma},\check{v})$ near the zero at (5.2). On the other hand if we can determine $\check{\Phi}$ from other considerations then the validity of (5.6) would provide further evidence for the postulate that in $\mathcal{N}=4$ supersymmetric string theories the Gauss-Bonnet term gives the complete correction to black hole entropy to first non-leading order.

6 Examples

In this section we shall describe several applications of the general procedure described in §2. Some of them will involve known cases and will provide a test for our procedure, while others will be new examples where we shall derive a set of constraints on certain dyon partition functions which have not yet been computed from first principles.

6.1 Dyons with unit torsion in heterotic string theory on T^6

We consider a dyon of charge (Q,P) in the heterotic string theory on T^6. Q and P take values in the Narain lattice Λ [40,41]. Let S^1 and $\widetilde{S}^1$ be two circles of T^6, each labelled by a coordinate with period 2π and let us denote by $n',\widetilde{n}$ the momenta along S^1 and $\widetilde{S}^1$, by $-w',-\widetilde{w}$ the fundamental string winding numbers along S^1 and $\widetilde{S}^1$, by $N',\widetilde{N}$ the Kaluza-Klein monopole charges associated with S^1 and $\widetilde{S}^1$, and by $-W'$, $-\widetilde{W}$ the H-monopole charges associated with S^1 and $\widetilde{S}^1$ [18]. Then in the four dimensional subspace consisting of charge vectors

$$Q = \begin{pmatrix} \widetilde{n} \\ n' \\ \widetilde{w} \\ w' \end{pmatrix}, \qquad P = \begin{pmatrix} \widetilde{W} \\ W' \\ \widetilde{N} \\ N' \end{pmatrix}, \tag{6.1.1}$$

the metric L takes the form

$$L = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{6.1.2}$$

where I_2 denotes 2×2 identity matrix. In this subspace we consider a three parameter family of charge vectors (Q, P) with

$$Q = \begin{pmatrix} 0 \\ m \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} K \\ J \\ 1 \\ 0 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}. \tag{6.1.3}$$

This has

$$Q^2 = -2m, \quad P^2 = 2K, \quad Q \cdot P = -J. \tag{6.1.4}$$

We shall identify this set of charge vectors as the set $\mathcal{A}$. As required, Q^2, P^2 and $Q \cdot P$ are independent linear functions of m, K and J so that for a pair of distinct values of (m, K, J) we get a pair of distinct values of $(Q^2, P^2, Q \cdot P)$. All the charge vectors in this family have unit torsion, i.e. if we express the charges as linear combinations $\sum Q_i e_i$ and $\sum P_i e_i$ of primitive basis elements e_i of the lattice Λ, then the torsion

$$r(Q, P) \equiv \gcd\{Q_i P_j - Q_j P_i\}, \tag{6.1.5}$$

is equal to 1. In this case it is known that Q^2, P^2 and $Q \cdot P$ are the complete set of T-duality invariants [42], i.e. beginning with a pair (Q, P) with unit torsion we can reach any other pair with unit torsion and same values of Q^2, P^2 and $Q \cdot P$ via a T-duality transformation. Since the set $\mathcal{A}$ contains all integer triplets $(Q^2/2, P^2/2, Q \cdot P)$ we conclude that the set $\mathcal{B}$ is the set of all (Q, P) with unit torsion. The corresponding partition function is known [1] – it is the inverse of the weight ten Igusa cusp form Φ_{10} of the full $Sp(2, \mathbb{Z})$ group.

We shall now examine how Φ_{10} satisfies the various constraints derived in the previous sections. First of all note that since S-duality transformation does not change the torsion r, the full $SL(2, \mathbb{Z})$ group is a symmetry of this set. Furthermore in this set $Q^2/2$, $P^2/2$ and $Q \cdot P$

are all quantized in integer units. Thus the partition function is invariant under translation of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ by arbitrary integer units. These correspond to symplectic transformations of the form

$$\begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & a_1 & a_3 \\ 0 & 1 & a_3 & a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad a_1, a_2, a_3 \in \mathbb{Z}. \tag{6.1.6}$$

Clearly each of these transformations belong to $Sp(2, \mathbb{Z})$ and is a symmetry of Φ_{10}.

Next we turn to the constraints from the wall crossing formula. In this case all the walls are related by S-duality transformation to the wall corresponding to the decay $(Q, P) \rightarrow (Q, 0) + (0, P)$. So it is sufficient to study the consequences of the wall crossing formula at this wall. Clearly Q^2 and P^2 given in (6.1.4) are uncorrelated. Furthermore in heterotic string theory on T^6 all Q's with a given Q^2 are related by T-duality transformation [43]. The same is true for P. Thus the subtleties mentioned below eq.(4.11) are absent, and the behaviour of $\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})$ near $\check{v} = 0$ is expected to be given by (4.9). In this case both the electric and the magnetic half-BPS partition functions are given by $\eta(\tau)^{-24}$ where η denotes the Dedekind function. Thus we have, as a consequence of the wall crossing formula,

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto \{\check{v}^2 \, (\eta(\check{\rho}))^{24} \, (\eta(\check{\sigma}))^{24} + \mathcal{O}(\check{v}^4)\}. \tag{6.1.7}$$

$\eta(\tau)^{24}$ transforms as a modular form of weight 12 under an $SL(2, \mathbb{Z})$ transformation. From eqs.(4.26) it follows that these $SL(2, \mathbb{Z})$ transformations may be regarded as the following symplectic transformations of $\check{\rho}$, $\check{\sigma}$, $\check{v}$

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & s \end{pmatrix}$$
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z}), \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{6.1.8}$$

Furthermore $\check{\Phi}$ should have weight $12 - 2 = 10$.

Let us now compare these with the known properties of Φ_{10}. $\Phi_{10}(\check{\rho}, \check{\sigma}, \check{v})$ is indeed known to have the factorization property (6.1.7). Furthermore since Φ_{10} transforms as a modular form of weight 10 under the full $Sp(2, \mathbb{Z})$ group, and since (6.1.8) are $Sp(2, \mathbb{Z})$ matrices, they represent symmetries of Φ_{10}. Thus we see that in this case the full set of symmetries of ϕ_m and ϕ_e lift to symmetries of $\check{\Phi}$. It is worth noting that the matrices given in (6.1.6) and (6.1.8) generate the full $Sp(2, \mathbb{Z})$ group. Thus in this case by assuming that the full modular groups of ϕ_e and ϕ_m lift to symmetries of the partition function we could determine the symmetries of the partition function.

Finally let us consider the constraints coming from the knowledge of black hole entropy. In this case the function $g(\tau)$ appearing in (5.4) is given by $\eta(\tau)^{24}$. Thus (5.6) takes the fom

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto (2v - \rho - \sigma)^{10} \left\{ v^2 \, \eta(\rho)^{24} \, \eta(\sigma)^{24} + \mathcal{O}(v^4) \right\}, \tag{6.1.9}$$

where $(\check{\rho}, \check{\sigma}, \check{v})$ and (ρ, σ, v) are related via eq.(5.7). The Siegel modular form $\Phi_{10}(\check{\rho}, \check{\sigma}, \check{v})$ satisfies these properties. In fact since (5.7) represents an $Sp(2, \mathbb{Z})$ transformation, the property (6.1.9) of $\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})$ follows from the factorization property (6.1.7). This however will not be the case in a more generic situation.

6.2 Dyons with unit torsion and even $Q^2/2$ in heterotic on T^6

We now consider again heterotic string theory on T^6, but choose the set $\mathcal{A}$ to be collection of (Q, P) of the form:

$$Q = \begin{pmatrix} 0 \\ 2m \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} K \\ J \\ 1 \\ 0 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}. \tag{6.2.1}$$

This has

$$Q^2 = -4m, \quad P^2 = 2K, \quad Q \cdot P = -J. \tag{6.2.2}$$

We note that all the charge vectors have $Q^2/2$ even. Since Q^2 is T-duality invariant, any other charge vector which can be obtained from this one by a T-duality transformation has $Q^2/2$ even. Thus the set $\mathcal{B}$ now consists of charge vectors which have even $Q^2/2$ and arbitrary integer values of $P^2/2$ and $Q \cdot P$. Since this set $\mathcal{B}$ is a subset of charges for which the spectrum was analyzed in §6.1 we do not expect to derive any new results. Nevertheless we have chosen this example as this will serve as a useful guide to our analysis in later sections.

We first note that the quantization conditions of Q^2, P^2 and $Q \cdot P$ imply the following periods of the partition function:

$$(\check{\rho}, \check{\sigma}, \check{v}) \to (\check{\rho} + a_1, \check{\sigma} + a_2, \check{v} + a_3), \quad a_1 \in \mathbb{Z}, \quad a_2 \in \frac{1}{2}\mathbb{Z}, \quad a_3 \in \mathbb{Z}. \tag{6.2.3}$$

The period along $\check{\sigma}$ is not an integer. We can remedy this by defining

$$Q_s = Q/2\,, \qquad \check{\sigma}_s = 4\check{\sigma}, \qquad \check{v}_s = 2\check{v}\,. \tag{6.2.4}$$

so that $Q_s^2/2$ and $Q_s\cdot P$ are now quantized in half integer units. The periods $(\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3)$ of the variables $(\check{\rho}, \check{\sigma}_s, \check{v}_s)$ conjugate to $(P_s^2/2, Q_s^2/2, Q_s\cdot P_s)$ are now integers, given by,

$$\widetilde{a}_1 \in \mathbb{Z}, \quad \widetilde{a}_2 \in 2\,\mathbb{Z}, \quad \widetilde{a}_3 \in 2\,\mathbb{Z}\,. \tag{6.2.5}$$

The dyon partition function in this case can be easily calculated from the one for §6.1 by taking into account the evenness of $Q^2/2$. This amounts to adding to the original partition function another term where $\check{\sigma}$ is shifted by 1/2. Thus we have

$$\frac{1}{\check{\Phi}(\check{\rho},\check{\sigma},\check{v})} = \frac{1}{2}\left(\frac{1}{\Phi_{10}(\check{\rho},\check{\sigma},\check{v})} + \frac{1}{\Phi_{10}(\check{\rho},\check{\sigma}+\frac{1}{2},\check{v})}\right)\,, \tag{6.2.6}$$

or, in terms of the rescaled variables,

$$\frac{1}{\check{\Phi}(\check{\rho},\check{\sigma},\check{v})} = \frac{1}{2}\left(\frac{1}{\Phi_{10}(\check{\rho},\frac{1}{4}\check{\sigma}_s,\frac{1}{2}\check{v}_s)} + \frac{1}{\Phi_{10}(\check{\rho},\frac{1}{4}\check{\sigma}_s+\frac{1}{2},\frac{1}{2}\check{v}_s)}\right)\,. \tag{6.2.7}$$

Let us determine the symmetries of this partition function. For this it will be useful to work in terms of the original unscaled variables $(\check{\rho},\check{\sigma},\check{v})$ and at the end go back to the rescaled variables. The first term on the right hand side of (6.2.6) has the usual $Sp(2,\ \mathbb{Z})$ symmetries acting on the variables $(\check{\rho},\check{\sigma},\check{v})$. However not all of these are symmetries of the second term. Given an $Sp(2,\ \mathbb{Z})$ matrix $\begin{pmatrix} a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\\ a_4 & b_4 & c_4 & d_4\end{pmatrix}$, it is a symmetry of the second term provided its action on $(\check{\rho},\check{\sigma},\check{v})$ can be regarded as an $Sp(2,\ \mathbb{Z})$ action $\begin{pmatrix} a_1' & b_1' & c_1' & d_1'\\ a_2' & b_2' & c_2' & d_2'\\ a_3' & b_3' & c_3' & d_3'\\ a_4' & b_4' & c_4' & d_4'\end{pmatrix}$ on $(\check{\rho},\check{\sigma}+\frac{1}{2},\check{v})$ followed by a translation on $\check{\sigma}$ by 1/2. Since a translation of $\check{\sigma}$ by 1/2 can be regarded as a symplectic transformation with the matrix $\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1/2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$, the above condition takes the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1/2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\\ a_4 & b_4 & c_4 & d_4\end{pmatrix} = \begin{pmatrix} a_1' & b_1' & c_1' & d_1'\\ a_2' & b_2' & c_2' & d_2'\\ a_3' & b_3' & c_3' & d_3'\\ a_4' & b_4' & c_4' & d_4'\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1/2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\,. \tag{6.2.8}$$

This gives

$$\begin{pmatrix} a'_1 & b'_1 & c'_1 & d'_1 \\ a'_2 & b'_2 & c'_2 & d'_2 \\ a'_3 & b'_3 & c'_3 & d'_3 \\ a'_4 & b'_4 & c'_4 & d'_4 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 - \frac{1}{2}b_1 \\ a_2 + \frac{1}{2}a_4 & b_2 + \frac{1}{2}b_4 & c_2 + \frac{1}{2}c_4 & d_2 + \frac{1}{2}(d_4 - b_2) - \frac{1}{4}b_4 \\ a_3 & b_3 & c_3 & d_3 - \frac{1}{2}b_3 \\ a_4 & b_4 & c_4 & d_4 - \frac{1}{2}b_4 \end{pmatrix} . \qquad (6.2.9)$$

The coefficients a_i, b_i, c_i and d_i are integers. Requiring that there exist integer a'_i, b'_i, c'_i and d'_i satisfying the above constraints we get further conditions on a_i, b_i, c_i and d_i. These take the following form:

$$a_4, b_4, c_4, b_1, b_3 \in 2\,\mathbb{Z}, \quad b_4 - 2(d_4 - b_2) \in 4\,\mathbb{Z}\,. \qquad (6.2.10)$$

