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MAGNETIC MONOPOLES, BOGOMOL'NYI BOUND AND SL(2,Z) INVARIANCE IN STRING THEORY

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ABSTRACT

We show that in heterotic string theory compactified on a six dimensional torus, the lower bound (Bogomol'nyi bound) on the dyon mass is invariant under the SL(2,Z) transformation that interchanges strong and weak coupling limits of the theory. Elementary string excitations are also shown to satisfy this lower bound. Finally, we identify specific monopole solutions that are related via the strong-weak coupling duality transformation to some of the elementary particles saturating the Bogomol'nyi bound, and these monopoles are shown to have the same mass and degeneracy of states as the corresponding elementary particles.

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Introduction

Following earlier ideas [1 − 10] we have proposed recently [11] that heterotic string theory compactified on a six dimensional torus may have an SL(2,Z) symmetry that exchanges electric and magnetic fields, and also the strong and weak coupling limits of the string theory. Existence of this symmetry demands that the theory must necessarily contain magnetically charged particles. Allowed values of electric and magnetic charges in this theory that are consistent with Dirac quantization condition were found, and the set of these allowed values was shown to be invariant under SL(2,Z) transformation [12]. This, however, does not establish that states whose quantum numbers are related by SL(2,Z) transformation have identical masses, − a necessary condition for SL(2,Z) invariance of the theory. This is the problem that we try to address in this paper.

Elementary string excitations carry only electric charge, and their masses are well known in the weak coupling limit of the theory. SL(2,Z) transform of these states carry both electric and magnetic charges in general, and must arise as soliton solutions in this theory. Thus in order to establish the SL(2,Z) invariance of the mass spectrum, we must compare the elementary particle masses at weak coupling to the soliton masses at strong coupling. In a generic theory, calculating soliton masses at strong coupling would have been an impossible task; however, since the theory under consideration has $N = 4$ supersymmetry, one can derive some results about the soliton masses in this theory that are not expected to receive any quantum corrections [13]. In particular, for a soliton carrying a given amount of electric and magnetic charges, one can derive a lower bound (known as the Bogomol'nyi bound) for the mass of the soliton. The bound is saturated for supersymmetric solitons, and the masses of such solitons are expected not to receive any quantum corrections. Thus one can compare these exact mass formulae as well as the lower bound on the soliton masses with the masses of the elementary string excitations and ask if they agree with the postulate of SL(2,Z) invariance of the theory. Although this would not prove that SL(2,Z) is a symmetry of the theory, this would provide a stringent test of this symmetry.

In this paper we show first that the Bogomol'nyi bound is invariant under SL(2,Z) transformation, and second, that the masses of the elementary string excitations also satisfy the Bogomol'nyi bound, with a subset of them saturating the bound. This implies that the elementary string excitations saturating the Bogomol'nyi bound, and the supersymmetric solitons whose quantum numbers are related to those of these elementary particles by SL(2,Z) transformation, have the same mass. We also identify the specific soliton solutions that are related by an SL(2,Z) transformation to some of the elementary string excitations saturating the Bogomol'nyi bound.

Some other aspects of SL(2,Z) invariance have been discussed in ref.[14].

Review

The low energy effective action describing ten dimensional heterotic string theory is given by

$$
\begin{aligned}
S = & \frac{1}{32\pi}\int d^{10}x\sqrt{-\det G_S^{(10)}}\, e^{-\Phi^{(10)}}\Big(R_S^{(10)} + G_S^{(10)MN}\partial_M\Phi^{(10)}\partial_N\Phi^{(10)} \\
& -\frac{1}{12}G_S^{(10)MM'}G_S^{(10)NN'}G_S^{(10)TT'}H^{(10)}_{MNT}H^{(10)}_{M'N'T'} - \frac{1}{8}G_S^{(10)MM'}G_S^{(10)NN'}F^{(10)I}_{MN}F^{(10)I}_{M'N'}\Big)
\end{aligned}
\tag{1}
$$

where

$$
F^{(10)I}_{MN} = \partial_M A^{(10)I}_N - \partial_N A^{(10)I}_M \tag{2}
$$

and

$$
H^{(10)}_{MNT} = \Big(\partial_M B^{(10)}_{NT} - \frac{1}{4}A^{(10)I}_M F^{(10)I}_{NT} + \text{ cyclic permutations of } M,N,T\Big) \tag{3}
$$

Here $\Phi^{(10)}$ is the dilaton field, $G^{(10)}_{SMN}$ denote ten dimensional σ-model metric, $B^{(10)}_{MN}$ denote the rank two antisymmetric tensor field, and $A^{(10)I}_M$ denote 16 $U(1)$ gauge fields. The superscript $^{(10)}$ indicates that we are dealing with ten dimensional fields, the indices M,N,T are ten dimensional Lorentz indices and run from 0 to 9, and the indices I denote 16 dimensional gauge indices and run from 1 to 16. Note that we have included only the abelian gauge fields in the effective action. For a generic toroidal compactification to four dimensions, all the non-abelian symmetry is spontaneously broken, and only the $U(1)$ gauge fields remain massless [15] [16].

