U-duality Invariant Dyon Spectrum in type II on T^6

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Abstract

We give a manifestly U-duality invariant formula for the degeneracy of 1/8 BPS dyons in type II string theory on T^6 for a U-duality invariant subset of charge vectors. Besides depending on the Cremmer-Julia invariant the degeneracy also depends on other discrete invariants of $E_{7(7)}(\mathbb{Z})$.

Type IIA string theory compactified on T^6 is known to have an $E_{7(7)}(\mathbb{Z})$ U-duality invariance [1]. Thus we would expect the spectrum of 1/8 BPS dyons in this theory to be invariant under this symmetry. *A priori* this would require us to make a U-duality transformation not only on the charges but also on the moduli. One can partially avoid this problem by using the index measured by the helicity trace B_{14} [2,3]. This is known to not change under continuous variation of the moduli. However it can in principle jump across walls of marginal stability on which the 1/8 BPS dyon may decay into a pair of half-BPS dyons. It was however shown in [4] that such walls of marginal stability are absent for states which have a positive value of the Cremmer-Julia invariant [5,6], – the unique quartic combination of the charges which is invariant under the continuous $E_{7(7)}$ transformation. Thus for such states one should be able to express the index B_{14} as a function of the charges alone, and furthermore this formula must be invariant under the discrete U-duality group $E_{7(7)}(\mathbb{Z})$ acting on the charges.

For a special class of charge vectors a formula for the index B_{14} has been derived in [7,8,4]. In principle we can use the U-duality invariance of the theory to extend the result to all other charge vectors which are U-dual to the ones analyzed earlier. Our goal in this paper is to express the formula for B_{14} as a manifestly U-duality invariant function of the charges. During this analysis we shall also find that the U-duality orbit of the charge vectors analyzed earlier do not cover the full set of allowed charges in the theory, and we shall be able to express this restriction in a manifestly U-duality invariant manner.

We begin by introducing some notations and conventions. Type IIA string theory compactified on T^6 has 28 U(1) gauge fields of which 12 arise in the NSNS sector and the rest arise in the RR sector. Thus a generic state is characterized by 28 electric and 28 magnetic charges which together transform in the **56** representation of the U-duality group $E_{7(7)}(\mathbb{Z})$ [1]. We shall denote this 56 dimensional charge vector by q_a ($1 \le a \le 56$). q_a's will be normalized so that $q_a \in \mathbb{Z}$ and transform into integer linear combinations of each other under $E_{7(7)}(\mathbb{Z})$. It is often useful to examine the transformation properties of the charges under a special subgroup of the U-duality group containing the T-duality group $SO(6,6;\mathbb{Z})$ and the electric-magnetic S-duality group $SL(2,\mathbb{Z})$. Under the $SL(2,\mathbb{Z})\times SO(6,6;\mathbb{Z})$ subgroup the charge vector transforms in the $(\mathbf{2},\mathbf{12})\oplus(\mathbf{1},\mathbf{32})$ representation. Of these the **(2,12)** component can be identified as the NSNS charges and the $(\mathbf{1},\mathbf{32})$ component can be identified as the RR charges. We shall denote the **(2,12)** part of q_a as $M_{i\alpha}$ with $1 \le i \le 12$, $1 \le \alpha \le 2$ and the $(\mathbf{1},\mathbf{32})$ component of q_a as N_s ($1 \le s \le 32$). We can identify the components M_{i1} as the electric charges Q_i in the NSNS sector and the components M_{i2} as the magnetic charges P_i in the NSNS sector.

Let us now note some important facts about $E_{7(7)}(\mathbb{Z})$ representations [9]:

1. If we have two vectors q_a and q'_a in the **56** representation then there is a unique anti-symmetric bilinear $\mathbb{L}_{ab} q_a q'_b$ which is a singlet of the continuous $E_{7(7)}$ group. We shall normalize $\mathbb{L}_{ab}$ such that the part of $\mathbb{L}_{ab} q_a q'_b$ involving the NSNS charges take the form $\epsilon_{\alpha\beta} L_{ij} M_{i\alpha} M'_{j\beta}$, where $\epsilon_{\alpha\beta}$ is a totally antisymmetric tensor with the $\epsilon_{12} = -\epsilon_{21} = 1$ and L_{ij} is the $O(6,6)$ invariant metric with 6 eigenvalues 1 and 6 eigenvalues -1.