On the other hand the requirement that the original matrix $\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$ is symplectic, together with the first set of conditions given in (6.2.10), can be used to show that b_2 and d_4 are both odd. As a result $(b_2 - d_4)$ is even, and hence b_4 must be a multiple of 4 in order to satisfy (6.2.10). Thus we have

$$a_4 = 2\widehat{a}_4, \quad b_4 = 4\widehat{b}_4, \quad c_4 = 2\widehat{c}_4, \quad b_1 = 2\widehat{b}_1, \quad b_3 = 2\widehat{b}_3, \quad \widehat{a}_4, \widehat{b}_4, \widehat{c}_4, \widehat{b}_1, \widehat{b}_3 \in \mathbb{Z}\,. \qquad (6.2.11)$$

This determines the subgroup of $Sp(2, \mathbb{Z})$ which leaves the individual terms in (6.2.6) invariant. To this we must add the additional element corresponding to $\check{\sigma} \to \check{\sigma} + \frac{1}{2}$ which exchanges the two terms in (6.2.6). This corresponds to the symplectic transformation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (6.2.12)$$

The full symmetry group is then generated by the matrices:

$$\begin{pmatrix} a_1 & 2\widehat{b}_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & 2\widehat{b}_3 & c_3 & d_3 \\ 2\widehat{a}_4 & 4\widehat{b}_4 & 2\widehat{c}_4 & d_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (6.2.13)$$

We can easily determine how these transformations act on the rescaled variables $(\check{\rho}, \check{\sigma}_s, \check{v}_s)$. This is done with the help of conjugation by the symplectic matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \qquad (6.2.14)$$

relating $(\check\rho,\check\sigma,\check v)$ to $(\check\rho,\check\sigma_s,\check v_s)$. This converts the generators given in (6.2.13) to

$$\begin{pmatrix} a_1 & \widehat{b}_1 & c_1 & 2d_1 \\ 2a_2 & b_2 & 2c_2 & 4d_2 \\ a_3 & \widehat{b}_3 & c_3 & 2d_3 \\ \widehat{a}_4 & \widehat{b}_4 & \widehat{c}_4 & d_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{6.2.15}$$

We now note that all the matrices appearing in (6.2.15) have the form

$$\begin{pmatrix} * & * & * & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix} \quad \text{mod } 2 , \tag{6.2.16}$$

with $*$ denoting an arbitrary integer subject to the condition that (6.2.16) describes a symplectic matrix. Furthetmore the set of matrices (6.2.16) are closed under matrix multiplication. Thus the group generated by the matrices (6.2.15) is contained in the group $\check G$ consisting of $Sp(2, \mathbb{Z})$ matrices of the form (6.2.16). It is in fact easy to show that the group generated by the matrices (6.2.15) is the whole of $\check G$, i.e. any element of $\check G$ given in (6.2.16) can be written as a product of the elements given in (6.2.15).

We shall now set aside this result for a while and study the implications of S-duality symmetry and the wall crossing formula on the partition function. The eventual goal is to test the conclusions drawn from the general arguments along the lines of §3 and §4 against the known results for $\check\Phi$ given above. It follows from (3.2) and (6.2.2) that in order that an S-duality transformation generated by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ takes an arbitrary element of the set $\mathcal{B}$ to another element of the set $\mathcal{B}$ we must have b even. Thus S-duality transformations which preserve the set $\mathcal{B}$ take the form:

$$Q \to Q'' = aQ + bP, \quad P \to P'' = cQ + dP, \quad a, c, d \in \mathbb{Z}, \quad b \in 2\,\mathbb{Z}, \quad ad - bc = 1 \,. \tag{6.2.17}$$

On the original variables $(\check\rho,\check\sigma,\check v)$ the associated transformation can be represented by the symplectic matrix (3.14). After conjugation by the matrix (6.2.14) we get the symplectic matrix acting on the rescaled variables $(\check\rho,\check\sigma_s,\check v_s)$:

$$\begin{pmatrix} d & \widetilde{b} & 0 & 0 \\ \widetilde{c} & a & 0 & 0 \\ 0 & 0 & a & -\widetilde{c} \\ 0 & 0 & -\widetilde{b} & d \end{pmatrix} , \quad a, \widetilde{b} \equiv b/2, d \in \mathbb{Z}, \quad \widetilde{c} \equiv 2c \in 2\,\mathbb{Z} \,. \tag{6.2.18}$$

This clearly has the form given in (6.2.16). Also the periodicities along the $(\check{\rho}, \check{\sigma}_s, \check{v}_s)$ directions, as given in (6.2.5), are represented by the symplectic transformation

$$\begin{pmatrix} 1 & 0 & \widetilde{a}_1 & \widetilde{a}_3 \\ 0 & 1 & \widetilde{a}_3 & \widetilde{a}_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \widetilde{a}_1 \in \mathbb{Z}, \quad \widetilde{a}_2, \widetilde{a}_3 \in 2\,\mathbb{Z}\,. \tag{6.2.19}$$

These also are of the form given in (6.2.16).

Next we turn to the information obtained from the wall crossing relations. Consider first the wall associated with decay $(Q,P) \to (Q,0) + (0,P)$, – this controls the behaviour of $\check{\Phi}$ near $\check{v} = 0$ via eq.(4.9). Since $Q^2 = -4m$ and $P^2 = 2K$ can vary independently inside the set $\mathcal{A}$, and since any two charge vectors of the same norm can be related by a T-duality transformation [43], there is no subtlety of the type described below (4.11). The inverse of the magnetic partition function ϕ_m entering (4.9) is the same as the one that appeared in (6.1.7):

$$\phi_m(\check{\rho}) = (\eta(\check{\rho}))^{24}\,. \tag{6.2.20}$$

The electric partition function gets modified from the corresponding expression given in (6.1.7) due to the fact that we are only including even $Q^2/2$ states. As a result the partition function now becomes $\frac{1}{2}\left(\eta(\check{\sigma})^{-24} + \eta\left(\check{\sigma} + \frac{1}{2}\right)^{-24}\right)$. Replacing $\check{\sigma}$ by $\check{\sigma}_s/4$ we get

$$\phi_e(\check{\sigma})^{-1} = \frac{1}{2}\left(\eta\left(\frac{\check{\sigma}_s}{4}\right)\right)^{-24} + \frac{1}{2}\left(\eta\left(\frac{\check{\sigma}_s}{4} + \frac{1}{2}\right)\right)^{-24}\,. \tag{6.2.21}$$

This leads to the following behaviour of $\check{\Phi}$ near $\check{v}_s = 0$:

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto \left[\check{v}_s^2\,\eta(\check{\rho})^{24}\left\{\left(\eta\left(\frac{\check{\sigma}_s}{4}\right)\right)^{-24} + \left(\eta\left(\frac{\check{\sigma}_s}{4} + \frac{1}{2}\right)\right)^{-24}\right\}^{-1} + \mathcal{O}(\check{v}_s^4)\right] \tag{6.2.22}$$

$\check{\Phi}$ given in (6.2.7) can be shown to satisfy this property.

$\phi_m(\check{\rho})$ given in (6.2.20) transforms as a modular form of weight 12 under

$$\check{\rho} \to \frac{\alpha\check{\rho} + \beta}{\gamma\check{\rho} + \delta}, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})\,. \tag{6.2.23}$$

On the other hand $\phi_e(\check{\sigma})$ given in (6.2.21) can be shown to transform as a modular form of weight 12 under

$$\check{\sigma}_s \to \frac{p\check{\sigma}_s + q}{r\check{\sigma}_s + s}, \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma^0(2)\,, \tag{6.2.24}$$

i.e. $SL(2, \mathbb{Z})$ matrices with q even. (6.2.23) and (6.2.24) can be represented as symplectic transformations of $(\check\rho, \check\sigma_s, \check v_s)$ generated by the $Sp(2, \mathbb{Z})$ matrices

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & s \end{pmatrix}, \quad q \in 2\,\mathbb{Z}, \quad \alpha, \beta, \gamma, \delta, p, r, s \in \mathbb{Z}\,. \tag{6.2.25}$$

We now note that these transformations fall in the class given in (6.2.16). Thus in this case the modular symmetries of the half-BPS partition function associated with pole at $\check v = 0$ are lifted to symmetries of the full partition function.

In this case there is one additional wall which is not related to the wall considered above by the $\Gamma^0(2)$ S-duality transformation (6.2.17) acting on the original variables. This corresponds to the decay $(Q, P) \to (Q - P, 0) + (P, P)$. Comparing this with (4.1) we see that here

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{6.2.26}$$

Following (4.14), (4.15) and the relationship (6.2.4) between the original variables and the rescaled variables we have

$$\check\rho' = \check\rho + \frac{1}{4}\check\sigma_s + \check v_s, \quad \check\sigma' = \frac{1}{4}\check\sigma_s, \quad \check v' = \frac{1}{2}\check v_s + \frac{1}{4}\check\sigma_s\,, \tag{6.2.27}$$

$$Q' = Q - P, \quad P' = P\,. \tag{6.2.28}$$

Thus the pole of the partition function is at $\check v_s + \frac{1}{2}\check\sigma_s = 0$. Furthermore since from the relations (6.2.2) we see that the allowed values of $(Q - P)^2/2 = J + K - 2m$ and $P^2/2 = K$ are uncorrelated and can take arbitrary integer values, it follows from (4.22) that at this zero $\check\Phi$ goes as

$$\check\Phi(\check\rho, \check\sigma_s, \check v_s) \propto (2\check v_s + \check\sigma_s)^2\, \phi_m\left(\check\rho + \frac{1}{4}\check\sigma_s + \check v_s; 1, 1\right)\, \phi_e\left(\frac{1}{4}\check\sigma_s; 1, 0\right) + \mathcal{O}\left((2\check v_s + \check\sigma_s)^4\right)\,. \tag{6.2.29}$$

$\phi_m(\tau; 1, 1)$ denotes the partition function of half-BPS states carrying charges (P, P), with τ being conjugate to the variable $P'^2/2 = P^2/2$. Thus we have $\phi_m(\tau; 1, 1) = (\eta(\tau))^{24}$. On the other hand $(\phi_e(\tau; 1, 0))^{-1}$ is the partition function of half BPS states carrying charges $(Q', 0) = (Q - P, 0)$ with τ being conjugate to $Q'^2/2 = (Q - P)^2/2$. Since $(Q - P)^2/2 = (-2m + K + J)$ can take arbitrary integer values, the corresponding partition function is also given by $\eta(\tau)^{-24}$. Thus we have near $(\check\sigma_s + 2\check v_s) = 0$

$$\check\Phi(\check\rho, \check\sigma_s, \check v_s) \propto \left\{(2\check v_s + \check\sigma_s)^2 \eta\left(\check\rho + \frac{1}{4}\check\sigma_s + \check v_s\right)^{24} \eta\left(\frac{\check\sigma_s}{4}\right)^{24} + \mathcal{O}((2\check v_s + \check\sigma_s)^4)\right\} \tag{6.2.30}$$

$\check{\Phi}$ given in (6.2.7) can be shown to satisfy this property.

$\phi_m(\tau; 1, 1)$ transforms as a modular form of weight 12 under $\tau \to (\alpha_1\tau+\beta_1)/(\gamma_1\tau+\delta_1)$ with $\alpha_1, \beta_1, \gamma_1, \delta_1 \in \mathbb{Z}$, $\alpha_1\delta_1 - \beta_1\gamma_1 = 1$. On the other hand $\phi_e(\tau; 1, 0)$ transforms as a modular form of weight 12 under $\tau \to (p_1\tau + q_1)/(r_1\tau + s_1)$ with $p_1, q_1, r_1, s_1 \in \mathbb{Z}$, $p_1 s_1 - q_1 r_1 = 1$. Using (4.29), (4.30) and (6.2.14) we see that the the action of these transformations on the variables $(\check{\rho}, \check{\sigma}_s, \check{v}_s)$ may be represented by the symplectic matrices

$$\begin{pmatrix} \alpha_1 & (\alpha_1 - 1)/2 & \beta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_1 & \gamma_1/2 & \delta_1 & 0 \\ \gamma_1/2 & \gamma_1/4 & (\delta_1 - 1)/2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & (1 - p_1)/2 & q_1 & -2q_1 \\ 0 & p_1 & -2q_1 & 4q_1 \\ 0 & 0 & 1 & 0 \\ 0 & r_1/4 & (1 - s_1)/2 & s_1 \end{pmatrix}. \tag{6.2.31}$$

By comparing with the matrices given in (6.2.16) we see however that the transformations (6.2.31) generates symmetries of the full partition function only after we impose the additional constraints

$$r_1, \gamma_1 \in 4\, \mathbb{Z}\,. \tag{6.2.32}$$

Thus here we encounter a case where only a subset of the symmetries of the partition function near a pole is lifted to a full symmetry of the partition function. By examining the details carefully one discovers that in this case the pole comes from the first term in (6.2.7). Whereas this term displays the full symmetry given in (6.2.31), requiring that the other term also transforms covariantly under this symmetry generates the additional restrictions given in (6.2.32).

Finally we turn to the constraint from black hole entropy. As in §6.1, in this case we have $g(\tau) = \eta(\tau)^{24}$ in (5.4). Thus (5.6) takes the form

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto (2v - \rho - \sigma)^{10} \left\{ v^2\, \eta(\rho)^{24}\, \eta(\sigma)^{24} + \mathcal{O}(v^4) \right\}, \tag{6.2.33}$$

where $(\check{\rho}, \check{\sigma}, \check{v})$ and (ρ, σ, v) are related via (5.7). $\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})$ given in (6.2.6) can be shown to satisfy this property. In fact the relevent pole of $\check{\Phi}^{-1}$ comes from the first term on the right hand side of (6.2.6). The location of the zeroes of Φ_{10} are given in (7.8), and it follows from this that the second term does not have a pole at $v = 0$.