We now compactify the theory on a 6 dimensional torus. Let us denote by m, n $(1 \le m, n \le 6)$ the six internal directions, and by μ, ν $(\mu, \nu = 0, 7, 8, 9)$ the four uncompactified directions. In terms of the ten dimensional fields, we define the four dimensional fields as follows:*

$$
\begin{aligned}
&\hat{G}_{mn} = G^{(10)}_{Smn}, \quad \hat{B}_{mn} = B^{(10)}_{mn}, \quad \hat{A}^I_m = A^{(10)I}_m, \quad \Phi = \Phi^{(10)} - \frac{1}{2}\ln\det\hat{G} \\
&A^m_\mu = \frac{1}{2}\hat{G}^{mn}G^{(10)}_{Sn\mu}, \quad A^{I+12}_\mu = -\frac{1}{2\sqrt{2}}A^{(10)I}_\mu + \frac{1}{\sqrt{2}}\hat{A}^I_m A^m_\mu, \\
&A^{m+6}_\mu = \frac{1}{2}B^{(10)}_{m\mu} - \hat{B}_{mn}A^n_\mu + \frac{1}{2\sqrt{2}}\hat{A}^I_m A^{I+12}_\mu \\
&G_{S\mu\nu} = G^{(10)}_{S\mu\nu} - G^{(10)}_{Sm\mu}G^{(10)}_{Sn\nu}\hat{G}^{mn}, \quad B_{\mu\nu} = B^{(10)}_{\mu\nu} - 4B^{(10)}_{mn}A^m_\mu A^n_\nu - 2(A^m_\mu A^{m+6}_\nu - \mu \leftrightarrow \nu) \\
&\qquad 1 \le m, n \le 6, \quad 1 \le I \le 16
\end{aligned}
\tag{4}
$$

where $\hat{G}^{mn}$ denotes the inverse matrix of $\hat{G}_{mn}$. The field strengths associated with the four dimensional gauge fields and the anti-symmetric tensor field are defined as

$$
F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu, \qquad 1 \le \alpha \le 28 \tag{5}
$$

and

$$
H_{\mu\nu\rho} = \Big(\partial_\mu B_{\nu\rho} + 2A^\alpha_\mu L_{\alpha\beta}F^\beta_{\nu\rho} + \text{ cyclic permutations of } \mu, \nu, \rho\Big) \tag{6}
$$

where L denotes the 28×28 matrix,

$$
L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix} \tag{7}
$$

The Einstein metric in 4 dimensions is obtained from the metric $G_{S\mu\nu}$ through the

* In writing down these relations, we have made a change of normalization from the one used in ref.[11] to the one used in ref.[12].

rescaling,

$$G_{\mu\nu} = e^{-\Phi} G_{S\mu\nu} \tag{8}$$

From now on, we shall choose the convention that all four dimensional indices will be raised and lowered with the Einstein metric. With this convention, we define the dual field strength

$$\tilde{F}^{\alpha\mu\nu} = \frac{1}{2}(\sqrt{-\det G})^{-1} \epsilon^{\mu\nu\rho\sigma} F^{\alpha}_{\rho\sigma} \tag{9}$$

The equations of motion of the anti-symmetric tensor field allow us to define a scalar field Ψ through the equation:

$$H^{\mu\nu\rho} = -(\sqrt{-\det G})^{-1} e^{2\Phi} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \Psi \tag{10}$$

We can combine the fields Φ and Ψ into a single complex scalar field λ:

$$\lambda = \Psi + i e^{-\Phi} \equiv \lambda_1 + i\lambda_2 \tag{11}$$

Finally, all information about the scalar fields $\hat{G}_{mn}$, $\hat{B}_{mn}$ and $\hat{A}^I_m$ may be included in a single 28 dimensional matrix M satisfying,

$$M^T = M, \quad M^T L M = L \tag{12}$$

M is defined as,

$$M = \begin{pmatrix} P & Q & R \\ Q^T & S & U \\ R^T & U^T & V \end{pmatrix} \tag{13}$$