2. The bilinear $q_a q_b$ can be decomposed into a **133** and a **1463** representation of $E_{7(7)}$. Let us focus on the **133** component of this bilinear. After suitable normalization one can ensure that all the 133 elements are integers and that they transform into integer linear combinations of each other under an $E_{7(7)}(\mathbb{Z})$ transformation. We denote by $\psi(q)$ the gcd of all these 133 elements:

$$\psi(q) \equiv \gcd(q \otimes q)_{\mathbf{133}} \,. \tag{1}$$

   Then $\psi(q)$ must remain invariant under an $E_{7(7)}(\mathbb{Z})$ transformation.

3. The trilinear $q_a q_b q_c$ decomposes into **56**, **6480** and **24320** representations of $E_{7(7)}$. Let us focus on the **56** part and denote these 56 components by $\widetilde{q}_a$:

$$\widetilde{q} \equiv (q \otimes q \otimes q)_{\mathbf{56}} \,. \tag{2}$$

   The normalization of $\widetilde{q}$ is fixed as follows. We can construct an $E_{7(7)}(\mathbb{Z})$ invariant $\Delta(q)$ as

$$\Delta(q) = \frac{1}{2}\, \mathbb{L}_{ab} \widetilde{q}_a q_b \,. \tag{3}$$

   $\Delta(q)$ is the well known Cremmer-Julia invariant. We shall normalize $\widetilde{q}_a$ such that $\Delta(q)$ has the following normalization

$$\Delta(q) = Q^2 P^2 - (Q \cdot P)^2 + \cdots \tag{4}$$

   where the inner products of Q and P are calculated using the $SO(6,6)$ invariant metric L_{ij} and $\cdots$ denotes terms involving RR charges. With this normalization of $\widetilde{q}_a$, we define

$$\chi(q) \equiv \gcd(q \wedge \widetilde{q}) = \gcd\{q_a \widetilde{q}_b - q_b \widetilde{q}_a\} \,. \tag{5}$$

   Since $\widetilde{q}_a$ transforms in the same way as q_a, the components of $\widetilde{q}_a$ transform into integer linear combinations of each other under $E_{7(7)}(\mathbb{Z})$. It follows from this that $\chi(q)$ is invariant under an $E_{7(7)}(\mathbb{Z})$ transformation.

We are now ready to state our result. First we shall state the restriction on the charges for which our result is valid. This restriction can be stated in two parts:

1. The charge vector q must be such that its **(1,32)** part can be removed by an $E_{7(7)}(\mathbb{Z})$ transformation. In other words it must be U-dual to a configuration where the D-brane charges vanish.[1]

2. We constrain our charge vector q such that

$$\psi(q) = 1\,. \tag{6}$$

Given a charge vector satisfying these two conditions, our result for the dyon spectrum can be expressed in terms of $\Delta(q)$ and $\chi(q)$ as follows. The index $d(q) \equiv (-1)^{\Delta(q)} B_{14} = (-1)^{Q\cdot P} B_{14}$ associated with the charge vector q is given by

$$d(q) = \sum_{s\in\mathbb{Z},\, 2s|\chi(q)} s\,\widehat{c}(\Delta(q)/s^2)\,, \tag{7}$$

where $\widehat{c}(u)$ is defined through the relations [10, 7]

$$-\,\vartheta_1(z|\tau)^2\,\eta(\tau)^{-6} \equiv \sum_{k,l} \widehat{c}(4k-l^2)\,e^{2\pi i(k\tau+lz)}\,. \tag{8}$$

$\vartheta_1(z|\tau)$ and $\eta(\tau)$ are respectively the odd Jacobi theta function and the Dedekind eta function. Since $\chi(q)$ and $\Delta(q)$ are $E_{7(7)}(\mathbb{Z})$ invariant, the above formula is manifestly $E_{7(7)}(\mathbb{Z})$ invariant.

We shall now give the proof of (7). Since we have restricted our charge vector q such that it can be rotated into purely NSNS charges, and since (7) is manifestly U-duality invariant, it is enough to prove (7) for charges belonging to the NSNS sector. In this case with the help of T-duality transformations we can bring the electric and magnetic components Q and P into a four dimensional subspace, represented by fundamental string winding and momentum along two circles S^1 and $\widehat{S}^1$ and Kaluza-Klein (KK) monopole charges and H-monopole charges along the same circles [11]. Using further S- and T-duality transformations we can bring the charges into the form [12, 13]

$$Q = \begin{pmatrix} 0 \\ n \\ 0 \\ 1 \end{pmatrix}, \quad P = \begin{pmatrix} Q_1 \\ J \\ Q_5 \\ 0 \end{pmatrix}, \quad n, Q_1, Q_5, J \in \mathbb{Z}, \quad Q_5 | J, Q_1\,. \tag{9}$$

[1] We do not know if this condition is trivial, i.e. whether it holds for all charge vectors for which 1/8 BPS dyon exists.