6.3 Dyons of torsion 2 in heterotic string theory on T^6

We consider again heterotic string theory on T^6 and take the set $\mathcal{A}$ to consist of charge vectors of the form

$$Q = \begin{pmatrix} 1 \\ 2m+1 \\ 1 \\ 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 2K+1 \\ 2J+1 \\ 1 \\ -1 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}. \tag{6.3.1}$$

This has

$$Q^2 = 4(m+1), \quad P^2 = 4(K-J), \quad Q \cdot P = 2(K+J-m+1). \tag{6.3.2}$$

Furthermore gcd$\{Q_iP_j - Q_jP_i\}$=2. Thus we have a family of charge vectors with torsion 2. It was shown in [42,20] that for $r=2$ there are three T-duality orbits for given $(Q^2, P^2, Q \cdot P)$ – in the first Q is twice a primitive lattice vector, in the second P is twice a primitive lattice vector and in the third both Q and P are primitive but $Q \pm P$ are twice primitive lattice vectors. The dyon charges given in (6.3.1) are clearly of the third kind. In the notation of [20] the discrete T-duality invariants of these charges are $(r_1 = 1, r_2 = 1, r_3 = 2, u_1 = 1)$. Note that as we vary m, J and K, $Q^2/2$ and $P^2/2$ take all possible even values and $Q \cdot P$ takes all possible values subject to the restriction that $Q \pm P$ are twice primitive lattice vectors. The latter condition requires $Q \cdot P$ to be even and $Q \cdot P - \frac{1}{2}Q^2 - \frac{1}{2}P^2$ to be a multiple of four. It now follows from the result of [42,20] that the T-duality orbit $\mathcal{B}$ of the set $\mathcal{A}$ consists of all the pairs (Q, P) with $(r_1 = 1, r_2 = 1, r_3 = 2, u_1 = 1)$ and even values of $Q^2/2$, $P^2/2$.

Since $Q^2/2$, $P^2/2$ and $Q \cdot P$ are all even and $Q^2 + P^2 + 2Q \cdot P$ is a multiple of 8, it is natural to introduce new charge vectors and variables

$$Q_s \equiv Q/2, \quad P_s \equiv P/2, \quad \check{\rho}_s \equiv 4\check{\rho}, \quad \check{\sigma}_s \equiv 4\check{\sigma}, \quad \check{v}_s \equiv 4\check{v}, \tag{6.3.3}$$

so that we have

$$\frac{1}{2}Q_s^2 = \frac{1}{2}(m+1), \quad \frac{1}{2}P_s^2 = \frac{1}{2}(K-J), \quad Q_s \cdot P_s = \frac{1}{2}(K+J-m+1), \tag{6.3.4}$$

quantized in half integer units subject to the constraint that

$$\frac{1}{2}Q_s^2 + \frac{1}{2}P_s^2 + Q_s \cdot P_s = K+1, \tag{6.3.5}$$

is an integer. Since $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$ are conjugate to $(P_s^2/2, Q_s^2/2, Q_s \cdot P_s)$, the partition function (2.1) will be periodic under

$$(\check{\rho}_s, \check{\sigma}_s, \check{v}_s) \to (\check{\rho}_s + 2, \check{\sigma}_s, \check{v}_s),\ (\check{\rho}_s, \check{\sigma}_s + 2, \check{v}_s),\ (\check{\rho}_s, \check{\sigma}_s, \check{v}_s + 2),\ (\check{\rho}_s + 1, \check{\sigma}_s + 1, \check{v}_s + 1). \tag{6.3.6}$$

The group generated by these transformations can be collectively represented by symplectic matrices of the form

$$\begin{pmatrix} 1 & 0 & \widetilde{a}_1 & \widetilde{a}_3 \\ 0 & 1 & \widetilde{a}_3 & \widetilde{a}_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3 \in \mathbb{Z}, \quad \widetilde{a}_1 + \widetilde{a}_2, \widetilde{a}_2 + \widetilde{a}_3, \widetilde{a}_1 + \widetilde{a}_3 \in 2\,\mathbb{Z}\,, \tag{6.3.7}$$

acting on the variables $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$. For future reference we note that the change of variables from $(\check{\rho}, \check{\sigma}, \check{v})$ to $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$ can be regarded as a symplectic transformation of the form

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}. \tag{6.3.8}$$

We now need to determine the subgroup of the S-duality group that leaves the set $\mathcal{B}$ invariant. If we did not have the restriction that $Q^2/2$ and $P^2/2$ are even, then this subgroup would consist of $SL(2, \mathbb{Z})$ matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ subject to the restriction $a + b \in 2\,\mathbb{Z} + 1$ and $c + d \in 2\,\mathbb{Z} + 1$ [20], – these conditions guarantee that the new charge vectors (Q'', P'') are each primitive and hence have the same set of discrete T-duality invariants $(r_1 = 1, r_2 = 1, r_3 = 2, u_1 = 1)$. We shall now argue that the same subgroup also leaves the set $\mathcal{B}$ invariant. For this we need to note that if we begin with a (Q, P) for which $Q^2/2$, $P^2/2$ and $Q \cdot P$ are all even then their S-duality transforms given in (3.2) will automatically have the same properties. Thus requiring the transformed pair (Q'', P'') to have even $Q''^2/2$ and $P''^2/2$, as is required for (Q'', P'') to belong to the set $\mathcal{B}$, does not put any additional restriction on the S-duality transformations. Since both Q and P are scaled by the same amount to get the rescaled charges Q_s and P_s, the S-duality group action on (Q_s, P_s) is identical to that on (Q, P) and hence its action on $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$ is identical to that on $(\check{\rho}, \check{\sigma}, \check{v})$. Using (3.14) we see that the representations of these symmetries as symplectic matrices are given by

$$\begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad a + c \in 2\,\mathbb{Z} + 1, \quad b + d \in 2\,\mathbb{Z} + 1\,, \tag{6.3.9}$$

acting on the variables $(\check{\rho}, \check{\sigma}, \check{v})$ and also on $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$.

Next we turn to the constraints from the wall crossing formula. We begin with the wall associated with the decay $(Q, P) \to (Q, 0) + (0, P)$, – this controls the behaviour of $\check{\Phi}$ at $\check{v} = 0$.

The analysis is straightforward. We note that both electric and magnetic partition functions involve summing over all possible even $Q^2/2$ and $P^2/2$ values. An analysis similar to the one leading to (6.2.21) give

$$\phi_e(\check{\sigma})^{-1} = \frac{1}{2}\left\{\eta\left(\frac{\check{\sigma}_s}{4}\right)^{-24} + \eta\left(\frac{\check{\sigma}_s+2}{4}\right)^{-24}\right\}, \tag{6.3.10}$$

and

$$\phi_m(\check{\rho})^{-1} = \frac{1}{2}\left\{\eta\left(\frac{\check{\rho}_s}{4}\right)^{-24} + \eta\left(\frac{\check{\rho}_s+2}{4}\right)^{-24}\right\}. \tag{6.3.11}$$

Thus we have

$$\check{\Phi}(\check{\rho},\check{\sigma},\check{v}) \sim \left[\check{v}_s^2\left\{\eta\left(\frac{\check{\sigma}_s}{4}\right)^{-24} + \eta\left(\frac{\check{\sigma}_s+2}{4}\right)^{-24}\right\}^{-1}\left\{\eta\left(\frac{\check{\rho}_s}{4}\right)^{-24} + \eta\left(\frac{\check{\rho}_s+2}{4}\right)^{-24}\right\}^{-1} + \mathcal{O}(\check{v}_s^4)\right], \tag{6.3.12}$$

near $\check{v}=0$. One can easily verify that the functions $\phi_e(\check{\sigma})$ and $\phi_m(\check{\rho})$ transform as modular forms of weight 12 under the transformation $\check{\sigma}_s \to (p\check{\sigma}_s+q)/(r\check{\sigma}_s+s)$ and $\check{\rho}_s \to (\alpha\check{\rho}_s+\beta)/(\gamma\check{\rho}_s+\delta)$ with $\begin{pmatrix} p & q \\ r & s\end{pmatrix} \in \Gamma^0(2)$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \Gamma^0(2)$. These can be regarded as symplectic transformations of the form

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & s \end{pmatrix},$$
$$\alpha\delta-\beta\gamma=1, \quad ps-qr=1, \quad p,r,s,\alpha,\gamma,\delta\in \mathbb{Z}, \quad q,\beta\in 2\,\mathbb{Z}, \tag{6.3.13}$$

acting on $(\check{\rho}_s,\check{\sigma}_s,\check{v}_s)$.

Next we consider the wall associated with the decay $(Q,P)\to((Q-P)/2,(P-Q)/2)+((Q+P)/2,(Q+P)/2)$. From (4.1) we see that the associated matrix can be taken to be

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{6.3.14}$$

According to (4.2) this controls the behaviour of $\check{\Phi}(\check{\rho},\check{\sigma},\check{v})$ near its zero at

$$\check{\rho}-\check{\sigma}=0\,. \tag{6.3.15}$$

Following the procedure outlined in eqs.(4.14)-(4.22) we can find the coefficient of $(\check{\rho}-\check{\sigma})^2$ in the expression for $\check{\Phi}$. One can see from (6.3.1) that in this case $\frac{1}{2}((Q+P)/2)^2$ and $\frac{1}{2}((Q-P)/2)^2$

can take all possible independent integer values $(K+1)$ and $(m-J)$ respectively. We find from (4.23) that the inverses of the relevant half-BPS partition functions are:

$$\phi_e(\tau; a_0, c_0) = \eta(2\tau)^{24}, \quad \phi_m(\tau; b_0, d_0) = \eta(2\tau)^{24}\,. \tag{6.3.16}$$

The factor of 2 in the argument of η is due to the fact that $Q'^2/2 = (d_0 Q - b_0 P)^2/2 = (Q-P)^2/4$ and $P'^2/2 = (-c_0 Q + a_0 P)^2/2 = (Q+P)^2/4$ entering in (4.23) are twice the usual integer normalized combinations $\frac{1}{8}(Q \pm P)^2$. This gives, from (4.14), (4.22) and (6.3.3)

$$\check\Phi(\check\rho, \check\sigma, \check v) \sim \left\{ (\check\rho_s - \check\sigma_s)^2\, \eta((\check\rho_s + \check\sigma_s - 2\check v_s)/4)^{24}\, \eta((\check\rho_s + \check\sigma_s + 2\check v_s)/4)^{24} + \mathcal{O}((\check\rho_s - \check\sigma_s)^4) \right\}, \tag{6.3.17}$$

near $\check\rho_s \simeq \check\sigma_s$. Since $\eta(2\tau)$ transforms covariantly under $\tau \to (\alpha\tau + \frac{1}{2}\beta)/(2\gamma\tau + \delta)$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$, both ϕ_e and ϕ_m have full $SL(2, \mathbb{Z})$ symmetry. Using (4.29), (4.30) and (6.3.8) to represent them as symplectic transformations on the variables $(\check\rho_s, \check\sigma_s, \check v_s)$ we get the following two sets of symplectic matrices:

$$\frac{1}{2}\begin{pmatrix} \alpha_1+1 & \alpha_1-1 & 2\beta_1 & 2\beta_1 \\ \alpha_1-1 & \alpha_1+1 & 2\beta_1 & 2\beta_1 \\ \gamma_1/2 & \gamma_1/2 & \delta_1+1 & \delta_1-1 \\ \gamma_1/2 & \gamma_1/2 & \delta_1-1 & \delta_1+1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} p_1+1 & -p_1+1 & 2q_1 & -2q_1 \\ -p_1+1 & p_1+1 & -2q_1 & 2q_1 \\ r_1/2 & -r_1/2 & s_1+1 & -s_1+1 \\ -r_1/2 & r_1/2 & -s_1+1 & s_1+1 \end{pmatrix},$$

$$\alpha_1, \beta_1, \gamma_1, \delta_1, p_1, q_1, r_1, s_1 \in \mathbb{Z}, \quad \alpha_1\delta_1 - \beta_1\gamma_1 = p_1 s_1 - q_1 r_1 = 1\,. \tag{6.3.18}$$

Next we turn to the wall corresponding to the decay $(Q,P) \to (Q-P, 0) + (P,P)$. This corresponds to the choice

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{6.3.19}$$

and is associated with the zero of $\check\Phi$ at

$$\check\sigma + \check v = 0\,. \tag{6.3.20}$$

Since $(Q-P)^2/8 = m - J$ and $P^2/4 = K - J$ can take independent integer values, we should be able to use (4.9), (4.10). The behaviour of $\check\Phi$ near this zero is however somewhat ambiguous since one of the decay products – the state carrying charge $(Q-P, 0)$ – is not a primitive dyon. As a result the index associated with this state is ambiguous.[11] Nevertheless if we go ahead

[11] For half-BPS states in $\mathcal{N}=2$ supersymmetric theories a modification of the wall crossing formula for such non-primitive decays has been suggested in [28]. It is not clear *a priori* how to modify it for the decays of quarter BPS dyons in $\mathcal{N}=4$ supersymmetric string theories. In §7 we shall propose a formula for the partition function of the states being studied in this section and examine it to find what the modification should be.