where,

$$\begin{aligned} P^{mn} =& \hat{G}^{mn}, \quad Q^m_{\ n} = \hat{G}^{mp}(\hat{B}_{pn} + \frac{1}{4}\hat{A}^I_p \hat{A}^I_n), \quad R^{mI} = \frac{1}{\sqrt{2}} \hat{G}^{mp} \hat{A}^I_p, \\ S_{mn} =& (\hat{G}_{mp} - \hat{B}_{mp} + \frac{1}{4}\hat{A}^I_m \hat{A}^I_p) \hat{G}^{pq} (\hat{G}_{qn} + \hat{B}_{qn} + \frac{1}{4}\hat{A}^J_q \hat{A}^J_n) \\ U_m^{\ I} =& \frac{1}{\sqrt{2}} (\hat{G}_{mp} - \hat{B}_{mp} + \frac{1}{4}\hat{A}^J_m \hat{A}^J_p) \hat{G}^{pq} \hat{A}^I_q, \quad V^{IJ} = \delta^{IJ} + \frac{1}{2}\hat{A}^I_p \hat{G}^{pq} \hat{A}^J_q \end{aligned} \tag{14}$$

and ^{T} denotes the transpose of a matrix. The equations of motion derived from the action (1) can be shown to be equivalent to those derived from the action [17] [12]

$$\begin{aligned} S =& \frac{1}{32\pi}\int d^4x\sqrt{-\det G}\Big(R - \frac{1}{2(\lambda_2)^2}G^{\mu\nu}\partial_\mu\lambda\partial_\nu\bar{\lambda} - \lambda_2\vec{F}^T_{\mu\nu}.LML.\vec{F}^{\mu\nu} \\ &+ \lambda_1\vec{F}^T_{\mu\nu}.L.\tilde{\vec{F}}^{\mu\nu} + \frac{1}{8}G^{\mu\nu}Tr(\partial_\mu ML\partial_\nu ML)\Big) \end{aligned} \tag{15}$$

where we have used vector notation to denote the 28 dimensional vector $F^\alpha_{\mu\nu}$. The equations of motion derived from the above action may be shown to be invariant under the SL(2,Z) transformation [10] [11]

$$\begin{aligned} &\lambda \to \frac{a\lambda+b}{c\lambda+d}, \quad G_{\mu\nu}\to G_{\mu\nu}, \quad M\to M, \quad \vec{F}_{\mu\nu}\to(c\lambda_1+d)\vec{F}_{\mu\nu}+c\lambda_2 ML.\tilde{\vec{F}}_{\mu\nu} \\ &\qquad a,b,c,d\in Z, \quad ad-bc=1 \end{aligned} \tag{16}$$

The electric and magnetic charges of a particle are defined in terms of the asymptotic values of the electric and magnetic fields as follows

$$\vec{F}_{0r}\simeq\frac{\vec{q}_{el}}{r^2}, \quad \tilde{\vec{F}}_{0r}\simeq\frac{\vec{q}_{mag}}{r^2} \tag{17}$$

where $r=\sqrt{(x^7)^2+(x^8)^2+(x^9)^2}$. The allowed spectrum of $(\vec{q}_{el},\vec{q}_{mag})$ in toroidally compactified heterotic string theory was calculated in ref.[12], and was found to be of the form

$$\vec{q}_{mag}=M^{(0)}L\vec{\beta}, \quad \vec{q}_{el}=\frac{1}{\lambda_2^{(0)}}(\vec{\alpha}+\lambda_1^{(0)}\vec{\beta}) \tag{18}$$

where $\lambda_1^{(0)}$, $\lambda_2^{(0)}$ and $M^{(0)}$ denote the asymptotic values of λ_1, λ_2 and M respectively, and $\vec{\alpha}$ and $\vec{\beta}$ are arbitrary vectors belonging to a lattice P which is even and self-dual with respect to the metric L.

Bogomol'nyi Bound and its SL(2,Z) Invariance

An explicit formula for the Bogomol'nyi bound on the mass of a dyon for toroidally compactified heterotic string theory was derived in ref.[9].⋆ We shall

⋆ Lower bound to magnetically charged black hole mass in supersymmetric theories was derived in ref.[18]. Invariance of the Bogomol'nyi bound for the mass of dyonic black holes under SL(2,R) transformation was shown in ref.[19].

first write down this formula, then reexpress it in terms of the charges $\vec{q}_{el}$, $\vec{q}_{mag}$ defined through eqs.(17), and finally show that it is invariant under the SL(2,Z) transformation of $\vec{q}_{el}$, $\vec{q}_{mag}$ induced from eq.(16).

