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T-DUALITY OF p-BRANES

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Abstract

We investigate possible existence of duality symmetries which exchange the Kaluza-Klein modes with the wrapping modes of a BPS saturated p-brane on a torus. Assuming the validity of the conjectured U-duality symmetries of type II and heterotic string theories and M-theory, we show that for a BPS saturated p-brane there is an SL(2,Z) symmetry that mixes the Kaluza-Klein modes on a $(p+1)$ dimensional torus $T^{(p+1)}$ with the wrapping modes of the p-brane on $T^{(p+1)}$. The field that transforms as a modular parameter under this SL(2,Z) transformation has as its real part the component of the $(p+1)$-form gauge field on $T^{(p+1)}$, and as its imaginary part the volume of $T^{(p+1)}$, measured in the metric that couples naturally to the p-brane.

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Solitonic p-dimensional extended objects (p-branes)[1, 2] have played an important role in the recent developments in string theory. Many of the BPS states in string theory that are required to exist by various dualities[3], can be interpreted as wrapping modes of p-branes on p-cycles of internal manifold. Even though at present the world-volume theory of p-branes is ill understood for $p \geq 2$, p-branes have provided useful insight into the non-perturbative dynamics of string theories[4]-[13]. Recent discovery[14] that p-branes carrying Ramond-Ramond (RR) charges have exact conformal field theoretic description as Dirichlet branes has opened up new avenues for testing various duality conjectures in string theory, and at the same time, has provided new insight into the dynamics of p-branes[15]-[27].

One of the early attempts to use p-branes in the study of duality symmetries in string theory was made in ref.[28]. There it was observed that the S-duality symmetry of the heterotic string theory compactified on a six dimensional torus can be interpreted as the T-duality of the theory of five-branes[29, 30], in the sense that it exchanges the Kaluza-Klein modes with the winding modes of the five-brane on the 6-torus. Subsequently, attempt was made, following earlier work of Duff and Lu[31], to show that this is a symmetry of the 5-brane world-volume theory, but this program ran into difficulties[32, 33].[3] By now, however, there has been mounting evidence for S-duality of heteroric string theory compactified on a 6-torus. Thus we can now turn the argument around and say that the complete theory has a symmetry which acts as the T-duality transformation of the 5-brane, even though the 5-brane world-volume theory itself may not be invariant under this transformation. This result might, in fact, point to a reformulation of the five-brane world-volume theory where this T-duality symmetry will be manifest.

In this paper we shall use similar reasoning to study the possible existence of T-duality symmetry that exchanges the Kaluza-Klein modes with the wrapping modes of a p-brane for general BPS saturated p-branes. Our starting point will be the various conjectured U-duality symmetries in various string theories. We shall assume these U-duality conjectures to be true, and then try to re-interprete some of the U-duality transformations as T-duality transformations on various p-branes.

For fundamental strings, T-duality transformation that inverts the volume of the internal torus, exchanges the Kaluza-Klein modes with the winding modes of the string. This makes sense, since for compactification on a k-dimensional torus, both the winding number vector and the momentum vector are k-dimensional, with integer entries. However, for p-branes the situation is somewhat more complicated. The wrapping modes of a p-brane on a k-torus ($k \geq p$) is described by a $\binom{k}{p}$ dimensional vector, whereas the Kaluza-Klein modes are still described by a k-dimensional charge vector. Thus it does not make sense to talk about a symmetry that exchanges the two vectors. However, the dimensionality of the two vectors coincide for $k = p + 1$, and hence in this case such an exchange symmetry does make sense. This is the case that we shall focus on in this paper. Our result can be stated as follows:

For a BPS saturated p-brane, invariant under half of the global supersymmetry generators in a D-dimensional theory (obtained by compactification of string/M- theory) with at least 16 global supersymmetry generators,[4] *there appears an SL(2,Z) symmetry when we compactify the theory on a* $(p+1)$ *dimensional torus. Let* $\mathcal{A}_{M_1\cdots M_{p+1}}$ *denote the* $(p+1)$*-form field strength in D-dimensions* $(0 \leq M_i \leq D-1)$*and* $\mathcal{G}_{MN}$ *denote the metric in*

[3] Later S-duality of the heterotic string theory was realised as the T-duality of the dual type IIA string theory[34, 35].

[4] For theories with less number of supersymmetries even the T-duality transformations involving fundamental strings get modified by quantum corrections[36, 37].