and assume the naive index that follows from tree level spectrum of elementary string states, we get the following factorization behaviour of $\check\Phi$:

$$\check\Phi(\check\rho,\check\sigma,\check v)\overset{?}{\sim}\left\{(\check\sigma_s+\check v_s)^2\phi_e\left(\frac{\check\sigma_s}{4};1,0\right)\,\phi_m\left(\frac{\check\rho_s+\check\sigma_s+2\check v_s}{4};1,1\right)+\mathcal{O}((\check\sigma_s+\check v_s)^4)\right\}\quad\text{for } \check v\simeq-\check\sigma\,, \tag{6.3.21}$$

with

$$\begin{aligned}\phi_e\left(\frac{\tau}{4};1,0\right) &= \frac{1}{4}\left\{\eta(\tau/4)^{-24}+\eta((\tau+1)/4)^{-24}+\eta((\tau+2)/4)^{-24}+\eta((\tau+3)/4)^{-24}\right\}^{-1},\\ \phi_m\left(\frac{\tau}{4};1,1\right) &= \frac{1}{2}\left\{\eta(\tau/4)^{-24}+\eta(\tau+2)/4)^{-24}\right\}^{-1}.\end{aligned} \tag{6.3.22}$$

$\phi_e(\tau/4)$ has duality symmetries of the form $\tau\to(p_2\tau+q_2)/(r_2\tau+s_2)$ with $\begin{pmatrix}p_2 & q_2\\ r_2 & s_2\end{pmatrix}\in\Gamma_0(2)$. On the other hand $\phi_m(\tau/4)$ has duality symmetries of the form $\tau\to(\alpha_2\tau+\beta_2)/(\gamma_2\tau+\delta_2)$ with $\begin{pmatrix}\alpha_2 & \beta_2\\ \gamma_2 & \delta_2\end{pmatrix}\in\Gamma^0(2)$. Using (4.29), (4.30) and (6.3.8) we find that the modular properties in this factorized limit correspond to the following symplectic transformations acting on $(\check\rho_s,\check\sigma_s,\check v_s)$

$$\begin{pmatrix}\alpha_2 & \alpha_2-1 & \beta_2 & 0\\ 0 & 1 & 0 & 0\\ \gamma_2 & \gamma_2 & \delta_2 & 0\\ \gamma_2 & \gamma_2 & \delta_2-1 & 1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1 & 1-p_2 & q_2 & -q_2\\ 0 & p_2 & -q_2 & q_2\\ 0 & 0 & 1 & 0\\ 0 & r_2 & 1-s_2 & s_2\end{pmatrix},$$

$$\alpha_2\delta_2-\beta_2\gamma_2=1=p_2s_2-r_2q_2,\quad \alpha_2,\gamma_2,\delta_2,p_2,q_2,s_2\in\mathbb{Z},\quad \beta_2,r_2\in 2\,\mathbb{Z}\,. \tag{6.3.23}$$

We can now try to see if all the symplectic transformation matrices (6.3.7), (6.3.9), (6.3.13), (6.3.18) and (6.3.23), representing possible symmetries of $\check\Phi$, fit into some subgroup of $Sp(2,\,\mathbb{Z})$ defined by some congruence condition. As it stands there does not seem to be a simple congruence subgroup of $Sp(2,\,\mathbb{Z})$ that fits all the matrices since some of these matrices do not even have integer entries. However if we restrict γ and r in (6.3.13) to be even, i.e. assume that only a $\Gamma(2)\times\Gamma(2)$ subgroup of the symmetry group $\Gamma^0(2)\times\Gamma^0(2)$ of the $\check v\to 0$ limit survives as a symmetry of the full partition function, and restrict γ_1 and r_1 in (6.3.18) to be multiples of 4, i.e. assume that only a $\Gamma_0(4)\times\Gamma_0(4)$ subgroup of the $\check\rho_s\to\check\sigma_s$ limit survives as a symmetry of the full partition function, then there is a simple congruence subgroup of $Sp(2,\,\mathbb{Z})$ into which all the matrices fit:

$$\begin{pmatrix}1+u & u & v & v\\ u & 1+u & v & v\\ w & w & 1+u & u\\ w & w & u & 1+u\end{pmatrix}\quad\text{mod } 2,\quad u,v,w=0,1\,. \tag{6.3.24}$$

We speculate that this could be the symmetry group of the dyon partition function under consideration.

Finally we turn to the constraint from black hole entropy. As in §6.1, in this case we have $g(\tau) = \eta(\tau)^{24}$ in (5.4). Thus (5.6) takes the form

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto (2v - \rho - \sigma)^{10} \left\{ v^2 \, \eta(\rho)^{24} \, \eta(\sigma)^{24} + \mathcal{O}(v^4) \right\}, \tag{6.3.25}$$

where (ρ, σ, v) and $(\check{\rho}, \check{\sigma}, \check{v})$ are related via (5.7).

Before concluding this section we would like to note that we can easily extend the analysis of this section to the complementary subset of torsion 2 dyons with Q, P primitive and $Q^2/2$ and $P^2/2$ odd. For this we consider six dimensional electric and magnetic charge vectors with metric

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & I_2 & 0 \end{pmatrix}, \tag{6.3.26}$$

and take the set $\mathcal{A}$ to be the collection of charge vectors (Q, P) with

$$Q = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2m+1 \\ 1 \\ 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 \\ 1 \\ 2K+1 \\ 2J+1 \\ 1 \\ -1 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}. \tag{6.3.27}$$

This has

$$\frac{Q^2}{2} = 2m + 1, \quad \frac{P^2}{2} = 2(K - J) + 1, \quad Q \cdot P = 2(K + J - m + 1). \tag{6.3.28}$$

Thus we have $Q^2/2$ and $P^2/2$ odd and $Q \cdot P$ even. Furthermore we still have the constraint that $Q^2 + P^2 + 2Q \cdot P$ is a multiple of 8. Thus with $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$ defined as in (6.3.3), the partition function is antiperiodic under $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s) \to (\check{\rho}_s + 2, \check{v}_s, \check{\sigma}_s), (\check{\rho}_s, \check{\sigma}_s + 2, \check{v}_s)$ and periodic under $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s) \to (\check{\rho}_s, \check{\sigma}_s, \check{v}_s + 2), (\check{\rho}_s + 1, \check{\sigma}_s + 1, \check{v}_s + 1)$. We can now repeat the analysis of this section for this set of dyons. The results are more or less identical except for some relative minus signs between the terms in the curly brackets in eqs.(6.3.10)-(6.3.12) and the second equation in (6.3.22).

6.4 Dyons in $\mathbb{Z}_2$ CHL orbifold with twisted sector electric charge

We now consider a $\mathbb{Z}_2$ CHL orbifold defined as follows [29, 30]. We begin with $E_8 \times E_8$ heterotic string theory on $T^4 \times S^1 \times \widetilde{S}^1$ with S^1 and $\widetilde{S}^1$ labelled by coordinates with period 4π and 2π respectively, and take a quotient of the theory by a $\mathbb{Z}_2$ symmetry that involves 2π shift along S^1 together with an exchange of the two E_8 factors. In the four dimensional subspace of charges given in (6.1.1), now the momentum n' along S^1 is quantized in units of $1/2$ whereas the Kaluza-Klein monopole charge N' along S^1 is quantized in units of 2 [9]. We shall take the set $\mathcal{A}$ to be consisting of charge vectors of the form

$$Q = \begin{pmatrix} 0 \\ m/2 \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} K \\ J \\ 1 \\ 0 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}\,. \tag{6.4.1}$$

For this state we have

$$Q^2 = -m, \quad P^2 = 2K, \quad Q \cdot P = -J\,. \tag{6.4.2}$$

As usual we denote by $\mathcal{B}$ the set of all (Q, P) which are related to the ones given in (6.4.2) by a T-duality transformation. Since $Q^2/2$, $P^2/2$ and $Q \cdot P$ are quantized in units of $1/2$, 1 and 1 respectively, $\check{\Phi}$ satisfies the periodicity conditions (3.16) with

$$a_1 \in \mathbb{Z}, \quad a_2 \in 2\,\mathbb{Z}, \quad a_3 \in \mathbb{Z}\,. \tag{6.4.3}$$

Comparison of (6.4.1) and (6.1.1) shows that the winding charge $-w'$ along S^1 is 1 for this state. Thus it represents a twisted sector state.

Our next task is to determine the subgroup of the S-duality group that leaves the set $\mathcal{B}$ invariant. In this case the full S-duality group is $\Gamma_0(2)$, generated by matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, d \in \mathbb{Z}$, $c \in 2\,\mathbb{Z}$, $ad - bc = 1$. It was shown in [18] that the set $\mathcal{B}$ is closed under the full S-duality group. Thus the full S-duality group must be a symmetry of the partition function.

We now turn to the constraints from the wall crossing formula. Consider first the wall associated with the decay $(Q, P) \to (Q, 0) + (0, P)$, – this in fact is the only case we need to analyze since all the walls are related to this one by S-duality transformation [13]. First of all note from (6.4.1), (6.4.2) that for a given $Q^2 = -m$ the charge vector $Q \in \mathcal{A}$ is fixed uniquely. Thus the index of half-BPS states with charge $(Q, 0)$ can be regarded as a function of Q^2. On the other hand for a given $P^2 = 2K$ there is a family of $P \in \mathcal{A}$ labelled by

J, but these can be transformed to the vector corresponding to $J = 0$ by the T-duality transformation matrix [18] $\begin{pmatrix} 1 & 0 & 0 & J \\ 0 & 1 & -J & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Thus the index of the charge vector $(0, P)$ can also be expressed as a function of P^2. Finally we see from (6.4.2) that the allowed values of Q^2 and P^2 are uncorrelated. Thus we can use eqs.(4.9), (4.10) to extract the behaviour of $\check{\Phi}$ near $\check{v} = 0$. The electric partition function can be calculated by examining the spectrum of twisted sector states in the heterotic string theory [44, 45, 46, 47]. On the other hand the magnetic partition function can be calculated by examing the spectrum of D1-D5 system in a dual type IIB description of the theory [18]. The results are

$$\phi_e(\check{\sigma}) = \eta(\check{\sigma})^8 \eta(\check{\sigma}/2)^8 \,, \qquad \phi_m(\check{\rho}) = \eta(\check{\rho})^8 \eta(2\check{\rho})^8 \,. \tag{6.4.4}$$

Eq.(4.9) then gives, near $\check{v} = 0$,

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto \left\{ \check{v}^2 \eta(\check{\sigma})^8 \eta(\check{\sigma}/2)^8 \eta(\check{\rho})^8 \eta(2\check{\rho})^8 + \mathcal{O}(\check{v}^4) \right\} \,. \tag{6.4.5}$$

$\phi_e(\check{\sigma})$ and $\phi_m(\check{\rho})$ transform as modular forms of weight 8 under

$$\check{\sigma} \to \frac{p\check{\sigma} + q}{r\check{\sigma} + s}, \quad p, r, s \in \mathbb{Z}, \quad q \in 2\,\mathbb{Z}, \quad ps - qr = 1 \,, \tag{6.4.6}$$

and

$$\check{\rho} \to \frac{\alpha\check{\rho} + \beta}{\gamma\check{\rho} + \delta}, \quad \alpha, \beta, \delta \in \mathbb{Z}, \quad \gamma \in 2\,\mathbb{Z}, \quad \alpha\delta - \beta\gamma = 1 \,. \tag{6.4.7}$$

The corresponding groups are $\Gamma^0(2)$ and $\Gamma_0(2)$ respectively. Thus from (3.14), (3.17), (4.26) we see that if (6.4.6) and (6.4.7) lift to symmetries of the full partition function then the partition function transforms as a modular form of weight 6 under the $Sp(2, \mathbb{Z})$ transformations of the form

$$\begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & a_1 & a_3 \\ 0 & 1 & a_3 & a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & s \end{pmatrix}, \tag{6.4.8}$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2), \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma^0(2), \quad a_1, a_3 \in \mathbb{Z}, \quad a_2 \in 2\,\mathbb{Z} \,. \tag{6.4.9}$$

All the $Sp(2, \mathbb{Z})$ matrices in (6.4.8) subject to the constraints (6.4.9) have the form

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & * & * & 1 \end{pmatrix} \quad \text{mod } 2\,. \tag{6.4.10}$$

Furthermore the set of matrices (6.4.10) are closed under matrix multiplication. Thus the group generated by the set of $Sp(2, \mathbb{Z})$ matrices (6.4.8) subject to the condition (6.4.9) is contained in the group $\check{G}$ of $Sp(2, \mathbb{Z})$ matrices (6.4.10).

All the symmetries listed in (6.4.8) are indeed symmetries of the dyon partition function of this model proposed in [6] and proved in [9]. Furthermore near $\check{v} = 0$ the partition function is known to have the factorization property given in (6.4.5) [13, 16, 17]. One question that one can ask is: do the matrices given in (6.4.8) generate the full symmetry group of the partition function (which is known in this case)? It turns out that the answer is no. This group does not include the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \tag{6.4.11}$$

since this is not of the form given in (6.4.10). This generates the transformation

$$\check{\rho} \to \frac{\check{\rho}}{(1-\check{v})^2 - \check{\sigma}\check{\rho}}, \quad \check{\sigma} \to \frac{\check{\sigma}}{(1-\check{v})^2 - \check{\sigma}\check{\rho}}, \quad \check{v} \to \frac{\check{\rho}\check{\sigma} + \check{v}(1-\check{v})}{(1-\check{v})^2 - \check{\sigma}\check{\rho}}\,, \tag{6.4.12}$$

and is known to be a symmetry of the partition function.[12]

Finally we turn to the constraints from black hole entropy. In this case the function $g(\tau)$ is given by [39, 44]:

$$g(\tau) = \eta(\tau)^8\, \eta(2\tau)^8\,. \tag{6.4.13}$$

Thus (5.6) takes the fom

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto (2v - \rho - \sigma)^6 \left\{ v^2\, \eta(\rho)^8 \eta(2\rho)^8\, \eta(\sigma)^8 \eta(2\sigma)^8 + \mathcal{O}(v^4) \right\}, \tag{6.4.14}$$

where (ρ, σ, v) and $(\check{\rho}, \check{\sigma}, \check{v})$ are related via (5.7). The dyon partition function of the $\mathbb{Z}_2$ CHL model is known to satisfy this property. In fact historically this is the property that was used to guess the form of the partition function [6].