Let us define,

$$T_{(m)\mu\nu} = \partial_\mu G^{(10)}_{m\nu} - \partial_\nu G^{(10)}_{m\mu} - H^{(10)}_{m\mu\nu} \tag{19}$$

We define $\tilde{T}_{(m)\mu\nu}$ through an equation analogous to eq.(9) and then define the charges Q_m, P_m through the asymptotic values of these fields as follows:

$$T_{(m)0r} \simeq \frac{Q_m}{r^2}, \quad \tilde{T}_{(m)0r} \simeq \frac{P_m}{r^2} \tag{20}$$

In terms of these charges, the Bogomol'nyi bound on the dyon mass may be expressed as [9]

$$m^2 \geq \frac{1}{64}\lambda_2^{(0)}(\hat{G}^{(0)mn}Q_mQ_n + \hat{G}^{(0)mn}P_mP_n) \equiv (m_0)^2 \tag{21}$$

provided $G_{\mu\nu}$ approaches $\eta_{\mu\nu}$ asymptotically. Here the superscript $^{(0)}$ denotes asymptotic values of various fields. Using eqs.(4), and the definition of Q_m, P_m given in eqs.(19), (20), we get,

$$\begin{aligned} Q_m =& 2q_{el}^{m+6} + 2(\hat{G}^{(0)}_{mn} + \hat{B}^{(0)}_{mn} + \frac{1}{4}\hat{A}^{(0)I}_m\hat{A}^{(0)I}_n)q^n_{el} - \sqrt{2}\hat{A}^{(0)I}_m q_{el}^{I+12} \\ P_m =& 2q_{mag}^{m+6} + 2(\hat{G}^{(0)}_{mn} + \hat{B}^{(0)}_{mn} + \frac{1}{4}\hat{A}^{(0)I}_m\hat{A}^{(0)I}_n)q^n_{mag} - \sqrt{2}\hat{A}^{(0)I}_m q_{mag}^{I+12} \end{aligned} \tag{22}$$

Substituting this in eq.(21) we get the following expression for the Bogomol'nyi bound m_0:

$$(m_0)^2 = \frac{\lambda_2^{(0)}}{16}\{\vec{q}^{\,T}_{el}.(LM^{(0)}L + L).\vec{q}_{el} + \vec{q}^{\,T}_{mag}.(LM^{(0)}L + L).\vec{q}_{mag}\} \tag{23}$$

Before testing SL(2,Z) invariance of m_0, let us note that for states carrying q^α_{el},

q^{α}_{mag} charge 0 for $1 \leq \alpha \leq 12$, we get,

$$(m_0)^2 = \frac{\lambda_2^{(0)}}{32}\{(\hat{A}_p^{(0)I} q_{el}^{I+12})\hat{G}^{(0)pq}(\hat{A}_q^{(0)J} q_{el}^{J+12}) + (\hat{A}_p^{(0)I} q_{mag}^{I+12})\hat{G}^{(0)pq}(\hat{A}_q^{(0)J} q_{mag}^{J+12})\} \tag{24}$$

This is precisely Osborn's formula [2] for the Bogomol'nyi bound on the monopole mass for a global $N = 4$ supersymmetric Yang-Mills theory. The fields $\hat{A}^I_m$ should be interpreted as Higgs fields in this case.†

Let us now study the SL(2,Z) transformation law of $(m_0)^2$. Using eqs.(16), (17) we see that under SL(2,Z) transformation,

$$\begin{aligned} \lambda_2^{(0)} &\to \frac{\lambda_2^{(0)}}{|c\lambda^{(0)} + d|^2} \\ \vec{q}_{el} &\to (c\lambda_1^{(0)} + d)\vec{q}_{el} + c\lambda_2^{(0)} M^{(0)} L.\vec{q}_{mag} \\ \vec{q}_{mag} &\to (c\lambda_1^{(0)} + d)\vec{q}_{mag} - c\lambda_2^{(0)} M^{(0)} L.\vec{q}_{el} \end{aligned} \tag{25}$$

Using eqs.(12) and (23) we get

$$m_0 \to m_0 \tag{26}$$

under the transformation (25). The above result implies that if we find two states whose quantum numbers are related by SL(2,Z) transformation, and if both of these states saturate the Bogomol'nyi bound, then their masses are automatically identical.

† All comparisons are made in the gauge where the asymptotic Higgs field is directed along a fixed direction in the gauge group, except along a Dirac string singularity.

Where do Known Monopole Solutions Fit in?

Eq.(18) gives the allowed spectrum of electric and magnetic charges of a monopole. We shall now try to analyse the asymptotic fields of various known monopole solutions [20] [9] [21] [22] and see where they fit in this list. Eq.(18), however, is not the most convenient starting point for this analysis, since the lattice P, to which the vectors $\vec{\alpha}$ and $\vec{\beta}$ belong, itself depends on $M^{(0)}$. In particular, if P_0 denotes the lattice P for $M^{(0)} = I$, then from eqs.(15), (17) and (18) we see that,

$$P = (LM^{(0)}L)^{-1}P_0 = M^{(0)}P_0 \tag{27}$$

where we have used eq.(12) to get the last relation in eq.(27). Using eq.(12) we see that P_0 is also an even, self-dual, Lorentzian lattice with metric L. We can now express $\vec{\alpha}$ and $\vec{\beta}$ as,

$$\vec{\alpha} = M^{(0)}\vec{\alpha}_0, \quad \vec{\beta} = M^{(0)}\vec{\beta}_0, \qquad \vec{\alpha}_0, \vec{\beta}_0 \in P_0 \tag{28}$$

