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SL(2, Z) DUALITY AND MAGNETICALLY CHARGED STRINGS

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ABSTRACT

Postulate of SL(2,Z) invariance of toroidally compactified heterotic string theory in four dimensions implies the existence of new string (dual string) states carrying both, electric and magnetic charges. In this paper we study the interaction between these dual strings. In particular, we consider scattering of two such strings in the limit where one string passes through the other string without touching it, and show that such a scattering leads to the exchange of a fixed amount of electric and magnetic charges between the two strings.

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1. Introduction

Following earlier ideas $[1-9]$ we have proposed recently $[10, 11]$ that the toroidally compactified heterotic string theory [12] [13] in four dimensions may be invariant under an SL(2,Z) group of transformations. These transformations mix electric and magnetic fields, and at the same time act non-trivially on the axion-dilaton field, thereby interchanging the strong and weak coupling limits of the theory. Further work in this direction was reported in ref.[14]. SL(2,R) symmetry was used in refs.[9][10][15] to generate magnetically charged black hole solutions in string theory.

In order that the full string theory has SL(2,Z) invariance, the theory must contain magnetically charged states. The allowed spectrum of electric and magnetic charges in the theory was computed in ref.[11]. A natural question to ask would be, 'where do these magnetically charged states come from?' A partial answer to this question was provided in ref.[10]. Following ref.[16], if we regard fundamental strings as solitons of the effective field theory (a description which is likely to hold for states representing long strings) then the dual strings may be constructed simply from the SL(2,Z) transform of these solitons.

In this paper we shall study the interaction between these dual strings, which also includes interaction between a dual string and an ordinary string. In particular, we show that the force between an infinitely long straight dual string and an ordinary test string parallel to the dual string vanishes. We then study the scattering of two closed dual strings when one of them passes through the other without touching it, and show that the result is an exchange of a fixed amount of electric and magnetic charge between the two strings, determined by the quantum numbers of the original string.

The plan of the paper is as follows. In sect.2 we give a brief review of SL(2, Z) invariance in toroidally compactified heterotic string theory, and also discuss the relationship between classical solutions in this theory and fundamental strings. In sect.3 we construct the magnetically charged dual string solutions by taking

SL(2,Z) transformation of the fundamental string solution. In sect.4 we calculate the force between a dual string and an ordinary test string parallel to it, and show that it vanishes. In sect.5 we study the result of adiabatically transporting a particle, carrying both, electric and magnetic charges, around a dual string and show that both these charges change as a result of this transport. In sect.6 we derive the same results by regarding the string as the boundary of a domain wall, and calculating the electric and magnetic charges exchanged between the particle and the domain wall as the particle passes through the wall. In sect.7 we use the results of sect.5 and 6 to study the scattering of two strings. We summarize our results in sect.8.

2. Review

We begin by giving a brief review of the duality invariance of the effective field theory and soliton solutions in this theory representing fundamental strings. We consider heterotic string theory with six of its ten dimensions compactified on a torus with constant background gauge and anti-symmetric tensor fields. For a generic compactification, the only massless bosonic fields in the theory are the metric $G_{\mu\nu}$, a complex scalar field λ representing the axion-dilaton system, a set of 28 gauge fields $A_\mu^{(\alpha)}$ $(1 \le \alpha \le 28)$ which we shall denote as a 28 dimensional vector $\vec{A}_\mu$, and a 28×28 matrix valued field M satisfying,

$$M^T = M, \qquad M^T L M = L \tag{2.1}$$

where

$$L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix} \tag{2.2}$$

I_n being the $n \times n$ identity matrix. In terms of these fields, the action is given by,

$$\begin{aligned} S =& \frac{1}{32\pi} \int d^4x \sqrt{-\det G} [R - \frac{1}{2(\lambda_2)^2} G^{\mu\nu} \partial_\mu \lambda \partial_\nu \bar{\lambda} - \lambda_2 \vec{F}^T_{\mu\nu} . LML . \vec{F}^{\mu\nu} \\ &+ \lambda_1 \vec{F}^T_{\mu\nu} . L . \tilde{\vec{F}}^{\mu\nu} + \frac{1}{8} G^{\mu\nu} Tr(\partial_\mu ML \partial_\nu ML)] \end{aligned} \tag{2.3}$$

where

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu \tag{2.4}$$

The equations of motion derived from this action are invariant under the SL(2,R) transformation [10]:

$$\begin{aligned} &\lambda \to \frac{a\lambda + b}{c\lambda + d}, \qquad \vec{F}_{\mu\nu} \to c\lambda_2 ML . \tilde{\vec{F}}_{\mu\nu} + (c\lambda_1 + d) \vec{F}_{\mu\nu} \\ &M \to M, \qquad G_{\mu\nu} \to G_{\mu\nu} \end{aligned} \tag{2.5}$$

where a, b, c, d are real numbers. However, quantum corrections due to instantons break the SL(2,R) invariance to at most SL(2,Z) invariance, for which,

$$a, b, c, d \in Z, \qquad ad - bc = 1 \tag{2.6}$$

The equations of motion derived from the action (2.3) has a string like classical solution [16], given by,

$$\begin{aligned} \lambda =& \frac{1}{2\pi i} \ln \frac{z}{r_0} \\ ds^2 =& - dt^2 + (dx^3)^2 - \frac{1}{2\pi} \ln \frac{r}{r_0} dz d\bar{z} \end{aligned} \tag{2.7}$$

This describes a fundamental string lying along the x^3 direction. $z = x^1 + ix^2$ denotes the complex coordinate transverse to the string. The core of the string in this case is located at $z = 0$. This solution can be shown to be invariant under eight of the sixteen global supersymmetry generators [16].

Note that the solution is sensible only in the region $r < r_0$, where r_0 is an arbitrary length scale. From physical consideration we see that r_0 should be taken to be of the order of the overall size of the closed string loop. Only for $r << r_0$ the string looks like a straight string, and eq.(2.7) gives a good description of the field configuration in this region. r_0 also contains information about the asymptotic value of λ_2; for closed string loops of the same size, r_0 takes different values for different asymptotic values of the field λ_2.

The solution has several zero modes. First of all there are two bosonic zero modes which simply correspond to shifting the location of the core of the string in the $x^1 - x^2$ plane. There are eight fermionic zero modes which correspond to supersymmetry transformation of the solution with the eight broken global supersymmetry generators. These supersymmetry generators are chiral with respect to the gamma matrices associated with the $t - x^3$ direction [16] [17], hence the corresponding fermionic zero modes are also chiral. In particular for the solution given in eq.(2.7), they turn out to be right chiral. Finally, there are 28 bosonic zero modes generated by $O(7, 23)$ deformation of the solution as discussed in refs.[10][17]. The parameters labelling the deformed solution may be identified as the charge per unit length carried by the string corresponding to the 28 gauge fields. To first order in the deformation parameters $q^{(I)}$ $(13 \leq I \leq 28)$, $p^{(m)}$, $l^{(m)}$ $(1 \leq m \leq 6)$, the solution deformed by these charge zero modes is given by,

$$
\begin{aligned}
\lambda =&\frac{1}{2\pi i}\ln\frac{z}{r_0}\\
ds^2 =&-dt^2+(dx^3)^2-\frac{1}{2\pi}\ln\frac{r}{r_0}dzd\bar{z}\\
F^{(I)}_{-zt} =&F^{(I)}_{-z3}=\frac{q^{(I)}}{z}\frac{1}{(\ln(r/r_0))^2}\quad\text{for } 13\le I\le 28\\
F^{(m)}_{-zt}-F^{(m+6)}_{-zt} =&F^{(m)}_{-z3}-F^{(m+6)}_{-z3}=\frac{\sqrt{2}p^{(m)}}{z}\frac{1}{(\ln(r/r_0))^2}\quad\text{for } 1\le m\le 6\\
F^{(m)}_{-zt}+F^{(m+6)}_{-zt} =&-(F^{(m)}_{-z3}+F^{(m+6)}_{-z3})=\frac{\sqrt{2}l^{(m)}}{z}\frac{1}{(\ln(r/r_0))^2}\quad\text{for } 1\le m\le 6\\
F^{(\alpha)}_{-\bar{z}t} =&F^{(\alpha)}_{-\bar{z}3}=0\quad\text{for } 1\le\alpha\le 28\\
M =&I
\end{aligned}
\tag{2.8}
$$

which generalizes the solution given in ref.[17] where only the parameters $q^{(I)}$ were present. Here

$$
\vec{F}_{\pm\mu\nu}=-ML.\vec{F}_{\mu\nu}\pm i\tilde{\vec{F}}_{\mu\nu}
\tag{2.9}
$$

$q^{(I)}$, $p^{(m)}$ measure the charge per unit length, as well as the current in the $-x^3$ direction associated with the gauge fields $A^{(I)}_\mu$ $(13\le I\le 28)$ and $(A^{(m)}_\mu - A^{(m+6)}_\mu)/\sqrt{2}$ $(1\le m\le 6)$ respectively, and $l^{(m)}$ measure the charge per unit length, and the current in the x^3 direction associated with the gauge field $(A^m_\mu + A^{(m+6)}_\mu)/\sqrt{2}$ $(1\le m\le 6)$.