This analysis can be easily generalized to the dyons of $\mathbb{Z}_N$ CHL orbifolds carrying twisted sector electric charges.

[12]This is the symmetry refered to as $g_3(1, 0)$ in [6] in a different representation.

6.5 Dyons in $\mathbb{Z}_2$ CHL model with untwisted sector electric charge

We again consider the $\mathbb{Z}_2$ CHL model introduced in §6.4, but now take the set $\mathcal{A}$ to consist of dyons with charge vectors

$$Q = \begin{pmatrix} 0 \\ (2m+1)/2 \\ 0 \\ -2 \end{pmatrix}, \qquad P = \begin{pmatrix} 2K+1 \\ J \\ 1 \\ 0 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}. \tag{6.5.1}$$

Since $w' = -2$ for this state, it represents an untwisted sector state. For this state we have

$$Q^2 = -2(2m+1), \quad P^2 = 2(2K+1), \quad Q \cdot P = -2J. \tag{6.5.2}$$

Note that Q and P are both primitive. Since $Q \cdot P$ is quantized in units of 2, we shall define

$$Q_s = \frac{Q}{\sqrt{2}}, \quad P_s = \frac{P}{\sqrt{2}}, \quad \check\rho_s = 2\check\rho, \quad \check\sigma_s = 2\check\sigma, \quad \check v_s = 2\check v. \tag{6.5.3}$$

Thus we have

$$Q_s^2 = -(2m+1), \quad P_s^2 = 2K+1, \quad Q_s \cdot P_s = -J. \tag{6.5.4}$$

Since $Q_s^2/2$ is quantized in units of 1/2, we expect the partition function to have $\check\sigma_s$ period 2. However except for an overall additive factor of 1/2, $Q_s^2/2$ is actually quantized in integer units. Thus the partition function has the additional property that it is odd under $\check\sigma_s \to \check\sigma_s + 1$. Similarly since P_s^2 is an odd integer, the partition function picks up a minus sign under $\check\rho_s \to \check\rho_s + 1$. We shall call these symmetries of $\check\Phi$. Finally since $Q_s \cdot P_s$ is quantized in integer units, the period in the $\check v_s$ direction is also unity. The corresponding symplectic transformations acting on $(\check\rho_s, \check\sigma_s, \check v_s)$ are of the form

$$\begin{pmatrix} 1 & 0 & \widetilde a_1 & \widetilde a_3 \\ 0 & 1 & \widetilde a_3 & \widetilde a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde a_1, \widetilde a_2, \widetilde a_3 \in \mathbb{Z}. \tag{6.5.5}$$

Under this transformation the partition function picks up a multiplier factor of $(-1)^{\widetilde a_1 + \widetilde a_2}$.

Our next task is to determine the subgroup of the S-duality group $\Gamma_0(2)$ that leaves the set $\mathcal{B}$ – defined as the T-duality orbit of $\mathcal{A}$ – invariant. For this let us apply the S-duality transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$ on the charge vector (6.5.1). This gives

$$Q' = aQ + bP = \begin{pmatrix} b(2K+1) \\ (2m+1)a/2 + bJ \\ b \\ -2a \end{pmatrix}, \qquad P' = cQ + dP = \begin{pmatrix} d(2K+1) \\ (2m+1)c/2 + dJ \\ d \\ -2c \end{pmatrix}. \tag{6.5.6}$$

We need to choose a, b, c, d such that (6.5.6) is inside the set $\mathcal{B}$, i.e. it can be brought to the form (6.5.1) after a T-duality transformation. The T-duality transformations acting within this four dimensional subspace are generated by matrices of the form [18]:

$$\begin{pmatrix} n_1 & -m_1 & & \\ -l_1 & k_1 & & \\ & & k_1 & l_1 \\ & & m_1 & n_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k_2 & & & -l_2 \\ & k_2 & l_2 & \\ & m_2 & n_2 & \\ -m_2 & & & n_2 \end{pmatrix}, \quad \begin{pmatrix} k_i & l_i \\ m_i & n_i \end{pmatrix} \in \Gamma_0(2)\,. \quad (6.5.7)$$

Now suppose b in (6.5.6) is even. Then we can apply a T-duality transformation on the charge vector given in (6.5.6) with the matrix

$$\begin{pmatrix} 1 & & & l_0 \\ & 1 & -l_0 & \\ & 0 & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} d & -2c & & \\ -b/2 & a & & \\ & & a & b/2 \\ & & 2c & d \end{pmatrix}, \qquad l_0 \equiv \frac{1}{2} bd(2K+1) - \frac{c}{2}\{(2m+1)a + 2bJ\}\,. \tag{6.5.8}$$

It is straightfoward to verify that this brings (6.5.6) back to the set $\mathcal{A}$ consisting of pairs of charge vectors of the form given in (6.5.1). This shows that a sufficient condition for (6.5.6) to lie in the set $\mathcal{B}$ is to have b even, i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$. Using (3.2) we can also see that this condition is necessary since acting on a pair (Q, P) with $Q^2/2$ odd, $P^2/2$ odd and $Q \cdot P$ even, an S-duality transformation produces a (Q', P') with odd $Q'^2/2$ only if b is even.

Thus we identify the subgroup $\Gamma(2)$ of the S-duality group $\Gamma_0(2)$ as the symmetry of the set $\mathcal{B}$. The overall scaling of Q and P does not change the symmetry group. Thus the quarter BPS dyon partition function associated with the set $\mathcal{B}$ must be invariant under the $\Gamma(2)$ S-duality symmetry. This in turn corresponds to symplectic transformations of the form

$$\begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix}, \qquad ad - bc = 1, \quad a, d \in \mathbb{Z}, \quad b, c \in 2\,\mathbb{Z}\,, \tag{6.5.9}$$

acting on $(\check\rho, \check\sigma, \check v)$ and also on $(\check\rho_s, \check\sigma_s, \check v_s)$.

Next we turn to the analysis of the constraints from wall crossing. First consider the wall corresponding to the decay $(Q, P) \to (Q, 0) + (0, P)$, and examine whether there are subtleties of kind mentioned below eq.(4.11) in applying eqs.(4.9), (4.10). For this we note that here $Q^2 = -2(2m+1)$ and $P^2 = 2(2K+1)$ are uncorrelated. For a given $Q^2 = -2(2m+1)$ there is a unique charge vector in the list given in (6.5.1). On the other hand even though for a given $P^2 = 2(2K+1)$ there is an infinite family of P labelled by J, they are all related by the T-duality transformation matrix $\begin{pmatrix} 1 & 0 & 0 & -J \\ 0 & 1 & J & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ to the vector $\begin{pmatrix} 2K+1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Thus there are no subtleties of the kind mentioned below (4.11) and we have

$$\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto \left\{ \check{v}^2 \, \phi_m(\check{\rho}) \phi_e(\check{\sigma}) + \mathcal{O}(\check{v}^4) \right\} \quad \text{for } \check{v} \simeq 0 \,. \tag{6.5.10}$$

The magnetic partition function is obtained from (6.4.4) by projection to odd values of $P^2/2$ followed by $\check{\rho} \to \check{\rho}_s/2$ replacement. This gives

$$\phi_m(\check{\rho})^{-1} = \frac{1}{2} \left\{ \eta(\check{\rho}_s/2)^{-8} \eta(\check{\rho}_s)^{-8} - \eta((\check{\rho}_s+1)/2)^{-8} \eta(\check{\rho}_s)^{-8} \right\} . \tag{6.5.11}$$

On the other hand the electric partition function can be calculated by analyzing the untwisted sector BPS spectrum of the fundamental heterotic string [44, 45, 46, 47]. After taking into account the fact that we are computing the partition function of odd $Q^2/2$ states only, and the $\check{\sigma} \to \check{\sigma}_s/2$ replacement, the result is

$$\begin{aligned} \phi_e^{-1}(\check{\sigma}) &= \frac{1}{2} \left(\psi_e(\check{\sigma}_s) - \psi_e(\check{\sigma}_s + 1) \right) , \\ \psi_e(\check{\sigma}_s) &= 8 \, \eta(\check{\sigma}_s/2)^{-24} \left[\frac{1}{2} \left(\vartheta_2(\check{\sigma}_s)^8 + \vartheta_3(\check{\sigma}_s)^8 + \vartheta_4(\check{\sigma}_s)^8 \right) - \vartheta_3(\check{\sigma}_s/2)^4 \vartheta_4(\check{\sigma}_s/2)^4 \right] . \end{aligned} \tag{6.5.12}$$

In (6.5.12) ψ_e describes the partition function before projecting on to the odd $Q^2/2$ sector [45].

$\phi_m(\check{\rho}_s)$ given in (6.5.11) transforms as a modular form of weight 8 under $\check{\rho}_s \to (\alpha \check{\rho}_s + \beta)/(\gamma \check{\rho}_s + \delta)$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2)$ with a multiplier $(-1)^\beta$. On the other hand $\phi_e(\check{\sigma})$ given in (6.5.12) can be shown to transform as a modular form of weight 8 under $\check{\sigma}_s \to (p\check{\sigma}_s + q)/(r\check{\sigma}_s + s)$ for $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma_0(2)$, with a multiplier $(-1)^q$. These duality symmetries correspond to the symplectic transformations

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & s \end{pmatrix} ,$$

$$\alpha\delta - \beta\gamma = 1, \quad ps - qr = 1, \quad \alpha, \beta, \delta, p, q, s \in \mathbb{Z}, \quad \gamma, r \in 2\,\mathbb{Z} \,, \tag{6.5.13}$$

acting on $(\check\rho_s, \check\sigma_s, \check v_s)$.

Since for the set $\mathcal{B}$ the S-duality group is $\Gamma(2)$, in this case there is another wall of marginal stability, associated with $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which cannot be related to the previous wall by an S-duality transformation. This corresponds to the decay $(Q,P) \to (Q-P,0)+(P,P)$ and controls the behaviour of $\check\Phi$ near $\check v + \check\sigma = 0$. As usual we first need to determine if there are any subtleties of the type mentioned below eq.(4.11). Eq.(6.5.1) shows that for a given $(Q-P)^2 = 4(K+J-m)$ there is an infinite family of $(Q-P) = \begin{pmatrix} -(2K+1) \\ (2m-2J+1)/2 \\ -1 \\ -2 \end{pmatrix}$ labelled by $2K$. However all of these can be related by T-duality transformation $\begin{pmatrix} 1 & 0 & 0 & K \\ 0 & 1 & -K & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ to the vector $\begin{pmatrix} -1 \\ (2m-2J-2K+1)/2 \\ -1 \\ -2 \end{pmatrix}$ which is determined completely in terms of $(Q-P)^2$. We have already seen earlier that all choices of P for a given P^2 are also related by T-duality transformations. Finally we note that in this case $(Q-P)^2/2$ and $P^2/2$ can take independent even and odd integer values respectively. It then follows that there are no subtleties of the kind mentioned below (4.11). After evaluating ϕ_e and ϕ_m by standard procedure we find that near $\check v + \check\sigma = 0$ $\check\Phi$ behaves as

$$\check\Phi(\check\rho, \check\sigma, \check v) \propto \left[\check v_s'^2 \, \left\{ \eta(\check\rho_s'/2)^{-8} - \eta((\check\rho_s'+1)/2)^{-8} \right\}^{-1} \eta(\check\rho_s')^8 \{\psi_e(\check\sigma_s') + \psi_e(\check\sigma_s'+1)\}^{-1} + \mathcal{O}(\check v_s'^4) \right] , \tag{6.5.14}$$

where

$$\check v_s' = \check v_s + \check\sigma_s, \quad \check\sigma_s' = \check\sigma_s, \quad \check\rho_s' = \check\rho_s + \check\sigma_s + 2\check v_s \, . \tag{6.5.15}$$

This has duality symmetry $\check\rho_s' \to (\alpha_1\check\rho_s' + \beta_1)/(\gamma_1\check\rho_s' + \delta_1)$ and $\check\sigma_s' \to (p_1\check\sigma_s' + q_1)/(r_1\check\sigma_s' + s_1)$ for $\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \in \Gamma_0(2)$ and $\begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} \in \Gamma_0(2)$ with a multiplier $(-1)^{\beta_1}$. We can express them as symplectic transformations acting on $(\check\rho_s, \check\sigma_s, \check v_s)$ using (4.29), (4.30). This gives

$$\begin{pmatrix} \alpha_1 & \alpha_1 - 1 & \beta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_1 & \gamma_1 & \delta_1 & 0 \\ \gamma_1 & \gamma_1 & \delta_1 - 1 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 1-p_1 & q_1 & -q_1 \\ 0 & p_1 & -q_1 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & r_1 & 1-s_1 & s_1 \end{pmatrix},$$

$$\alpha_1\delta_1 - \beta_1\gamma_1 = 1 = p_1 s_1 - r_1 q_1, \quad \alpha_1, \beta_1, \delta_1, p_1, q_1, s_1 \in \mathbb{Z}, \quad \gamma_1, r_1 \in 2\,\mathbb{Z}\,. \tag{6.5.16}$$

As usual, we would like to know if there is a natural subgroup of $Sp(2, \mathbb{Z})$ defined by some congruence relation into which all the $Sp(2, \mathbb{Z})$ matrices (6.5.5), (6.5.9), (6.5.13) and (6.5.16) fit. There is indeed such a subgroup defined as the collection of matrices of the form

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{mod } 2\,. \tag{6.5.17}$$

It is natural to speculate that (6.5.17) is the symmetry group of the partition function under consideration.