D-dimensions, which couple naturally to the p-brane. If $(D-p-1), \ldots (D-1)$ denote the compact directions, and $\mathcal{G}_{\{(D-p-1)\ldots(D-1)\}}$ denotes the determinant of the $(p+1)\times(p+1)$ matrix $\mathcal{G}_{mn}$ $((D-p-1)\le m,n\le (D-1))$, then

$$\tau \equiv \mathcal{A}_{(D-p-1)\ldots(D-1)} + i\sqrt{\mathcal{G}_{\{(D-p-1)\ldots(D-1)\}}}\,, \tag{1}$$

transforms under the SL(2,Z) transformation as

$$\tau \to \frac{p\tau+q}{r\tau+s}\,, \tag{2}$$

where

$$ps-qr=1, \qquad\qquad p,q,r,s\in Z\,. \tag{3}$$

Let p_m denote the momenta in the internal directions, which couple to $\mathcal{G}_{m\mu}$, normalized so that p_m are integers. Also, let w_m denote the winding number of the p-brane on p-cycles of the internal torus $T^{(p+1)}$ which couple to $\epsilon^{mm_1\ldots m_p}\mathcal{A}_{m_1\ldots m_p\mu}$, normalized so that w_m are integers. Then the SL(2,Z) transformation acts on these charge vectors as

$$\begin{pmatrix} p_m \\ w_m \end{pmatrix} \to \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} p_m \\ w_m \end{pmatrix}\,. \tag{4}$$

The $(D-p-1)$ dimensional canonical Einstein metric remains invariant under this duality transformation.

We shall now verify this result for diverse p-branes appearing in type II and heterotic string theories, as well as in the M-theory. Note, however, that not all of these results are independent, since many of the p-branes that we shall be analyzing can be reinterpreted as wrapping modes of higher branes on internal tori.

1. Elementary strings in type II / heterotic string theory on T^2:

In this case the SL(2,Z) transformation described above coincides with the usual T-duality transformation in string theory.

2. Neveu-Schwarz (NS) 5-brane in Heterotic / Type IIA theory compactified on T^6:

This case has been discussed in ref.[28] and the resulting SL(2,Z) coincides with the S-duality transformation in the four dimensional string theory. We shall work out this case here in some detail.

Let us denote by $G^{(NS5)}$ the metric that couples to the Neveu-Schwarz five-brane, and by $G^{(ST)}$ the metric that couples to the fundamental string. Also. let Φ be the ten dimensional dilaton. Then[38]

$$G^{(NS5)}_{MN} = e^{-\frac{2\Phi}{3}} G^{(ST)}_{MN}\,. \tag{5}$$

(We are choosing a normalization convention where the string theory effective action involving NS sector fields is multiplied by a factor of $e^{-2\Phi}$ when written in terms of the string metric.) One can check the correctness of the above equation by noting that for $G^{(ST)}_{MN}=\eta_{MN}$, the 5-brane world-volume is proportional to $e^{-2\Phi}\sim 1/g_{st}^2$; as expected for a soliton carrying magnetic charge under an NS sector gauge field. In this case the 6-form field $\mathcal{A}$ can be identified to the field $\widetilde{B}$ that is dual to the two form field B_{MN} (in the sense that $d\widetilde{B}\sim\ ^*(dB)$, where * denotes Hodge dual). Thus

$$\begin{aligned}\tau &= \widetilde{B}_{456789} + i\sqrt{G^{(NS5)}_{\{456789\}}} \\ &= \widetilde{B}_{456789} + ie^{-2\Phi}\sqrt{G^{(ST)}_{\{456789\}}}\,. \end{aligned} \tag{6}$$

Thus $Re(\tau)$ can be identified with the four dimensional axion field, and $Im(\tau)$ can be identified to $\exp(-2\Phi^{(4)})$, where $\Phi^{(4)}$ is the four dimensional dilaton field. This shows that τ is precisely the S field of the four dimensional theory. In other words, the T-duality of the five brane can be identified to the S-duality of the four dimensional theory. As shown in ref.[28], the transformation laws (4) of p_m and w_m also coincide with the corresponding transformation laws of the charges under S-duality transformation.