Eq.(18) may now be rewritten as,

$$\vec{q}_{mag} = L\vec{\beta}_0, \quad \vec{q}_{el} = \frac{1}{\lambda_2^{(0)}} M^{(0)}(\vec{\alpha}_0 + \lambda_1^{(0)}\vec{\beta}_0) \tag{29}$$

The lattice P_0 to which $\vec{\alpha}_0$ and $\vec{\beta}_0$ belong is now independent of $M^{(0)}$. We shall call $\vec{\alpha}_0$ and $\vec{\beta}_0$ electric and magnetic charge vectors respectively.

Let us now consider the BPS monopole solution in string theory discussed in refs [20] [9] in the gauge where asymptotically the Higgs field is directed along a fixed direction in the gauge space, except along a Dirac string singularity. With the normalization convention that we have chosen, the asymptotic values of various fields are given by,

$$
\begin{aligned}
&B^{(10)}_{\mu\nu} \simeq 0, \quad G^{(10)}_{S\mu\nu} \simeq Diag(-1, e^{2\phi_0}, e^{2\phi_0}, e^{2\phi_0}), \quad \Phi^{(10)} \simeq 2\phi_0, \quad \partial_\mu G^{(10)}_{m\nu} - \partial_\nu G^{(10)}_{m\mu} \simeq 0 \\
&H^{(10)}_{m0r} \simeq 0, \quad H^{(10)}_{mij} \simeq 8C\delta_{m,1}\epsilon_{ijk}\frac{x^k}{r^3}, \quad F^{(10)I}_{0r} \simeq 0, \quad F^{(10)I}_{ij} \simeq -4\delta^{I,1}\epsilon_{ijk}\frac{x^k}{r^3} \\
&B^{(10)}_{mn} \simeq 0, \quad A^{(10)I}_m \simeq 4C\delta_{m,1}\delta^{I,1}, \quad G^{(10)}_{Smn} \simeq Diag(e^{2\phi_0}, 1, 1, 1, 1, 1) \\
&\qquad 7 \le i, j \le 9, \quad 1 \le m, n \le 6, \quad 1 \le I \le 16
\end{aligned}
\tag{30}
$$

Using eq.(4) we see that ϕ_0 denotes the asymptotic value of the four dimensional dilaton field Φ. Using eqs.(8) and (11) we get,

$$
G_{\mu\nu} \simeq Diag(-e^{-\phi_0}, e^{\phi_0}, e^{\phi_0}, e^{\phi_0}), \quad \lambda^{(0)}_2 = e^{-\phi_0} \tag{31}
$$

We now scale the internal coordinate x^1 by e^{ϕ_0}, the time coordinate x^0 by $e^{-\phi_0/2}$, and the space coordinates x^7, x^8, x^9 by $e^{\phi_0/2}$, so that asymptotically $G^{(10)}_{Smn}$ approaches δ_{mn} and $G_{\mu\nu}$ approaches $\eta_{\mu\nu}$. In this new coordinate system the various transformed fields are given by,

$$
\begin{aligned}
&B^{(10)}_{\mu\nu} \simeq 0, \quad G^{(10)}_{S\mu\nu} \simeq Diag(-e^{\phi_0}, e^{\phi_0}, e^{\phi_0}, e^{\phi_0}), \quad \Phi^{(10)} \simeq 2\phi_0, \quad \partial_\mu G^{(10)}_{m\nu} - \partial_\nu G^{(10)}_{m\mu} \simeq 0 \\
&H^{(10)}_{m0r} \simeq 0, \quad H^{(10)}_{mij} \simeq 8C\delta_{m,1}e^{-\phi_0}\epsilon_{ijk}\frac{x^k}{r^3}, \quad F^{(10)I}_{0r} \simeq 0, \quad F^{(10)I}_{ij} \simeq -4\delta^{I,1}\epsilon_{ijk}\frac{x^k}{r^3} \\
&B^{(10)}_{mn} \simeq 0, \quad A^{(10)I}_m \simeq 4Ce^{-\phi_0}\delta_{m,1}\delta^{I,1}, \quad G^{(10)}_{Smn} \simeq \delta_{mn}
\end{aligned}
\tag{32}
$$

Using eqs.(4) and (32) we can find the asymptotic values of various four dimensional fields. Here we only list those which are asymptotically non-trivial:

$$
\begin{aligned}
&F^\alpha_{0r} \simeq 0, \quad \tilde F^\alpha_{0r} \simeq \delta_{\alpha,13}\frac{\sqrt{2}}{r^2} \\
&\hat A^I_m \simeq 4C\delta^{I,1}\delta_{m,1}e^{-\phi_0}, \qquad 1 \le \alpha \le 28, \quad 1 \le m \le 6, \quad 1 \le I \le 16
\end{aligned}
\tag{33}
$$