Finally we turn to the constraint from black hole entropy. Since we are considering $\mathbb{Z}_2$ CHL orbifold, the function $g(\tau)$ appearing in the coefficient of the Gauss-Bonnet term in the effective action is the same as the one in §6.4:

$$g(\tau) = \eta(\tau)^8\,\eta(2\tau)^8\,. \tag{6.5.18}$$

Thus (5.6) takes the fom

$$\check{\Phi}(\check{\rho},\check{\sigma},\check{v}) \propto (2v-\rho-\sigma)^6\,\{v^2\,\eta(\rho)^8\eta(2\rho)^8\,\eta(\sigma)^8\eta(2\sigma)^8 + \mathcal{O}(v^4)\}\,, \tag{6.5.19}$$

where (ρ,σ,v) and $(\check{\rho},\check{\sigma},\check{v})$ are related via (5.7).

7 A proposal for the partition function of dyons of torsion two

In this section we shall consider the set of dyons described in §6.3, carrying charge vectors (Q,P) with torsion 2, Q, P primitive and $Q^2/2$, $P^2/2$ even, and propose a form of the partition function that satisfies all the constraints derived in §6.3. The proposed form of the partition function is

$$\begin{aligned}\frac{1}{\check{\Phi}(\check{\rho},\check{\sigma},\check{v})} \quad = \quad & \frac{1}{16}\Bigg[\frac{1}{\Phi_{10}(\check{\rho},\check{\sigma},\check{v})} + \frac{1}{\Phi_{10}(\check{\rho},\check{\sigma}+\frac{1}{2},\check{v})} + \frac{1}{\Phi_{10}(\check{\rho}+\frac{1}{2},\check{\sigma},\check{v})} \\ & + \frac{1}{\Phi_{10}(\check{\rho}+\frac{1}{2},\check{\sigma}+\frac{1}{2},\check{v})} + \frac{1}{\Phi_{10}(\check{\rho}+\frac{1}{4},\check{\sigma}+\frac{1}{4},\check{v}+\frac{1}{4})} \\ & + \frac{1}{\Phi_{10}(\check{\rho}+\frac{1}{4},\check{\sigma}+\frac{3}{4},\check{v}+\frac{1}{4})} + \frac{1}{\Phi_{10}(\check{\rho}+\frac{3}{4},\check{\sigma}+\frac{1}{4},\check{v}+\frac{1}{4})}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Phi_{10}(\check\rho+\frac{3}{4},\check\sigma+\frac{3}{4},\check v+\frac{1}{4})}+\frac{1}{\Phi_{10}(\check\rho+\frac{1}{2},\check\sigma+\frac{1}{2},\check v+\frac{1}{2})}\\
&+\frac{1}{\Phi_{10}(\check\rho+\frac{1}{2},\check\sigma,\check v+\frac{1}{2})}+\frac{1}{\Phi_{10}(\check\rho,\check\sigma+\frac{1}{2},\check v+\frac{1}{2})}+\frac{1}{\Phi_{10}(\check\rho,\check\sigma,\check v+\frac{1}{2})}\\
&+\frac{1}{\Phi_{10}(\check\rho+\frac{3}{4},\check\sigma+\frac{3}{4},\check v+\frac{3}{4})}+\frac{1}{\Phi_{10}(\check\rho+\frac{3}{4},\check\sigma+\frac{1}{4},\check v+\frac{3}{4})}\\
&+\frac{1}{\Phi_{10}(\check\rho+\frac{1}{4},\check\sigma+\frac{3}{4},\check v+\frac{3}{4})}+\frac{1}{\Phi_{10}(\check\rho+\frac{1}{4},\check\sigma+\frac{1}{4},\check v+\frac{3}{4})}\Bigg]\\
&+\Bigg[\frac{1}{\Phi_{10}(\check\rho+\check\sigma+2\check v,\check\rho+\check\sigma-2\check v,\check\sigma-\check\rho)}\\
&+\frac{1}{\Phi_{10}(\check\rho+\check\sigma+2\check v+\frac{1}{2},\check\rho+\check\sigma-2\check v+\frac{1}{2},\check\sigma-\check\rho+\frac{1}{2})}\Bigg]\,. \qquad (7.1)
\end{aligned}
$$

The index $d(Q,P)$ is computed from this partition function using the formula

$$
\begin{aligned}
d(Q,P) &= \frac{1}{4}\,(-1)^{Q\cdot P+1}\int_{\mathcal{C}} d\check\rho_s d\check\sigma_s d\check v_s\, e^{-i\frac{\pi}{4}(\check\sigma_s Q^2+\check\rho_s P^2+2\check v_s Q\cdot P)}\,\frac{1}{\check\Phi(\check\rho,\check\sigma,\check v)}\,,\\
&(\check\rho_s,\check\sigma_s,\check v_s)\equiv(4\check\rho,4\check\sigma,4\check v)\,, \qquad (7.2)
\end{aligned}
$$

where the contour $\mathcal{C}$ is defined by fixing the imaginary parts of $\check\rho_s$, $\check\sigma_s$, $\check v_s$ to appropriate values depending on the domain of the moduli space in which we want to compute the index, and the real parts span the unit cell defined by the periodicity condition

$$
(\check\rho_s,\check\sigma_s,\check v_s)\to(\check\rho_s+2,\check\sigma_s,\check v_s),\,(\check\rho_s,\check\sigma_s+2,\check v_s),\,(\check\rho_s,\check\sigma_s,\check v_s+2),\,(\check\rho_s+1,\check\sigma_s+1,\check v_s+1)\,. \qquad (7.3)
$$

The overall multiplicative factor of 1/4 in (7.2) accounts for the fact that the unit cell defined by eqs.(7.3) has volume 4. The factor of 1/4 in the exponent in (7.2) accounts for the replacement of $(\check\rho,\check\sigma,\check v)$ by $(\check\rho_s/4,\check\sigma_s/4,\check v_s/4)$ in (2.2). Note that the sixteen terms inside the first square bracket in (7.1) together gives the partition function of dyons of unit torsion subject to the constraint that $Q^2/2$, $P^2/2$, $Q\cdot P$ are even and $Q^2+P^2-2Q\cdot P$ is a multiple of 8. The second term is new and reflects the effect of considering states with torsion two.

We shall now check that this formula satisfies all the constraints derived in §6.3. We begin with the S-duality transformations. The first term inside the first square bracket $(\Phi_{10}(\check\rho,\check\sigma,\check v))^{-1}$ is S-duality invariant since S-duality transformation can be regarded as an $Sp(2,\,\mathbb{Z})$ transformation on $(\check\rho,\check\sigma,\check v)$. The other terms inside the first square bracket have the form $(\Phi_{10}(\check\rho+b_1,\check\sigma+b_2,\check v+b_3))^{-1}$ for appropriate choices of (b_1,b_2,b_3). Since these shifts may be represented as symplectic transformations of $(\check\rho,\check\sigma,\check v)$, S-duality transformation (6.3.9) will change this term to $(\Phi_{10}(\check\rho + b_1'', \check\sigma + b_2'', \check v + b_3''))^{-1}$ with

$$\begin{pmatrix} 1 & 0 & b_1'' & b_3'' \\ 0 & 1 & b_3'' & b_2'' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & b_1 & b_3 \\ 0 & 1 & b_3 & b_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix},$$
$$a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad a + c \in 2\,\mathbb{Z} + 1, \quad b + d \in 2\,\mathbb{Z} + 1\,. \tag{7.4}$$

One finds that under such a transformation the sixteen triplets (b_1, b_2, b_3) appearing in the sixteen terms inside the first square bracket get permuted up to integer shifts which are symmetries of Φ_{10}. This proves the S-duality invariance of the first 16 terms.

To test the S-duality invariance of the second term we define

$$\check\rho' = (\check\rho + \check\sigma + 2\check v), \quad \check\sigma' = (\check\rho + \check\sigma - 2\check v), \quad \check v' = (\check\sigma - \check\rho)\,. \tag{7.5}$$

It is easy to verify that the effect of S-duality transformation (6.3.9) on the $(\check\rho', \check\sigma', \check v')$ variables is represented by the symplectic matrix

$$\begin{pmatrix} (a+b+c+d)/2 & (a+b-c-d)/2 & 0 & 0 \\ (a-b+c-d)/2 & (a-b-c+d)/2 & 0 & 0 \\ 0 & 0 & (a-b-c+d)/2 & -(a-b+c-d)/2 \\ 0 & 0 & -(a+b-c-d)/2 & (a+b+c+d)/2 \end{pmatrix}. \tag{7.6}$$

Given the conditions (7.4) on a, b, c, d, this is an $Sp(2, \mathbb{Z})$ transformation. Thus the first term inside the second square bracket in (7.1), given by $(\Phi_{10}(\check\rho', \check\sigma', \check v'))^{-1}$, is manifestly S-duality invariant. The second term involves a shift of $(\check\rho', \check\sigma', \check v')$ by $(1/2, 1/2, 1/2)$. One can easily check that this commutes with the symplectic transformation (7.6) up to integer shifts in $(\check\rho', \check\sigma', \check v')$. Thus the second term is also S-duality invariant.

We now turn to the wall crossing formulæ. First consider the wall associated with the decay $(Q, P) \to (Q, 0) + (0, P)$. The jump in the index across this wall is controlled by the residue of the pole at $\check v = 0$. In order to evaluate this residue it will be most convenient to choose the unit cell over which the integration (7.2) is performed to be $-1 \le \Re(\check\rho_s) < 1$, $-1 \le \Re(\check\sigma_s) < 1$ and $-\frac{1}{2} \le \Re(\check v_s) < \frac{1}{2}$ so that the image of the pole at $\check v_s = 0$ under $(\check\rho_s, \check\sigma_s, \check v_s) \to (\check\rho_s \pm 1, \check\sigma_s \pm 1, \check v_s \pm 1)$ is outside the unit cell, – otherwise we would need to include the contribution from this pole as well. Using (7.2) we see that the change in the index across this wall is given by

$$\Delta d(Q, P) = \frac{1}{4}\,(-1)^{Q\cdot P+1} \int_{iM_1-1}^{iM_1+1} d\check\rho_s \int_{iM_2-1}^{iM_2+1} d\check\sigma_s \oint d\check v_s\, e^{-i\frac{\pi}{4}(\check\sigma_s Q^2 + \check\rho_s P^2 + 2\check v_s Q\cdot P)}\, \frac{1}{\check{\tilde\Phi}(\check\rho, \check\sigma, \check v)}\,, \tag{7.7}$$

where $\oint$ denotes the contour around $\check{v}_s = 0$ and M_1, M_2 are large positive numbers. Now the poles in (7.1) can be found from the known locations of the zeroes in $\Phi_{10}(x,y,z)$:

$$
\begin{aligned}
& n_2(xy - z^2) + jz + n_1 y - m_1 x + m_2 = 0 \\
& m_1, n_1, m_2, n_2 \in \mathbb{Z}, \quad j \in 2\,\mathbb{Z} + 1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,.
\end{aligned} \tag{7.8}
$$

Using this we find that the poles in (7.1) at $\check{v}_s = 0$ can come from the first four terms inside the first square bracket. There is no pole at $\check{v}_s = 0$ from the terms in the second square bracket. The residue at the pole can be calculated by using the fact that

$$
\Phi_{10}(x,y,z) \simeq -4\pi^2\, z^2\, \eta(x)^{24}\eta(y)^{24} + \mathcal{O}(z^4)\,, \tag{7.9}
$$

near $z = 0$. This gives

$$
\check{\Phi}(\check{\rho},\check{\sigma},\check{v}) = -4\pi^2\,\check{v}_s^2 \left\{\eta\left(\frac{\check{\sigma}_s}{4}\right)^{-24} + \eta\left(\frac{\check{\sigma}_s+2}{4}\right)^{-24}\right\}^{-1} \left\{\eta\left(\frac{\check{\rho}_s}{4}\right)^{-24} + \eta\left(\frac{\check{\rho}_s+2}{4}\right)^{-24}\right\}^{-1} + \mathcal{O}(\check{v}_s^4)\,. \tag{7.10}
$$

Substituting this into (7.7) and using the convention that the $\check{v}_s$ contour encloses the pole clockwise, we get

$$
\begin{aligned}
\Delta d(Q,P) \;=\; & \frac{1}{16}\,(-1)^{Q\cdot P+1}\, Q\cdot P \int_{iM_1-1}^{iM_1+1} d\check{\rho}_s\, \left\{\eta(\check{\rho}_s/4)^{-24} + \eta((\check{\rho}_s+2)/4)^{-24}\right\}\, e^{-i\pi\check{\rho}_s\, P^2/4} \\
& \int_{iM_2-1}^{iM_2+1} d\check{\sigma}_s\, \left\{\eta(\check{\sigma}_s/4)^{-24} + \eta((\check{\sigma}_s+2)/4)^{-24}\right\}\, e^{-i\pi\check{\sigma}_s\, Q^2/4}\,.
\end{aligned} \tag{7.11}
$$

We now want to compare this with the general wall crossing formula given in (4.7). Here the relevant half-BPS partition functions are to be computed with Q^2 and P^2 restricted to be even. This gives

$$
\begin{aligned}
d_h(Q,0) &= \int_{iM-1/4}^{iM+1/4} d\tau \left\{\eta(\tau)^{-24} + \eta\left(\tau+\frac{1}{2}\right)^{-24}\right\} e^{-i\pi\tau Q^2} \\
&= \frac{1}{4}\int_{4iM-1}^{4iM+1} d\check{\sigma}_s\, \left\{\eta(\check{\sigma}_s/4)^{-24} + \eta((\check{\sigma}_s+2)/4)^{-24}\right\}\, e^{-i\pi\check{\sigma}_s\, Q^2/4}, \\
d_h(0,P) &= \int_{iM-1/4}^{iM+1/4} d\tau \left\{\eta(\tau)^{-24} + \eta\left(\tau+\frac{1}{2}\right)^{-24}\right\} e^{-i\pi\tau P^2} \\
&= \frac{1}{4}\int_{4iM-1}^{4iM+1} d\check{\rho}_s\, \left\{\eta(\check{\rho}_s/4)^{-24} + \eta((\check{\rho}_s+2)/4)^{-24}\right\}\, e^{-i\pi\check{\rho}_s\, P^2/4}\,,
\end{aligned} \tag{7.12}
$$

for some large positive number M. Using this we can rewrite (7.11) as

$$\Delta d(Q,P) = (-1)^{Q\cdot P+1}\, Q\cdot P\, d_h(Q,0)\, d_h(0,P)\,, \tag{7.13}$$

in agreement with the wall crossing formula.