3. Supermembrane in 11 dimensional supergravity compactified on T^3:

In this case the field $\mathcal{A}$ corresponds to the three form gauge potential C_{MNP} of the 11 dimensional supergravity, and the metric $\mathcal{G}$ corresponds to the metric $G^{(SG)}$ of the supergravity theory. Thus

$$\tau = C_{89(10)} + i\sqrt{G^{(SG)}_{\{89(10)\}}}\,. \tag{7}$$

We can regard this as a type IIA theory in (9+1) dimensions, spanned by the coordinates $x^0, \ldots x^9$, compactified on a two dimensional torus spanned by the coordinates x^8 and x^9. The relationship between the fields in the supergravity theory and the type II theory are given by[35]

$$C^{(SG)}_{89(10)} = B_{89}\,, \tag{8}$$

and

$$\begin{aligned} G^{(SG)}_{(10)(10)} &= e^{\frac{4\Phi}{3}}\,, \\ G^{(SG)}_{MN} &= e^{-\frac{2\Phi}{3}} G^{(ST)}_{MN} \qquad \text{for} \qquad 0 \le M, N \le 9\,. \end{aligned} \tag{9}$$

Thus,

$$\tau = B_{89} + i\sqrt{G^{(ST)}_{\{89\}}}\,. \tag{10}$$

This shows that τ is the usual modular parameter associated with the SL(2,Z) T-duality transformation of type IIA compactified on a two dimensional torus.

One can also verify that the transformation of the charges under the T-duality transformation of the 11-dimensional membrane coincides with their transformation under the usual T-duality transformation of the IIA theory compactified on T^2. To see this note that from the point of view of the IIA theory, p_8 and p_9 denote the Kaluza-Klein momenta along 8 and 9 directions respectively, whereas p_{10} denotes the $A^{(1)}_\mu$ charge, where we denote by $A^{(p)}$ the p-form gauge field arising in the RR sector of the type II theory. (p is even for IIB and odd for IIA.) On the other hand, w_9 and $-w_8$ denote the winding number along 8 and 9 directions respectively, and w_{10} denotes the $A^{(3)}_{89\mu}$ charge. Under the SL(2,Z) T-duality transformation of the membrane, $\vec{p}$ mixes with $\vec{w}$. This is precisely the way the SL(2,Z) T-duality transformation of the fundamental string acts on these charges. Thus we see that in this case the T-duality transformation associated with the supermembrane in the 11 dimensional supergravity compactified on T^3 can be identified with the usual T-duality transformation of the type IIA theory compactified on T^2.

4. 5-brane in 11-dimensional supergravity compactified on T^6:

In this case $\mathcal{A}$ can be identified with the 6-form gauge potential $\widetilde{C}$ in the 11-dimensional theory that is dual to C, in the sense that $d\widetilde{C} \sim {}^*(dC)$. $\mathcal{G}$ is still the metric $G^{(SG)}$ of the 11-dimensional supergravity. Thus

$$\tau = \widetilde{C}_{56789(10)} + i\sqrt{G^{(SG)}_{\{56789(10)\}}}\,. \tag{11}$$

We can again regard this as the type IIA theory compactified on T^5. We now have

$$\widetilde{C}_{56789(10)} = A^{(5)}_{56789}\,. \tag{12}$$

Using eqs.(9), (11) and (12) we get

$$\tau = A^{(5)}_{56789} + ie^{-\Phi}\sqrt{G^{(ST)}_{\{56789\}}}\,. \tag{13}$$

We want to show that the SL(2,Z) transformation acting on this modular parameter can indeed be identified with one of the known duality transformations in string theory. To do this let us make a string theoretic T-duality transformation that inverts the radii of all the circles labelled by coordinates $x^5, \ldots x^9$. This has the following effects:

1. It converts type IIA theory to type IIB theory.

2. It converts $A^{(5)}_{56789}$ to $A^{(0)}$.

3. It converts $e^{-\Phi}\sqrt{G^{(ST)}_{\{56789\}}}$ to $e^{-\Phi}$.

Thus this transformation converts τ to

$$A^{(0)} + ie^{-\Phi}\,. \tag{14}$$

The SL(2,Z) transformation acting on the above modular parameter is clearly the S-duality transformation of the 10-dimensional type IIB theory, which has been conjectured to be an exact symmetry of this theory. The transformation of the various charges can also be shown to work out correctly. Thus we see that the SL(2,Z) T-duality symmetry of the 5-brane in 11-dimensional supergravity compactified on T^6 follows as a consequence of the already conjectured dualities involving string theory and supergravity theory.