In particular, note that non-trivial $H^{(10)}_{1ij}$ is induced solely by $\tilde F^{13}_{0r}$. Comparing with eq.(17) we get,

$$
\vec q_{el} = 0, \quad q^\alpha_{mag} = \delta^{\alpha,13}\sqrt{2} \tag{34}
$$

Since these solutions do not carry any electric charge, they are valid solutions only for $\lambda_1 = 0$.⋆ Comparison with eq.(29) gives,

$$\vec{\alpha}_0 = 0, \quad \beta_0^\alpha = -\delta^{\alpha,13}\sqrt{2} \tag{35}$$

Note that $\vec{\alpha}_0$ and $\vec{\beta}_0$ are even with respect to the inner product metric L, as is required by the quantization condition. Also note that here allowed values of $\vec{\alpha}_0$ and $\vec{\beta}_0$ are quantized, but the constant C is arbitrary.

Next let us turn to the H monopole solutions [20] [21] [22] for which all fields become asymptotically trivial except for the field strength associated with the antisymmetric tensor field. The only non-trivial asymptotic field component is given by,

$$H^{(10)}_{mij} \simeq Q\delta_{m,1}\epsilon_{ijk}\frac{x^k}{r^3} \tag{36}$$

where Q is some parameter. Using eqs.(4), (17), and (29) we get,

$$\vec{\alpha}_0 = 0, \quad \beta_0^\alpha = -\frac{1}{2}Q\delta_{\alpha,1} \tag{37}$$

Again notice that $\vec{\alpha}_0$ and $\vec{\beta}_0$ are even with respect to the inner product metric L (in fact both $\vec{\alpha}_0^T.L.\vec{\alpha}_0$ and $\vec{\beta}_0^T.L.\vec{\beta}_0$ vanish.) The requirement that $\vec{\beta}_0$ lies on the lattice P_0 gives rise to the quantization condition on Q, as discussed in ref.[22].

Where do Elementary String Excitations Fit in?

We now try to see whether the elementary string excitations satisfy the Bogomol'nyi bound, and, if they do, then which are the excitations that saturate the bound. Since elementary string excitations do not carry any magnetic charge, we rewrite eq.(23) for particles carrying electric charge only:

$$(m_0)^2 = \frac{\lambda_2^{(0)}}{16}\vec{q}_{el}^{\,T}.(LM^{(0)}L + L).\vec{q}_{el} \tag{38}$$

We can simplify the above expression by using the observation of ref.[16] that the physics remains invariant under a simultaneous change of the background $M^{(0)}$

⋆ Dyon solutions in this theory can also be constructed [9].

and the lattice of electric charge vectors of the form:

$$M^{(0)} \to \Omega M^{(0)} \Omega^T, \quad \vec{q}_{el} \to \Omega \vec{q}_{el} \tag{39}$$

with Ω satisfying,

$$\Omega L \Omega^T = L \tag{40}$$

Under this transformation

$$P \to \Omega P, \quad P_0 \to L\Omega L P_0 \tag{41}$$

Let us choose $\Omega = \Omega_0$ such that $\Omega_0 M^{(0)} \Omega_0^T = I$, and denote all the transformed variables by putting a hat on top of them. In this case,

$$\hat{M}^{(0)} = I \tag{42}$$

and,

$$\hat{\vec{q}}_{el} = \frac{1}{\lambda_2^{(0)}} \hat{\vec{\alpha}}_0, \quad \hat{\vec{\alpha}}_0 \equiv L\Omega_0 L \vec{\alpha}_0 \in \hat{P}_0 \equiv L\Omega_0 L P_0 \tag{43}$$

Eq.(38) may now be rewritten as,

$$(m_0)^2 = \frac{\lambda_2^{(0)}}{16} \hat{\vec{q}}_{el}^{\,T}.(I+L).\hat{\vec{q}}_{el} = \frac{1}{8\lambda_2^{(0)}} (\hat{\vec{\alpha}}_{0R})^2 \tag{44}$$

where,

$$\hat{\vec{\alpha}}_{0{R \atop L}} \equiv \frac{1}{2}(I \pm L)\hat{\vec{\alpha}}_0 \tag{45}$$

We now turn to the mass formula for elementary string excitations [23]. This takes a simple form in terms of the vector $\hat{\vec{\alpha}}_0$:

$$m^2 = \frac{1}{8\lambda_2^{(0)}} \{(\hat{\vec{\alpha}}_{0R})^2 + 2N_R - 1\} = \frac{1}{8\lambda_2^{(0)}} \{(\hat{\vec{\alpha}}_{0L})^2 + 2N_L - 2\} \tag{46}$$