The collective excitation of the string may be described by making the parameters labelling these deformations functions of t and x^3. In particular, when we make $q^{(I)}$, $p^{(m)}$ and $l^{(m)}$ functions of t and x^3, charge conservation implies that,

$$
(\partial_t-\partial_3)q^{(I)}=(\partial_t-\partial_3)p^{(m)}=(\partial_t+\partial_3)l^{(m)}=0
\tag{2.10}
$$

therby showing that $q^{(I)}$ and $p^{(m)}$ denote left moving coordinates and $l^{(m)}$ denote right moving coordinates.

The set of all the collective excitations of the string can easily be seen to be in one to one correspondence with the dynamical degrees of freedom of the fundamental string in the static gauge. This leads to the hypothesis that the quantization of these collective coordinates will reproduce the full spectrum of states in the string theory. Although formally this may be an exact result after taking into account the correction to the action due to the higher derivative terms, in practice the usefulness of this hypothesis is limited to states associated with long strings.

3. Dual Strings

In ref.[11] we have indicated that the allowed spectrum of electric and magnetic charges in string theory is consistent with the SL(2,Z) invariance of the theory. This, however, does not answer the question as to where the magnetically charged states, that are necessary for SL(2,Z) invariance of the spectrum, come from. In this section we shall try to partially answer this question.

The answer in fact lies in the hypothesis stated at the end of the last section. Since according to this hypothesis, string states can be regarded as collective excitations of a classical solution in the effective field theory, the magnetically charged string states must come from the collective excitations of the SL(2,Z) transform of this classical solution. To make this more concrete, let us first write down the SL(2,Z) transform of the solution given in eq.(2.7) by the element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\begin{aligned} \lambda = & \frac{a \ln(z/r_0) + 2\pi i b}{c \ln(z/r_0) + 2\pi i d} \\ ds^2 = & - dt^2 + (dx^3)^2 - \frac{1}{2\pi} \ln(r/r_0) dz d\bar{z} \end{aligned} \tag{3.1}$$

Note that as we go around the string, λ changes to

$$(\tilde{a}\lambda + \tilde{b})/(\tilde{c}\lambda + \tilde{d}) \tag{3.2}$$

where,

$$\tilde{g} \equiv \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} T \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix} \tag{3.3}$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.4}$$

The zero modes of this solution can be constructed in the following way. Instead of trying to write down the deformed solution directly, we can simply take the zero mode deformation of the solution (2.7), and take the SL(2,Z) transform of it. SL(2,Z) invariance of the equations of motion will imply that the transformed configuration is also a solution of the equations of motion, and hence denotes the zero mode deformation of the solution given in eq.(3.1). Quantization of these zero modes should, in turn, produce the magnetically charged strings required for duality invariance of the theory, at least those corresponding to long strings. We shall not explicitly display the deformed solution here.

Note that the electrically and magnetically charged string states in a theory with a given asymptotic value of λ are not related to each other by SL(2,Z) transformation. Instead, the magnetically charged particles in this theory are related by SL(2, Z) transformation to the purely electrically charged particles in a theory with a different asymptotic value of λ.

There is one question that must be addressed before we conclude this section. So far we have discussed SL(2,Z) invariance of the theory in the case where the fermionic background fields have been set to zero. But the true SL(2,Z) invariance of the theory requires SL(2, Z) invariance of the equations of motion even in the presence of fermionic background fields. In particular, this is necessary if we want to construct the fermionic zero modes of the SL(2,Z) transformed solution. We shall now give an indirect proof of the SL(2, Z) invariance of the equations of motion after the inclusion of the fermionic fields. This is done by comparing the dimensionally reduced heterotic string theory to the $N = 4$ Poincare supergravity theory coupled to abelian gauge field multiplets as discussed in ref.[8]. It can be shown that the bosonic part of the action given in eqs.(4.18), (4.26) of ref.[8] is identical to the action given in eq.(2.3) after we make the following identification of fields[⋆]

$$M = UOO^TU^{-1}, \qquad \frac{i}{\lambda} = \frac{\phi_1 - \phi_2}{\phi_1 + \phi_2} \tag{3.5}$$

and a redefinition of the gauge fields $\vec{F} \to U\vec{F}$. Here U is a matrix that diagonalizes L:

$$U^{-1}LU = \begin{pmatrix} I_6 & & \\ & -I_6 & \\ & & -I_{16} \end{pmatrix} \tag{3.6}$$