In this subspace the T-duality invariant metric L takes the form

$$L = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{10}$$

so that we have

$$Q^2 = 2n, \quad P^2 = 2Q_1Q_5, \quad Q \cdot P = J\,. \tag{11}$$

Also since $Q_5|Q_1, J$, the torsion associated with the pair (Q, P) is [14]

$$\gcd(Q \wedge P) = Q_5\,. \tag{12}$$

In a suitable duality frame and up to signs, we can interprete Q as representing n units of fundamental string winding charge and 1 unit of momentum along S^1 and P as representing Q_1 units of NS 5-brane wrapped on $T^4 \times S^1$, J units of KK monopole charge associated with S^1 and Q_5 units of KK monopole charge associated with $\widehat{S}^1$. Here T^4 denotes the four torus along directions other than those labelled by S^1 and $\widehat{S}^1$.

Let us now investigate the meaning of the constraint (6) on (Q, P). $\psi(q)$ is the gcd of the components of the **133** representation in the bilinear $q_a q_b$. Now under $SL(2,\mathbb{Z}) \times O(6,6;\mathbb{Z})$ the **133** representation of $E_{7(7)}(\mathbb{Z})$ decomposes into **(3,1)** +**(1,66)** + **(2,32′)**. If the original q_a's have vanishing components corresponding to the RR directions then the $(\mathbf{2}, \mathbf{32'})$ component vanishes and the **(3,1)** and **(1,66)** components correspond to, respectively

$$\{Q^2/2, P^2/2, Q \cdot P\} \quad \text{and} \quad \{Q_iP_j - Q_jP_i\}\,. \tag{13}$$

Using (11) and (12) we now see that for the charge vector (9) we have

$$\psi(q) = \gcd\{n, Q_1Q_5, J, Q_5\} = \gcd\{n, Q_5\}\,, \tag{14}$$

since Q_1 and J are divisible by Q_5. The condition (6) now gives

$$\gcd\{n, Q_5\} = 1\,. \tag{15}$$

Our goal is to prove (7) for the charge vector given in (9). A formula for the index B_{14} for unit torsion dyons has been written down in [7, 8, 4], but since the charge vector (9) has torsion Q_5, we cannot use the results of [7,8,4] directly. We shall now show however that by a suitable U-duality transformation we can map the charge vector (9) to another charge vector of unit torsion, and then use the result of [4] to compute $d(q)$. For this we recall that the

U-duality group of type IIA string theory compactified on T^4 contains a string-string duality transformation that exchanges the fundamental string with an NS 5-brane wrapped on T^4. Applying this duality transformation on the charge vector (9) exchanges the quantum number n representing fundamental string winding charge along S^1 with the quantum number Q_1 representing NS 5-brane winding charge along $T^4 \times S^1$. Thus the new charge vectors Q' and P' are of the form

$$Q' = \begin{pmatrix} 0 \\ Q_1 \\ 0 \\ 1 \end{pmatrix}, \quad P' = \begin{pmatrix} n \\ J \\ Q_5 \\ 0 \end{pmatrix}, \quad n, Q_1, Q_5, J \in \mathbb{Z}, \quad Q_5 | J, Q_1 \,. \tag{16}$$

For this we have

$$Q'^2 = 2Q_1, \quad P'^2 = 2nQ_5, \quad Q' \cdot P' = J \,, \tag{17}$$

and

$$\gcd\{Q' \wedge P'\} = \gcd(n, J, Q_5) = 1 \,, \tag{18}$$

using (15). Due to (18) the new charge vectors (Q', P') belong to the class for which the degeneracy of states was analyzed in [4]. The result for the degeneracy may be stated as

$$d'(q) = \sum_{s | Q'^2/2, P'^2/2, Q'\cdot P'} s\, \widehat{c}\left((Q'^2 P'^2 - (Q' \cdot P')^2)/s^2\right) = \sum_{s|Q_1, nQ_5, J} s\, \widehat{c}((4nQ_1Q_5 - J^2)/s^2) \,. \tag{19}$$