In the above expression $(\hat{\vec{\alpha}}_{0R})^2$ and $(\hat{\vec{\alpha}}_{0L})^2$ denote the internal momentum contributions, N_R and N_L denote the oscillator contributions, and -1 and -2 denote the ghost contributions in the right and the left sectors respectively. (In our notation the right hand sector is the world-sheet supersymmetric sector.) GSO projection requires N_R to be at least 1/2, since we need a factor of $\psi^M_{-1/2}$ to create the lowest mass state in the Neveu-Schwarz sector.⋆ Eq.(46) then gives,

$$m^2 \geq \frac{1}{8\lambda_2^{(0)}}(\hat{\vec{\alpha}}_{0R})^2 \tag{47}$$

which is the same bound as eq.(44). The elementary particle states saturating the Bogomol'nyi bound have $N_R = 1/2$, but, as we can see from eq.(46), N_L is not fixed for these states.

Monopole Solutions Conjugate to Elementary Particles Saturating the Bogomol'nyi Bound

We shall now indicate how to identify the monopole solutions which have quantum numbers related via the SL(2,Z) transformation $\lambda \to -1/\lambda$ to those of the elementary particles saturating the Bogomol'nyi bound. Invariance of the Bogomol'nyi bound under SL(2,Z) transformation will then automatically tell us that these states have the same mass. We concentrate on the transformation $\lambda \to -1/\lambda$, since this transformation sends $\hat{\vec{\alpha}}_0$ to $-\hat{\vec{\beta}}_0$ and $\hat{\vec{\beta}}_0$ to $\hat{\vec{\alpha}}_0$ [12], and hence, acting on the purely electrically charged states, produces purely magnetically charged states for $\lambda_1^{(0)} = 0$.

We have seen that the elementary string excitations saturating the Bogomol'nyi bound has $N_R = 1/2$, but N_L is unrestricted. We shall now analyze the three cases separately: $N_L = 0$, $N_L = 1$ and $N_L \geq 2$.

Case I: $N_L = 0$. Here

$$m^2 = \frac{1}{8\lambda_2^{(0)}}(\hat{\vec{\alpha}}_{0R})^2 = \frac{1}{8\lambda_2^{(0)}}\{(\hat{\vec{\alpha}}_{0L})^2 - 2\} \tag{48}$$

⋆ Since the Ramond sector states are degenerate with the Neveu-Schwarz sector states, we do not need to analyze the mass formula in the Ramond sector separately.

so that,

$$(\hat{\vec{\alpha}})^2 \equiv (\hat{\vec{\alpha}}_{0L})^2 - (\hat{\vec{\alpha}}_{0R})^2 = 2 \tag{49}$$

In this case the only ten dimensional Lorentz index of the state comes from the oscillator $\psi^M_{-1/2}$ in the right hand sector. Together with the Ramond sector states, these states form massive vector supermultiplets of $N = 4$ supersymmetry algebra. We shall now show that each of these states may be interpreted as belonging to an $SU(2)$ gauge multiplet, that has become massive due to the spontaneous breaking of the $SU(2)$ gauge symmetry. To do this we again use the trick of changing the lattice P_0 at the cost of changing the background $M^{(0)}$ as described in eq.(39). This time we look for a transformation matrix ω such that,

$$\tilde{\vec{\alpha}}_0 \equiv L\omega L\hat{\vec{\alpha}}_0 \tag{50}$$

has,

$$\tilde{\vec{\alpha}}_{0R} \equiv \frac{1}{2}(I + L)\tilde{\vec{\alpha}}_0 = 0 \tag{51}$$

We define,

$$\tilde{M}^{(0)} = \omega\omega^T \tag{52}$$

so that eq.(48) may now be rewritten as,

$$m^2 = \frac{1}{16\lambda_2^{(0)}}\tilde{\vec{\alpha}}_0^T.(\tilde{M}^{(0)} + L).\tilde{\vec{\alpha}}_0 \tag{53}$$

We can interpret $\tilde{\vec{\alpha}}_0$ as the new electric charge vector lying on the lattice $\tilde{P}_0 = L\omega L\hat{P}_0$, and $\tilde{M}^{(0)}$ as the new background value of M.

Let us now note that if, keeping the lattice $\tilde{P}_0$ fixed, we had set $\tilde{M}^{(0)}$ to I, then m^2 would vanish. This, in turn, shows that these states may be interpreted as otherwise massless states, which have acquired mass due to the background $\tilde{M}^{(0)}$. More specifically, these states may be interpreted as $SU(2)$ gauge bosons and their superpartners, which have acquired mass due to spontaneous breaking of SU(2) by the background $\tilde{M}^{(0)}$. The charged generators of this $SU(2)$ group correspond to the vectors $\pm\tilde{\vec{\alpha}}_0$ on the lattice $\tilde{P}_0$.