In eq.(3.5), the right hand sides of the equations contain variables appearing in ref.[8], whereas the left hand sides of the equations contain variables appearing in eq.(2.3). Since the bosonic part of the two actions are identical, we have a strong evidence that the two theories are indeed the same. We shall proceed with the assumption that this is the case. This assumption is further supported by the fact that both theories have local $N = 4$ supersymmetry.

In ref.[8] it was shown that the gauge field equations of motion are invariant under SL(2,R) transformation even after including the fermionic fields in this theory. This result, combined with an earlier result of Gaillard and Zumino [6] shows that all the field equations must be invariant under the SL(2,R) transformation. This establishes the SL(2,R) invariance of the full set of equations of motion of the dimensionally reduced heterotic string theory. Similar argument has been advanced previously by Schwarz [14].

⋆ Partial results to this effect have been obtained previously in refs.[18].

4. Force Exerted by a Dual String on an Ordinary String

We now begin our study of the interaction between dual strings, and also between a dual string and an ordinary string. The first quantity we would like to compute is the force exerted by an infinitely long straight dual string on an ordinary test string kept parallel to itself some distance away. A similar computation for two ordinary strings parallel to each other had yielded the answer that the net force between such strings vanish [16] [17].

We begin by writing down the action of a test string in the presence of a background axion-dilaton-gravitational field:

$$S_{string} = \int d^2\xi(\sqrt{-\gamma}G_{S\mu\nu}(X)\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu + \epsilon^{\alpha\beta}B_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu) + \dots \tag{4.1}$$

where ξ^α and $\gamma_{\alpha\beta}$ denote the coordinates and metric on the string world-sheet respectively, X^μ denote the coordinates of the string, Φ denotes the dilaton field, and $G_{S\mu\nu}$ denotes the string metric. $\dots$ denotes terms involving background gauge fields and world sheet fermionic fields, which we are setting to zero for the present analysis. The relation between the fields appearing here and those in eq.(2.3) are given by,

$$\begin{aligned} G_{S\mu\nu} =& e^{\Phi}G_{\mu\nu}, \qquad e^{-\Phi} = \lambda_2 \\ G^{\sigma\sigma'}\partial_{\sigma'}\lambda_1 =& \frac{1}{2}(\sqrt{-\det G})^{-1}e^{-2\Phi}\epsilon^{\mu\nu\rho\sigma}(\partial_\mu B_{\nu\rho} + \frac{1}{2}\vec{A}^T_\mu.L.\vec{F}_{\nu\rho}) \end{aligned} \tag{4.2}$$

Let us now consider the test string lying along the 3 direction,

$$X^0 = \xi^0, \qquad X^3 = \xi^1 \tag{4.3}$$

with the gauge choice,

$$\sqrt{-\gamma}\gamma^{\alpha\beta} = \eta^{\alpha\beta} \tag{4.4}$$

If $Z = X^1 + iX^2$ denotes the complex coordinate transverse to the string, then the equation of motion of Z that follows from the action (4.1) in the background (3.1) is given by (with the help of eqs.(4.2))

$$D^\alpha D_\alpha Z + \Gamma^Z_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu - i\frac{\partial_{\bar Z}\lambda}{\lambda_2} G^{Z\bar Z} = 0 \tag{4.5}$$

The second term, which in this gauge is given by $\Gamma^Z_{33} - \Gamma^Z_{tt}$ vanishes, as can easily be seen by computing the Christoffel symbol from the metric given in eq.(3.1). The last term also vanishes, since λ given in eq.(3.1) is an analytic function of z. The net result is that the equation of motion of the coordinate Z looks like,

$$D^\alpha D_\alpha Z = 0 \tag{4.6}$$

showing that there is no net force exerted on the test string.

The same analysis can also be repeated for the case where the dual string carries electric and magnetic charge density (these are the solutions deformed by the charge zero modes). As in the case of ref.[17], in this case the net magnetic and electric forces exerted on the test string cancel each other.