Our task now is to show that (7) reduces to $d'(q)$ given in (19) for the choice of charge vectors given in (9), or equivalently (16). For definiteness we shall work with the charge vector given in (9). First of all we note from (4) that here $\Delta(q) = 4nQ_1Q_5 - J^2$. Thus (7) takes the form

$$d(q) = \sum_{2s|\chi(q)} s\, \widehat{c}((4nQ_1Q_5 - J^2)/s^2) \,. \tag{20}$$

We shall now analyze the condition on s imposed by the requirement $2s|\chi(q)$. For this we first need to construct the vector $\widetilde{q}_a$ appearing in the expression (5) of $\chi(q)$. Now $\widetilde{q}_a$ is a trilinear in q_a transforming in the **56** representation of $E_{7(7)}(\mathbb{Z})$. Using the fact that **56** decomposes into a **(2,12)** and a **(1,32)** representation of $SL(2,\mathbb{Z}) \times O(6,6;\mathbb{Z})$ and that we have set the RR components of q to zero, we see that the only non-zero component of $\widetilde{q}_a$ must be the **(2,12)** components constructed from Q_i and P_i. Up to a normalization the unique trilinear combination of Q_i's and P_i's transforming in the **(2,12)** representation is

$$\begin{pmatrix} \widetilde{Q} \\ \widetilde{P} \end{pmatrix} = \begin{pmatrix} Q^2 P - (Q\cdot P) Q \\ -P^2 Q + (Q \cdot P) P \end{pmatrix} . \tag{21}$$

This is in fact also the correct normalization since with this choice $(\widetilde{Q}, \widetilde{P})$ satisfies the constraint given in (3), (4):

$$\frac{1}{2}\,\mathbb{L}_{ab}\widetilde{q}_a q_b = \frac{1}{2}\epsilon_{\alpha\beta}L_{ij}\widetilde{M}_{i\alpha}M_{j\beta} = \frac{1}{2}\left(\widetilde{Q}\cdot\ P - \widetilde{P}\cdot Q\right) = Q^2P^2 - (Q\cdot P)^2\,. \tag{22}$$

Using (9) and (21) we get

$$\widetilde{Q} = \begin{pmatrix} 2nQ_1 \\ nJ \\ 2nQ_5 \\ -J \end{pmatrix}, \qquad \widetilde{P} = \begin{pmatrix} JQ_1 \\ J^2 - 2Q_1Q_5 n \\ JQ_5 \\ -2Q_1Q_5 \end{pmatrix}, \tag{23}$$

and hence

$$\begin{aligned} \chi(q) &= \gcd\{q_a\widetilde{q}_b - \widetilde{q}_a q_b\} = \gcd\{Q_i\widetilde{Q}_j - Q_j\widetilde{Q}_i, P_i\widetilde{P}_j - P_j\widetilde{P}_i, Q_i\widetilde{P}_j - P_j\widetilde{Q}_i\} \\ &= \gcd\{2nQ_1, 2nJ, 2nQ_5, 2JQ_1, 2JQ_5, 2Q_1Q_5, 2J^2\}\,. \end{aligned} \tag{24}$$

In the last line of (24) we have dropped terms which contain one of the terms appearing inside { } as factors. We now use the fact that Q_5 divides J and Q_1. In this case $2Q_5$ is a factor of every term inside { } in the right hand side of (24) and we can write

$$\chi(q) = 2Q_5\ \gcd\{n, Q_1, J\}\,. \tag{25}$$

For configurations with $\psi(q) = 1$, (15) gives $\gcd\{n, Q_5\} = 1$. As a result we can drop factors of Q_5 from the Q_1 and J term inside { } in (25) and express this equation as

$$\chi(q) = 2Q_5\ \gcd\{n, Q_1/Q_5, J/Q_5\} = 2\gcd\{nQ_5, Q_1, J\}\,. \tag{26}$$

Using this we may express (20) as

$$d(q) = \sum_{s|nQ_5,Q_1,J} s\,\widehat{c}((4nQ_1Q_5 - J^2)/s^2)\,. \tag{27}$$

This in in precise agreement with (19), thereby proving (7).

It will be interesting to try to relax the constraint $\psi(q) = 1$ on the dyon charges. One could try to guess the answer following the approach of [15] or try to determine the spectrum directly by analyzing the D1-D5-KK monopole system along the line of [16].

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