Next we consider the wall associated with the decay $(Q,P) \to ((Q-P)/2, (P-Q)/2) + ((Q+P)/2, (Q+P)/2)$. The associated pole of the partition function is at $\check\rho = \check\sigma$. With the help of (7.9) we find that this pole arises from the first term inside the second square bracket in (7.1), and near this pole

$$\check\Phi(\check\rho,\check\sigma,\check v) = -\frac{\pi^2}{4}(\check\sigma_s - \check\rho_s)^2\, \eta((\check\rho_s+\check\sigma_s+2\check v_s)/4)\, \eta((\check\rho_s+\check\sigma_s-2\check v_s)/4) + \mathcal{O}\left((\check\sigma_s-\check\rho_s)^4\right)\,. \tag{7.14}$$

In order to compute the change in the index as we cross this wall, we change variables to

$$\check\rho'_s = (\check\rho_s+\check\sigma_s+2\check v_s)/2, \quad \check\sigma'_s = (\check\rho_s+\check\sigma_s-2\check v_s)/2, \quad \check v'_s = (\check\sigma_s-\check\rho_s)/2\,. \tag{7.15}$$

The periodicity properties (7.3) on $(\check\rho_s,\check\sigma_s,\check v_s)$ take the form

$$(\check\rho'_s,\check\sigma'_s,\check v'_s) \to (\check\rho'_s+2,\check\sigma'_s,\check v'_s),\ (\check\rho'_s,\check\sigma'_s+2,\check v'_s),\ (\check\rho'_s,\check\sigma'_s,\check v'_s+2),\ (\check\rho'_s+1,\check\sigma'_s+1,\check v'_s+1)\,. \tag{7.16}$$

We choose the unit cell in the $(\Re(\check\rho'_s), \Re(\check\sigma'_s), \Re(\check v'_s))$ to be $-1\le \Re(\check\rho'_s) < 1$, $-1 \le \Re(\check\sigma'_s) < 1$ and $-\frac{1}{2} \le \Re(\check v'_s) < \frac{1}{2}$. Since the jacobian of the transformation associated with (7.15) is unity, the change in the index across the wall is given by an expression analogous to (7.7)

$$\Delta d(Q,P) = \frac{1}{4}\,(-1)^{Q\cdot P+1}\int_{iM'_1-1}^{iM'_1+1} d\check\rho'_s \int_{iM'_2-1}^{iM'_2+1} d\check\sigma'_s \oint d\check v'_s\, e^{-i\frac{\pi}{4}(\check\sigma'_s Q'^2+\check\rho'_s P'^2+2\check v'_s Q'\cdot P')}\,\frac{1}{\check\Phi(\check\rho,\check\sigma,\check v)}\,, \tag{7.17}$$

where

$$Q' = \frac{Q-P}{\sqrt{2}}, \qquad P' = \frac{Q+P}{\sqrt{2}}\,. \tag{7.18}$$

Substituting (7.14) into (7.17) we get

$$\begin{aligned}\Delta d(Q,P) &= \frac{1}{4}\,(-1)^{Q\cdot P+1}\, Q'\cdot P' \int_{iM'_1-1}^{iM'_1+1} d\check\rho'_s\, \eta(\check\rho'_s/2)^{-24}\, e^{-i\pi\check\rho'_s P'^2/4} \\ &\qquad \int_{iM'_2-1}^{iM'_2+1} d\check\sigma'_s\, \eta(\check\sigma'_s/2)^{-24}\, e^{-i\pi\check\sigma'_s Q'^2/4}\,. \end{aligned} \tag{7.19}$$

On the other hand now the indices of the half-BPS decay products carrying charges

$$(Q_1,P_1) = ((Q-P)/2, (P-Q)/2), \qquad (Q_2,P_2) = ((Q+P)/2, (Q+P)/2)\,, \tag{7.20}$$

are given by

$$d_h(Q_1,P_1) = \int_{iM-1/2}^{iM+1/2} d\tau (\eta(\tau))^{-24} e^{-i\pi\tau((Q-P)/2)^2} = \frac{1}{2}\int_{2iM-1}^{2iM+1} d\check\sigma'_s\, \eta(\check\sigma'_s/2)^{-24}\, e^{-i\pi\check\sigma'_s Q'^2/4}\,, \tag{7.21}$$

and

$$d_h(Q_2,P_2) = \int_{iM-1/2}^{iM+1/2} d\tau (\eta(\tau))^{-24} e^{-i\pi\tau((Q+P)/2)^2} = \frac{1}{2}\int_{2iM-1}^{2iM+1} d\check\rho'_s\, \eta(\check\rho'_s/2)^{-24}\, e^{-i\pi\check\rho'_s P'^2/4}\,. \tag{7.22}$$

Using these results and the identities

$$Q_1\cdot P_2 - Q_2\cdot P_1 = Q'\cdot P', \quad (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1} = (-1)^{(Q-P)^2/2-P^2+Q\cdot P} = (-1)^{Q\cdot P}\,, \tag{7.23}$$

we can express (7.19) as

$$\Delta d(Q,P) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\,(Q_1\cdot P_2 - Q_2\cdot P_1)\, d_h(Q_1,P_1)\, d_h(Q_2,P_2)\,, \tag{7.24}$$

in agreement with the wall crossing formula.

Next consider the decay $(Q,P)\to(Q-P,0)+(P,P)$. This is controlled by the pole at $\check\sigma+\check v=0$. To analyze this contribution we define

$$Q' = Q-P, \quad P' = P \tag{7.25}$$

and

$$\check\rho'_s = \check\rho_s + \check\sigma_s + 2\check v_s, \quad \check\sigma'_s = \check\sigma_s, \quad \check v'_s = \check v_s + \check\sigma_s\,, \tag{7.26}$$

so that $\check\rho_s P^2 + \check\sigma_s Q^2 + 2\check v_s Q\cdot P = \check\rho'_s P'^2 + \check\sigma'_s Q'^2 + 2\check v'_s Q'\cdot P'$. In terms of these variables the periods are

$$(\check\rho'_s,\check\sigma'_s,\check v'_s) \to (\check\rho'_s+2,\check\sigma'_s,\check v'_s),\ (\check\rho'_s,\check\sigma'_s+1,\check v'_s),\ (\check\rho'_s,\check\sigma'_s,\check v'_s+2)\,. \tag{7.27}$$

The behaviour of $\check\Phi$ near $\check v'_s = 0$ can be found by the usual procedure and result is

$$\begin{aligned}\check\Phi(\check\rho,\check\sigma,\check v)^{-1} \;=\; & -\frac{1}{4\pi^2\check v'^2_s}\left\{\eta\left(\frac{\check\rho'_s}{4}\right)^{-24} + \eta\left(\frac{\check\rho'_s+2}{4}\right)^{-24}\right\} \\ & \times\left\{\eta\left(\frac{\check\sigma'_s}{4}\right)^{-24} + \eta\left(\frac{\check\sigma'_s+1}{4}\right)^{-24} + \eta\left(\frac{\check\sigma'_s+2}{4}\right)^{-24} + \eta\left(\frac{\check\sigma'_s+3}{4}\right)^{-24}\right\} \\ & -\frac{1}{\pi^2\check v'^2_s}\left\{\eta\left(\frac{\check\rho'_s}{4}\right)^{-24} + \eta\left(\frac{\check\rho'_s+2}{4}\right)^{-24}\right\}\eta(\check\sigma'_s)^{-24} + \mathcal{O}\left(\check v'^0_s\right)\,. \end{aligned} \tag{7.28}$$

Note that the first set of terms represent correctly the factorization behaviour given in (6.3.21), but the second set of terms are extra. Thus the wall crossing formula gets modified for the decay into non-primitive states. Using (7.28) we can compute the jump in the index across the wall

$$\begin{aligned}\Delta d(Q,P) =& \frac{1}{16}(-1)^{Q\cdot P+1}\, Q'\cdot P' \int_{iM_1'-1}^{iM_1'+1} d\check\rho_s' \left\{\eta\left(\frac{\check\rho_s'}{4}\right)^{-24} + \eta\left(\frac{\check\rho_s'+2}{4}\right)^{-24}\right\} e^{-i\pi\check\rho_s' P'^2/4} \\ & \int_{iM_2'-1/2}^{iM_2'+1/2} d\check\sigma_s' \left\{\eta\left(\frac{\check\sigma_s'}{4}\right)^{-24} + \eta\left(\frac{\check\sigma_s'+1}{4}\right)^{-24}\right. \\ & \qquad \left. +\eta\left(\frac{\check\sigma_s'+2}{4}\right)^{-24} + \eta\left(\frac{\check\sigma_s'+3}{4}\right)^{-24}\right\} e^{-i\pi\check\sigma_s' Q'^2/4} \\ & +\frac{1}{4}(-1)^{Q\cdot P+1}\, Q'\cdot P' \int_{iM_1'-1}^{iM_1'+1} d\check\rho_s' \left\{\eta\left(\frac{\check\rho_s'}{4}\right)^{-24} + \eta\left(\frac{\check\rho_s'+2}{4}\right)^{-24}\right\} e^{-i\pi\check\rho_s' P'^2/4} \\ & \quad \times \int_{iM_2'-1/2}^{iM_2'+1/2} d\check\sigma_s'\, \eta(\check\sigma_s')^{-24}\, e^{-i\pi\check\sigma_s' Q'^2/4} . \end{aligned} \tag{7.29}$$

Defining

$$(Q_1, P_1) = (Q-P, 0), \quad (Q_2, P_2) = (P, P)\,, \tag{7.30}$$

we can express (7.29) as

$$\Delta d(Q,P) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}(Q_1\cdot P_2 - Q_2\cdot P_1)\left\{d_h\left(Q_1,P_1\right) + d_h\left(\frac{1}{2}Q_1, \frac{1}{2}P_1\right)\right\} d_h\left(Q_2,P_2\right) . \tag{7.31}$$

The second term is extra compared to (4.7); it represents the effect of non-primitivity of the final state dyons.

Finally let us turn to the analysis of the black hole entropy. For this we need to identify the zeroes of $\check\Phi$ at $\check\rho\check\sigma - \check v^2 + \check v = 0$ and show that $\check\Phi$ has the behaviour given in (6.3.25) near this pole. This is easily done using (7.1) and the locations of the zeroes of Φ_{10} given in (7.8). One finds that the only term that has a zero at the desired location is the first term inside the first square bracket in (7.1). Furthermore this term is proportional to the dyon partition function $1/\Phi_{10}(\check\rho, \check\sigma, \check v)$ of the unit torsion states discussed in §6.1. Thus this term clearly will have the desired factorization property given in (6.3.25).