5. Adiabatic Transport of a Charged Particle Around a Dual String

We shall now consider the effect of adiabatically transporting a point particle carrying magnetic charge $\vec Q_m$ and electric charge $\vec Q_e$ around a dual string.⋆ Although the results of this and the next section may be derived by starting from the known results about the interaction of a dyon with an axionic domain wall [20] and then making a duality transformation of the results, we shall carry out the analysis explicitly, since this gives a physical understanding of the interaction mechanism. As was shown in ref.[11] the allowed spectrum of $(\vec Q_m, \vec Q_e)$ is

$$(\vec Q_m, \vec Q_e) = (M^{(0)} L\vec\beta,\ (\vec\alpha + \lambda_1^{(0)}\vec\beta)/\lambda_2^{(0)}) \tag{5.1}$$

where $\lambda_1^{(0)}$, $\lambda_2^{(0)}$ and $M^{(0)}$ are the asymptotic values of the fields λ_1, λ_2 and M respectively, and $\vec\alpha$, $\vec\beta$ are 28 dimensional vectors belonging to an even, self dual

⋆ Similar phenomena associated with the usual $R \to 1/R$ duality transformation was discussed in ref.[19].

lattice P with the metric L. Let us now consider a dual string that is a transform of the ordinary string by the group element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $\tilde{g} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ be the corresponding element defined in eq.(3.3). As we adiabatically transport the particle around the string, we expect $\vec{\alpha}$, $\vec{\beta}$ to remain fixed since they take discrete values. After we go around the string once, the background value of the field λ changes to

$$\lambda' = \frac{\tilde{a}\lambda + \tilde{b}}{\tilde{c}\lambda + \tilde{d}} \tag{5.2}$$

which, in turn, implies that the electric and magnetic charge vectors of the particle change to

$$(\vec{Q}'_m, \vec{Q}'_e) = (M^{(0)} L\vec{\beta}, \frac{1}{\lambda_2'^{(0)}}(\vec{\alpha} + \lambda_1'^{(0)}\vec{\beta})) \tag{5.3}$$

However, the background seen by the particle is now different from the one that was seen by it before. As a result, we should not directly compare $(\vec{Q}'_m, \vec{Q}'_e)$ with $(\vec{Q}_m, \vec{Q}_e)$. Instead, we need to make a duality transformation of the fields and the charges by the element,

$$\tilde{g}^{-1} = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix} \tag{5.4}$$

to state the results in the original coordinate system. This transformation sends λ' back to λ, and $(\vec{\alpha}, \vec{\beta})$ to $(\vec{\alpha}', \vec{\beta}')$ given by [11]

$$\begin{pmatrix} \vec{\alpha}' \\ -\vec{\beta}' \end{pmatrix} = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix} \begin{pmatrix} \vec{\alpha} \\ -\vec{\beta} \end{pmatrix} \tag{5.5}$$

Thus the final electric and magnetic charges of the particle are given by:

$$(\vec{Q}''_m, \vec{Q}''_e) = (M^{(0)} L\vec{\beta}', \frac{1}{\lambda_2^{(0)}}(\vec{\alpha}' + \lambda_1^{(0)}\vec{\beta}')) \tag{5.6}$$

The conservation of electric and magnetic charge implies that the charge lost by the particle must be deposited on the string, but the above analysis does not show explicitly how it happens. Also, for a realistic string, once the SL(2,Z) symmetry is broken by the instanton corrections, the field around a string is not given by eq.(3.1), but remains equal to its vacuum value $\lambda^{(0)}$ in most of the region of space, and changes quickly to $\lambda'^{(0)} = (\tilde{a}\lambda^{(0)} + \tilde{b})/(\tilde{c}\lambda^{(0)} + \tilde{d})$ across a thin domain wall bounded by the string. We shall now derive eq.(5.6) using this more realistic picture of strings, which will also clarify the charge exchange mechanism between the string and the domain wall.