5. RR p-brane in 10-dimensional type II theory compactified on $T^{(p+1)}$:

Type II theory contains p-branes carrying charge under the RR $(p+1)$-form field $A^{(p+1)}_{M_1\ldots M_{p+1}}$. p is even for the type IIA theory and odd for the type IIB theory. Here $\mathcal{G}$ corresponds to the metric $G^{(RRp)}$ that couples to the RR p-brane, and $\mathcal{A}$ corresponds to the RR gauge potential $A^{(p+1)}$. The relationship between $G^{(RRp)}$ and the string metric $G^{(ST)}$ is given by

$$G^{(RRp)}_{MN} = e^{-\frac{2\Phi}{p+1}} G^{(ST)}_{MN}\,. \tag{15}$$

The correctness of this equation can be checked by noting that for $G^{(ST)}_{MN} = \eta_{MN}$, the world-volume of the RR p-brane is proportional to $e^{-\Phi}$. This is in agreement with the fact that solitons carrying RR charge have their world-volume action proportional to g_{st}^{-1}. Thus now

$$\begin{aligned}\tau &= A^{(p+1)}_{(9-p)\ldots 9} + i\sqrt{G^{(RRp)}_{\{(9-p)\ldots 9\}}}\,, \\ &= A^{(p+1)}_{(9-p)\ldots 9} + ie^{-\Phi}\sqrt{G^{(ST)}_{\{(9-p)\ldots 9\}}}\,. \end{aligned} \tag{16}$$

Again in order to identify the SL(2,Z) transformation on this modular parameter with one of the known duality transformations, let us make a string T-duality transformation that inverts the radii of all the circles labelled by $x^{(9-p)}, \ldots x^9$. This gives a final theory that is always IIB, and transforms τ to

$$A^{(0)} + ie^{-\Phi}\,. \tag{17}$$

Thus the SL(2,Z) transformation on this modular parameter can again be identified to the S-duality of the 10-dimensional type IIB theory. The transformation on the charges can be shown to work out as before.

Note that formally, for $p=-1$, the SL(2,Z) T-duality transformation can be directly identified with the S-duality of the 10-dimensional type IIB theory. Thus in a formal sense, the S-duality of the type IIB theory can be interpreted as the SL(2,Z) T-duality associated with the BPS saturated -1 branes!

6. 0-branes in 9-dimensional heterotic / type II theory compactified on S^1:

When we compactify the 10-dimensional heterotic or type II theory on a circle, we get some BPS states which cannot be regarded as the wrapping modes of a higher dimensional brane. These are the Kaluza-Klein modes and couple to the gauge field components

$$\mathcal{A}_M \equiv G^{(ST)}_{9M}/G^{(ST)}_{99} \qquad \text{for} \qquad 0 \le M \le 8\,. \tag{18}$$

According to the general result that we have stated about T-duality of p-branes, we should get an SL(2,Z) duality symmetry when we further compactify the theory on a circle, which will exchange the BPS states carrying $\mathcal{A}_M$ charges with the states carrying momenta along the new compact direction. From the way it acts on the charges, it is clear that this SL(2,Z) is nothing but the SL(2,Z) representing the global diffeomorphism group of the 2-torus. We shall now see that the modular parameter, calculated according to the prescription given before, works out correctly. For this we note that the metric $\mathcal{G}_{MN}$ that couples naturally to the Kaluza-Klein modes is given by,

$$\mathcal{G}_{MN} = (G^{(ST)}_{99})^{-1}(G^{(ST)}_{MN} - G^{(ST)}_{M9}(G^{(ST)}_{99})^{-1}G^{(ST)}_{N9}), \qquad 0 \le M, N \le 8\,. \tag{19}$$

The overall normalization of the metric is determined by the requirement that the world-line action of the Kaluza-Klein mode for $G^{(ST)}_{MN} = \eta_{MN}$ $(0 \le M, N \le 8)$ must be proportional to its mass $1/R_9 \sim (G^{(ST)}_{99})^{-1/2}$. Thus

$$\tau = \mathcal{A}_8 + i\sqrt{\mathcal{G}_{88}} = \frac{G^{(ST)}_{98}}{G^{(ST)}_{99}} + i\frac{\sqrt{G^{(ST)}_{88}G^{(ST)}_{99} - (G^{(ST)}_{98})^2}}{G^{(ST)}_{99}}\,. \tag{20}$$

This is precisely the modular parameter of the two torus labelled by the coordinates x^8 and x^9. This shows that the SL(2,Z) associated with the diffeomorphism group of the 2-torus also can be interpreted as the T-duality associated with the 0-branes.

To summarize, we have shown through various examples that for a BPS saturated p-brane invariant under half of the global supersymmetry generators in a theory with at least 16 global supersymmetry generators, we can define a T-duality transformation that exchanges the wrapping modes of a p-brane on a $p+1$ dimensional torus and the Kaluza-Klein modes. This generalises the usual T-duality transformation in string theory that exchanges the string winding modes with the Kaluza-Klein modes, and points to a kind of democracy between all p-branes[4], whether elementary or composite.

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