Our proposal for the dyon partition function can be easily generalized to the torsion 2, primitive Q, P and odd $Q^2/2$, $P^2/2$ dyons discussed at the end of §6.3. This requires changing the signs of appropriate terms in (7.1) so that the partition function is odd under $\check\rho \to \check\rho + \frac{1}{2}$ and also under $\check\sigma \to \check\sigma + \frac{1}{2}$. The result is

$$\begin{aligned}
\frac{1}{\check\Phi(\check\rho,\check\sigma,\check v)} &= \frac{1}{16}\Bigg[\frac{1}{\Phi_{10}(\check\rho,\check\sigma,\check v)} - \frac{1}{\Phi_{10}(\check\rho,\check\sigma+\frac{1}{2},\check v)} - \frac{1}{\Phi_{10}(\check\rho+\frac{1}{2},\check\sigma,\check v)} \\
&+\frac{1}{\Phi_{10}(\check\rho+\frac{1}{2},\check\sigma+\frac{1}{2},\check v)} + \frac{1}{\Phi_{10}(\check\rho+\frac{1}{4},\check\sigma+\frac{1}{4},\check v+\frac{1}{4})} \\
&-\frac{1}{\Phi_{10}(\check\rho+\frac{1}{4},\check\sigma+\frac{3}{4},\check v+\frac{1}{4})} - \frac{1}{\Phi_{10}(\check\rho+\frac{3}{4},\check\sigma+\frac{1}{4},\check v+\frac{1}{4})} \\
&+\frac{1}{\Phi_{10}(\check\rho+\frac{3}{4},\check\sigma+\frac{3}{4},\check v+\frac{1}{4})} + \frac{1}{\Phi_{10}(\check\rho+\frac{1}{2},\check\sigma+\frac{1}{2},\check v+\frac{1}{2})} \\
&-\frac{1}{\Phi_{10}(\check\rho+\frac{1}{2},\check\sigma,\check v+\frac{1}{2})} - \frac{1}{\Phi_{10}(\check\rho,\check\sigma+\frac{1}{2},\check v+\frac{1}{2})} + \frac{1}{\Phi_{10}(\check\rho,\check\sigma,\check v+\frac{1}{2})} \\
&+\frac{1}{\Phi_{10}(\check\rho+\frac{3}{4},\check\sigma+\frac{3}{4},\check v+\frac{3}{4})} - \frac{1}{\Phi_{10}(\check\rho+\frac{3}{4},\check\sigma+\frac{1}{4},\check v+\frac{3}{4})} \\
&-\frac{1}{\Phi_{10}(\check\rho+\frac{1}{4},\check\sigma+\frac{3}{4},\check v+\frac{3}{4})} + \frac{1}{\Phi_{10}(\check\rho+\frac{1}{4},\check\sigma+\frac{1}{4},\check v+\frac{3}{4})}\Bigg] \\
&+\Bigg[\frac{1}{\Phi_{10}(\check\rho+\check\sigma+2\check v,\check\rho+\check\sigma-2\check v,\check\sigma-\check\rho)} \\
&-\frac{1}{\Phi_{10}(\check\rho+\check\sigma+2\check v+\frac{1}{2},\check\rho+\check\sigma-2\check v+\frac{1}{2},\check\sigma-\check\rho+\frac{1}{2})}\Bigg]\,. \qquad (7.32)
\end{aligned}$$

This together with (7.1) exhausts all the dyons of torsion two with Q, P primitive since there are no dyons of this type with $Q^2/2$ even, $P^2/2$ odd or vice versa. To see this we note that since $(Q\pm P)$ are $2\times$ primitive vectors, $(Q\pm P)^2/2$ must be multiples of four. Taking the sum and difference we find that $(Q^2+P^2)/2$ and $Q\cdot P$ must be even.

Since (7.1) and (7.32) contains information about all the torsion two dyons with primitive (Q,P), the full partition function for such dyons in obtained by taking the sum of these two functions. This gives the result quoted in (1.23).

Given this result on torsion two dyons in string theory we can go to appropriate gauge theory limit to extract information about torsion two dyons in gauge theories as in [48, 42]. For simplicity we shall consider $SU(3)$ gauge theories. If we denote by α_1 and α_2 the two simple roots of $SU(3)$, then, since the metric L reduces to the negative of the Cartan metric of the gauge group, we have

$$\alpha_1^2 = -2, \qquad \alpha_2^2 = -2, \qquad \alpha_1\cdot\alpha_2 = 1\,. \qquad (7.33)$$

Let us now consider a dyon of charge vectors (Q, P) with

$$Q = \alpha_1 - \alpha_2, \qquad P = \alpha_1 + \alpha_2 \,. \tag{7.34}$$

This has torsion 2. Furthermore both Q and P are primitive. Thus this falls in the class of dyons analyzed in this section. In fact, since

$$\frac{Q^2}{2} = -3, \qquad \frac{P^2}{2} = -1, \qquad Q \cdot P = 0 \,, \tag{7.35}$$

the index of these quarter BPS dyons in gauge theory must be contained in (7.32). We shall first show that the 16 terms inside the first square bracket in (7.32) do not contribute to the index of the dyons described in (7.34) in any domain in the moduli space. For this we note that the index computed from these terms is identical to the index of dyons of torsion 1 with appropriate constraints on Q^2, P^2 and $Q\cdot P$. In absence of these constraints the index is known to reproduce the index of unit torsion gauge theory dyons correctly [48], – these are dyons of charge (α_1, α_2) or ones related to it by $SL(2, \mathbb{Z})$ S-duality transformation:

$$(Q, P) = (a\alpha_1 + b\alpha_2, c\alpha_1 + d\alpha_2), \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \,. \tag{7.36}$$

Such dyons will always have $Q^2P^2 - (Q \cdot P)^2 = 3$, and hence can never give a state of the form given in (7.34) which has $Q^2P^2 - (Q \cdot P)^2 = 12$. This in turn shows that the 16 terms inside the first square bracket in (7.32) can never give a non-vanishing contribution to the index of a gauge theory state with charge vector given in (7.34).

Thus the only possible contribution to the index of the dyons with charges $(\alpha_1 - \alpha_2, \alpha_1 + \alpha_2)$ can come from the two terms inside the second square bracket in (7.32). In fact when $Q^2/2$ and $P^2/2$ are odd then both terms give equal contribution; so we can just calculate the contribution from the first term and multiply it by a factor of 2. Equivalently we could use (1.23) where only the first term is present with a factor of 2. Defining

$$\check\rho' = \check\rho + \check\sigma + 2\check v, \quad \check\sigma' = \check\rho + \check\sigma - 2\check v, \quad \check v' = \check\sigma - \check\rho \,, \tag{7.37}$$

we can identify the relevant term in $1/\check\Phi(\check\rho, \check\sigma, \check v)$ as $2/\Phi_{10}(\check\rho', \check\sigma', \check v')$. As usual the contribution of this term to the index depends on the choice of the integration contour, which in turn is determined by domain in the moduli space in which we want to compute the index. Equivalently we can say that in different domains we need to use different Fourier series expansion of $1/\check\Phi$. Now the index of a charge vector of the type given in (7.35) will come from a term in the expansion of $1/\check\Phi$ of the form

$$e^{-2i\pi(\check\rho + 3\check\sigma)} \,. \tag{7.38}$$

Using (7.37) this takes the form

$$e^{-2i\pi(\check\rho'+\check\sigma'+\check v')}\,. \tag{7.39}$$

Thus in whichever domain the Fourier expansion of $1/\check\Phi$ contains a term of the form (7.39) we have a non-vanishing index for the dyon in (7.35) with the index being equal to $(-1)^{Q\cdot P+1}$ times the coefficient of this term. Since here $Q\cdot P=0$, the index is -2 times the coefficient of (7.39) in the Fourier expansion of $1/\Phi_{10}(\check\rho',\check\sigma',\check v')$. Now from the analysis of partition function of torsion one dyons (see *e.g.* [48]) we know that this expansion indeed has a term of the form (7.39) for one class of choices of contour; these are the contours for which

$$\Im(\check\rho'), \Im(\check\sigma') >> -\Im(\check v') >> 0\,. \tag{7.40}$$

Using (4.4), or equivalently by an $SL(2, \mathbb{R})$ transformation of the results of [48], one can figure out the domain in the moduli space in which this choice of contour is the correct one. It turns out to be the domain bounded by the walls associated with the decays of (Q,P) into

$$((Q-P)/2,(P-Q)/2)+((Q+P)/2,(Q+P)/2), \quad (-P,P)+(P+Q,0), \quad (Q-P,0)+(P,P)\,. \tag{7.41}$$

In this domain $1/\Phi_{10}$ has to be first expanded in powers of $e^{2\pi i\check\rho'}$ and $e^{2\pi i\check\sigma'}$ and then each coefficient needs to be expanded in powers of $e^{-2\pi i\check v'}$. (7.39) is the leading term in this expansion and its coefficient is 1. As a result the index of the dyons is -2. This agrees with the results of [31,32,33,34] where it was shown that in an appropriate domain in the moduli space dyons of torsion r has index $(-1)^{r-1}r$, since these dyonic states are obtained by tensoring the basic supermultiplet with a state of spin $(r-1)/2$.

Using a string junction picture [49,50] ref. [31] also showed that the dyon considered above exists in a domain of the moduli space bounded by three walls of marginal stability, – one asscociated with the decay into $(\alpha_1,\alpha_1)+(-\alpha_2,\alpha_2)$, the second associated with the decay into $(2\alpha_1,0)+(-\alpha_1-\alpha_2,\alpha_1+\alpha_2)$ and the third one associated with the decay into $(-2\alpha_2,0)+(\alpha_1+\alpha_2,\alpha_1+\alpha_2)$. These are precisely the walls listed in (7.41). Since the gauge theory dyons cease to exist outside these walls, the index computed in gauge theory jumps by 2 across these three walls of marginal stability. Does this agree with the prediction of the proposed dyon partition function in string theory? We can calculate the change in the index associated with these decays using the standard formula (4.7) for decay into primitive dyons [35] and the modified formula (7.31) for decay into non-primitive dyons since the proposed partition function satisfies these relations. We find a jump in the index equal to 2 across each of these walls as predicted by the gauge theory results.

8 Reverse applications

In our analysis so far we have used the information on half BPS partition function to extract information about quarter BPS partition function. However we can turn this around. If the quarter BPS partition function is known then we can use it to extract information about the half-BPS partition function by first identifying an appropriate wall on which one of the decay products is the half BPS state under consideration and then studying the behaviour of the quarter BPS partition function near the pole that controls the jump in the index at this particular wall.

As an example we can consider the $\mathbb{Z}_2$ CHL model of §6.4. The decay $(Q,P) \to (Q,0) + (0,P)$ is controlled by the behaviour of $\check{\Phi}$ near $\check{v} = 0$. Thus if we did not know the spectrum of magnetically charged half BPS states in this theory, we could study the behaviour of $\check{\Phi}$ near $\check{v} = 0$ to get this information. In this case since all the other walls are related to this one by S-duality transformation, this is the only independent information we can get. However for more complicated models there can be more information.

To illustrate this we shall consider the example of the $\mathbb{Z}_6$ CHL model [51, 52] mentioned in footnote 9. Our set $\mathcal{A}$ consists of charge vectors of the form

$$Q = \begin{pmatrix} 0 \\ m/6 \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} K \\ J \\ 1 \\ 0 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}\,, \tag{8.1}$$

as in (6.4.1). We now consider the decay associated with the matrix

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}. \tag{8.2}$$

From (4.1) we see that this corresponds to the decay

$$(Q,P) \to (M_0, 2M_0) + (N_0, 3N_0)\,, \quad M_0 \equiv 3Q - P, \quad N_0 \equiv -2Q + P\,. \tag{8.3}$$

The charge vectors M_0 and N_0 are not related to Q or P by a T-duality transformation since they correspond to charges that are triple and double twisted respectively. Furthermore the dyon charges $(M_0, 2M_0)$ and $(N_0, 3N_0)$ cannot be related by S-duality group $\Gamma_1(6)$ to either a purely electric or a purely magnetic state whose index is known. On the other hand the partition function of quarter BPS states of the type given in (8.3) is known [11, 18]. Thus the latter can be used to extract information about the partition function of these half BPS states.

From (4.2) it follows that the relevant zero of $\check{\Phi}$ we need to examine is at

$$6\check{\rho} + \check{\sigma} + 5\check{v} = 0\,. \tag{8.4}$$

The zeroes of $\check{\Phi}$ have been classified in [9,18]. For a generic $\mathbb{Z}_N$ model $\check{\Phi}$ has double zeroes at

$$\begin{aligned} & n_2(\check{\sigma}\check{\rho} - \check{v}^2) + j\check{v} + n_1\check{\sigma} - \check{\rho}m_1 + m_2 = 0 \\ & m_1 \in N\,\mathbb{Z}, \quad n_1, m_2, n_2 \in \mathbb{Z}, \quad j \in 2\,\mathbb{Z}+1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,. \end{aligned} \tag{8.5}$$

For the $N = 6$ model, taking

$$m_1 = -6, \quad n_1 = 1, \quad m_2 = n_2 = 0, \quad j = 5\,, \tag{8.6}$$

we see that $\check{\Phi}$ indeed has a zero at (8.4). Thus by examining the known expression for $\check{\Phi}$ near this zero we can determine the half-BPS partition functions of interest. This can be done in a straightforward manner following the general procedure described in [9, 18].

We note in passing that in an arbitrary $\mathbb{Z}_N$ model with the set $\mathcal{A}$ chosen as

$$Q = \begin{pmatrix} 0 \\ m/N \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} K \\ J \\ 1 \\ 0 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}\,, \tag{8.7}$$

the walls of marginal stability are controlled by matrices $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ subject to the conditions [13]

$$a_0 d_0 - b_0 c_0 = 1, \quad a_0, b_0, c_0, d_0 \in \mathbb{Z}, \quad c_0 d_0 \in N\,\mathbb{Z}\,. \tag{8.8}$$

According to our hypothesis this decay will be controlled by a double zero of $\check{\Phi}$ at

$$\check{\rho} c_0 d_0 + \check{\sigma} a_0 b_0 + \check{v}(a_0 d_0 + b_0 c_0) = 0\,. \tag{8.9}$$

This corresponds to the choice

$$m_1 = -c_0 d_0, \quad n_1 = a_0 b_0, \quad m_2 = n_2 = 0, \quad j = a_0 d_0 + b_0 c_0\,, \tag{8.10}$$

in eq.(8.5). We now see that the m_i's, n_i's and j given in (8.10) satisfies all the constraints mentioned in (8.5) as a consequence of (8.8). Thus our proposal that the decay associated with the matrix $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ is always controlled by the zero at (8.9) is at least consistent with the locations of the zeroes of $\check{\Phi}$ for $\mathbb{Z}_N$ orbifold models.

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