6. Dynamics of Domain Wall Penetration

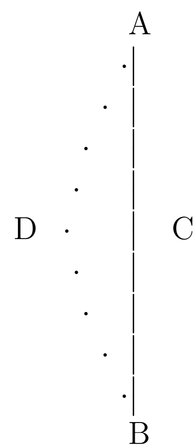

Fig. 1. Picture of a string

Let us consider the picture of the string depicted in Fig.1. The points A and B in this figure denote the points at which the string intersects the plane of the paper. The line C connecting the two points represents the intersection of the domain wall (bounded by the string) with the plane of the paper. On the right side of the wall C the field λ takes the value $\lambda^{(0)}$, whereas on the left side of the wall the field λ takes the value $\lambda'^{(0)}$. Finally, the line D represents the intersection of another fictitious wall bounded by the string with the plane of the paper. This fictitious wall is characterised by the choice of two different coordinate systems on the two sides of the wall, related by the SL(2,Z) transformation with the group element $\tilde{g}$ such that on the left hand side of the wall D the field λ takes value $\lambda^{(0)}$. In other words, the field λ takes value $\lambda^{(0)}$ on the right side of C and the left side of D, but takes the value $\lambda'^{(0)}$ in the region bounded by C and D. Note that C represents a real domain wall with finite thickness, and λ changes continuously from $\lambda^{(0)}$ to $\lambda'^{(0)}$ as we cross the wall, whereas D denotes an infinitely thin fictitious wall, and arises only because we choose to use two different coordinate systems on the two sides of the wall. As we shall see, the results of passage of a particle through these two walls are completely different. In both cases, however, as the particle crosses the wall, it deposits certain amount of electric and magnetic charge on the wall, which ultimately flows back to the string.

In studying the passage of charged particles through these walls, we shall use the method used by Sikivie [20] for studying the passage of charged particles through an axionic domain wall. First let us consider the passage of the particle through C. Both, inside, and outside the wall, the gauge fields satisfy the equation of motion:

$$D_\mu(\lambda_2 ML\vec{F}^{\mu\nu} - \lambda_1 \vec{\tilde{F}}^{\mu\nu}) = 0 \tag{6.1}$$

Let us now ignore all time derivatives (assuming that the motion of the particle is slow) and define,

$$\vec{F}^{i0} = \vec{E}^i, \qquad \vec{\tilde{F}}^{i0} = \vec{B}^i \tag{6.2}$$

We shall also assume that the energy per unit area of the wall is small compared to M_{Pl}^3, so that we can ignore the gravitational field produced by the wall. At the same time, the thickness of the wall is taken to be small compared to the overall size of the string, so that the variation of the gravitational field due to the string across the wall is small. The equation of motion (6.1) and the Bianchi identity now takes the form:

$$\begin{aligned} D_i(\lambda_2 ML\vec{E}^i - \lambda_1\vec{B}^i) =&0, \qquad \epsilon^{ijk}D_j(\lambda_2 ML\vec{B}_k + \lambda_1\vec{E}_k) = 0 \\ D_i\vec{B}^i =&0, \qquad \epsilon^{ijk}D_j\vec{E}_k = 0 \end{aligned} \tag{6.3}$$

Let us now denote by $\vec{B}_\perp$ ($\vec{E}_\perp$) and $\vec{B}'_\perp$ ($\vec{E}'_\perp$) the components of the magnetic (electric) fields perpendicular to the domain wall on the right and the left side of

C respectively. Similarly, we denote by $\vec{B}_{||}$ ($\vec{E}_{||}$) and $\vec{B}'_{||}$ ($\vec{E}'_{||}$) the components of magnetic (electric) fields parallel to the wall on the two sides of the wall. In the thin wall approximation, the variation of various fields in directions parallel to the wall are small compared to that in directions perpendicular to the wall. Eq.(6.3) then gives

$$
\begin{aligned}
\vec{B}'_{\perp} &= \vec{B}_{\perp}, \quad \lambda_2'^{(0)} M^{(0)} L \vec{E}'_{\perp} - \lambda_1'^{(0)} \vec{B}'_{\perp} = \lambda_2^{(0)} M^{(0)} L \vec{E}_{\perp} - \lambda_1^{(0)} \vec{B}_{\perp} \\
\vec{E}'_{||} &= \vec{E}_{||}, \quad \lambda_2'^{(0)} M^{(0)} L \vec{B}'_{||} + \lambda_1'^{(0)} \vec{E}'_{||} = \lambda_2^{(0)} M^{(0)} L \vec{B}_{||} + \lambda_1^{(0)} \vec{E}_{||}
\end{aligned}
\tag{6.4}
$$

Thus the total induced magnetic charge on the wall is given by,

$$
\Delta \vec{Q}_m = \frac{1}{4\pi} \int (\vec{B}'_{\perp} - \vec{B}_{\perp}) d^2 S = 0 \tag{6.5}
$$

On the other hand, the total induced electric charge is given by,

$$
\Delta \vec{Q}_e = \frac{1}{4\pi} \int (\vec{E}'_{\perp} - \vec{E}_{\perp}) d^2 S \tag{6.6}
$$

and is non-zero in general.