The monopoles related to these charged particles by $\lambda \to -1/\lambda$ transformation are characterized by zero electric charge vector, and magnetic charge vector $\tilde{\vec{\alpha}}_0$. These are precisely the BPS monopoles associated with the spontaneous breaking of this particular $SU(2)$, constructed in refs.[20][9]. It is also known [2] [9] that these monopoles belong to the massive vector supermultiplet of the $N = 4$ supersymmetry algebra. Hence we see that there is an exact one to one correspondence between the elementary particle states corresponding to $N_R = 1/2$, $N_L = 0$, and the monopole states whose quantum numbers are related to these by the SL(2,Z) transformation $\lambda \to -1/\lambda$. The masses of these monopoles and the elementary particle states are also identical, since they both saturate the Bogomol'nyi bound, and this bound has already been shown to be invariant under the SL(2,Z) transformation.

We should note, however, that the solutions in refs.[20][9] are constructed as a power series expansion in the scale of breaking of the SU(2) symmetry, which, in the present case, corresponds to a power series expansion in $(\hat{\vec{\alpha}}_{0R})^2$. Thus, the explicit form of the solution can be written down only for small $(\hat{\vec{\alpha}}_{0R})^2$; but we expect that the general features of the solution, e.g. partially unbroken supersymmetry, mass, and the degeneracy of states, will remain unchanged even for finite $(\hat{\vec{\alpha}}_{0R})^2$.

Case II: $N_L = 1$. In this case eq.(46) takes the form:

$$m^2 = \frac{1}{8\lambda_2^{(0)}}(\hat{\vec{\alpha}}_{0R})^2 = \frac{1}{8\lambda_2^{(0)}}(\hat{\vec{\alpha}}_{0L})^2 \tag{54}$$

so that,

$$(\hat{\vec{\alpha}}_0)^2 \equiv (\hat{\vec{\alpha}}_{0L})^2 - (\hat{\vec{\alpha}}_{0R})^2 = 0 \tag{55}$$

Thus the conjugate monopoles in this case will be characterized by zero electric charge vector and magnetic charge vector $\hat{\vec{\alpha}}_0$ with zero norm. Although one to one correspondence between monopoles and elementary particle states has not been established in this case, there are certainly known examples of such monopoles. These are the H-monopole solutions (monopoles carrying purely anti-symmetric tensor field charge) discussed in eqs.(36), (37), and also the Kaluza-Klein type of monopoles which are related to these H-monopole solutions via the usual $R \to 1/R$ (or more general $O(6,22;Z)$) duality transformation [20].

Case III: $N_L \geq 2$. In this case,

$$m^2 = \frac{1}{8\lambda_2^{(0)}}(\hat{\vec{\alpha}}_{0R})^2 = \frac{1}{8\lambda_2^{(0)}}\big((\hat{\vec{\alpha}}_{0L})^2 + 2N_L - 2\big) \tag{56}$$

so that,

$$(\hat{\vec{\alpha}}_0)^2 = (\hat{\vec{\alpha}}_{0L})^2 - (\hat{\vec{\alpha}}_{0R})^2 = 2 - 2N_L \leq -2 \tag{57}$$

The monopoles conjugate to these have magnetic charge vector $\hat{\vec{\alpha}}_0$ with negative norm with respect to the metric $-L$. There are no known monopole solutions with this quantum number. This, however, is not surprising, since, as we shall argue now, construction of such monopole solutions will probably involve massive string fields in a non-trivial way. To see this, let us note that in the two previous cases, there is a limit $((\hat{\vec{\alpha}}_{0R})^2 \to 0)$ in which the monopole mass vanishes. Such monopoles must be constructed out of nearly massless fields. On the other hand, in this case, there is no limit in which the monopole is massless, since from eq.(56) we see that $m^2 \geq (1/4\lambda_2^{(0)})$. Hence there is no reason why one should be able to construct such solutions purely in terms of nearly massless fields. Thus it appears that the only way to construct these monopole solutions would be to look for exact conformal field theories.

Conclusion

To summarize, in this paper we have identified the monopole solutions related via the SL(2,Z) transformation $\lambda \to -1/\lambda$ to some of the elementary string excitations, and have shown that these monopoles have the same mass and degeneracy of states as the elementary string excitations. Furthermore, we have shown that both, the dyon solutions, and the elementary excitations in string theory satisfy a lower bound to their masses, and this lower bound is invariant under the SL(2,Z) transformation. These results provide a further support to the conjecture that SL(2,Z) might be an exact symmetry of heterotic string theory compactified on a six dimensional torus.

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