We now consider a particle with charge $(\vec{Q}_m, \vec{Q}_e)$ approaching the wall C from the right. The electromagnetic fields due to the particle in the absence of the domain wall C are given by $\vec{E}^i = \vec{Q}_e r^i / r^3$, $\vec{B}^i = \vec{Q}_m r^i / r^3$. In order to calculate the total induced charge on the wall, we need to calculate the electric and magnetic fields that are obtained by solving eqs.(6.4) and then use eqs.(6.5), (6.6). This is done using the method of images. Let P denote the position of the incoming particle at a given instant of time and Q be its image point. We assume that the field to the right side of C is reproduced by the original particle at the point P and a fictitious charge $(\vec{q}_m^{(1)}, \vec{q}_e^{(1)})$ placed at the point Q. On the other hand, the field to the left side of C is assumed to be given by the original particle, together with a fictitious charge $(\vec{q}_m^{(2)}, \vec{q}_e^{(2)})$ placed at the point P. The boundary conditions

(6.4) then give,

$$\begin{aligned}
\vec{Q}_m + \vec{q}_m^{(2)} =& - (\vec{q}_m^{(1)} - \vec{Q}_m) \\
\vec{Q}_e + \vec{q}_e^{(2)} =& \vec{q}_e^{(1)} + \vec{Q}_e \\
\lambda_2'^{(0)} M^{(0)} L.(\vec{q}_e^{(2)} + \vec{Q}_e) - \lambda_1'^{(0)}(\vec{q}_m^{(2)} + \vec{Q}_m) =& \lambda_2^{(0)} M^{(0)} L.(-\vec{q}_e^{(1)} + \vec{Q}_e) - \lambda_1^{(0)}(-\vec{q}_m^{(1)} + \vec{Q}_m) \\
\lambda_2'^{(0)} M^{(0)} L.(\vec{q}_m^{(2)} + \vec{Q}_m) + \lambda_1'^{(0)}(\vec{q}_e^{(2)} + \vec{Q}_e) =& \lambda_2^{(0)} M^{(0)} L.(\vec{q}_m^{(1)} + \vec{Q}_m) + \lambda_1^{(0)}(\vec{q}_e^{(1)} + \vec{Q}_e)
\end{aligned} \tag{6.7}$$

Explicit expressions for the image charges can be found by solving these four equations. We are, however, interested in computing the total electric and magnetic charges induced on the wall. Using eqs.(6.5), (6.6) and (6.7), these are given by,

$$\begin{aligned}
\Delta\vec{Q}_m =& \frac{1}{2}(\vec{q}_m^{(1)} + \vec{q}_m^{(2)}) = 0 \\
\Delta\vec{Q}_e =& \frac{1}{2}(\vec{q}_e^{(1)} + \vec{q}_e^{(2)}) \\
=& \frac{(\lambda_2^{(0)})^2 - (\lambda_2'^{(0)})^2 - (\lambda_1^{(0)} - \lambda_1'^{(0)})^2}{(\lambda_2^{(0)} + \lambda_2'^{(0)})^2 + (\lambda_1'^{(0)} - \lambda_1^{(0)})^2}\vec{Q}_e + \frac{2\lambda_2^{(0)}(\lambda_1'^{(0)} - \lambda_1^{(0)})}{(\lambda_2^{(0)} + \lambda_2'^{(0)})^2 + (\lambda_1'^{(0)} - \lambda_1^{(0)})^2} M^{(0)} L.\vec{Q}_m
\end{aligned} \tag{6.8}$$

Note that the total induced charge on the wall is independent of the distance of the particle from the wall as long the particle is close enough so that we can regard the wall as infinite. As the particle approaches closer and closer to the wall, the total induced charge gets concentrated at the point of impact.

Let us assume that after passing through the wall the particle emerges with charge $(\vec{Q}'_m, \vec{Q}'_e)$. A similar analysis now shows that the total charge induced on the wall is given by,

$$\begin{aligned}
\Delta\vec{Q}'_m =& 0 \\
\Delta\vec{Q}'_e =& \frac{(\lambda_2'^{(0)})^2 - (\lambda_2^{(0)})^2 - (\lambda_1^{(0)} - \lambda_1'^{(0)})^2}{(\lambda_2^{(0)} + \lambda_2'^{(0)})^2 + (\lambda_1'^{(0)} - \lambda_1^{(0)})^2}\vec{Q}'_e + \frac{2\lambda_2'^{(0)}(\lambda_1^{(0)} - \lambda_1'^{(0)})}{(\lambda_2^{(0)} + \lambda_2'^{(0)})^2 + (\lambda_1'^{(0)} - \lambda_1^{(0)})^2} M^{(0)} L.\vec{Q}'_m
\end{aligned} \tag{6.9}$$

This result can be interpreted by saying that as the particle penetrates the domain wall, it exchanges charge with the wall. Charge conservation then gives

$$\begin{aligned}\vec{Q}_e + \Delta\vec{Q}_e =& \vec{Q}'_e + \Delta\vec{Q}'_e \\ \vec{Q}_m + \Delta\vec{Q}_m =& \vec{Q}'_m + \Delta\vec{Q}'_m\end{aligned} \tag{6.10}$$

Eqs.(6.8), (6.9) and (6.10) gives,

$$\begin{aligned}\vec{Q}'_e =& \frac{\lambda_2^{(0)}}{\lambda_2^{\prime(0)}}\vec{Q}_e + \frac{1}{\lambda_2^{\prime(0)}}(\lambda_1^{\prime(0)} - \lambda_1^{(0)})M^{(0)}L\vec{Q}_m \\ \vec{Q}'_m =& \vec{Q}_m\end{aligned} \tag{6.11}$$

Finally, using eq.(5.1), we get

$$\vec{Q}'_m = M^{(0)}L\vec{\beta}, \qquad \vec{Q}'_e = \frac{1}{\lambda_2^{\prime(0)}}(\vec{\alpha} + \lambda_1^{\prime(0)}\vec{\beta}) \tag{6.12}$$

Let us now analyse the effect of crossing the fictitious wall D on the electric and magnetic charges of the particle. Let $\vec{F}'_{\mu\nu}$ and $\vec{F}''_{\mu\nu}$ be the electromagnetic fields on the right and the left sides of this fictitious wall. Then from eq.(2.5) we see that the boundary condition across this wall is given by,

$$\vec{F}'_{\mu\nu} = (\tilde{c}\lambda_1^{(0)} + \tilde{d})\vec{F}''_{\mu\nu} + \tilde{c}\lambda_2^{(0)}M^{(0)}L.\vec{\tilde{F}}''_{\mu\nu} \tag{6.13}$$

which gives,

$$\begin{aligned}\vec{E}'_{\perp,||} =& (\tilde{c}\lambda_1^{(0)} + \tilde{d})\vec{E}''_{\perp,||} + \tilde{c}\lambda_2^{(0)}M^{(0)}L\vec{B}''_{\perp,||} \\ \vec{B}'_{\perp,||} =& (\tilde{c}\lambda_1^{(0)} + \tilde{d})\vec{B}''_{\perp,||} - \tilde{c}\lambda_2^{(0)}M^{(0)}L\vec{E}''_{\perp,||}\end{aligned} \tag{6.14}$$

and the reverse relations

$$\begin{aligned}\vec{E}''_{\perp,||} =& (-\tilde{c}\lambda_1^{\prime(0)} + \tilde{a})\vec{E}'_{\perp,||} - \tilde{c}\lambda_2^{\prime(0)}M^{(0)}L\vec{B}'_{\perp,||} \\ \vec{B}''_{\perp,||} =& (-\tilde{c}\lambda_1^{\prime(0)} + \tilde{a})\vec{B}'_{\perp,||} + \tilde{c}\lambda_2^{\prime(0)}M^{(0)}L\vec{E}'_{\perp,||}\end{aligned} \tag{6.15}$$

The total induced electric and magnetic charges on the wall are given by $\int(\vec{E}''_\perp - \vec{E}'_\perp)d^2S/4\pi$ and $\int(\vec{B}''_\perp - \vec{B}'_\perp)d^2S/4\pi$ respectively. As before, we first consider a particle carrying charge $(\vec{Q}'_m, \vec{Q}'_e)$ approaching the wall from the right, and calculate the total induced charge $(\Delta\vec{Q}'_m, \Delta\vec{Q}'_e)$ on the wall using eqs.(6.15). Then we assume that after passing through the wall, the particle carries charges $(\vec{Q}''_m, \vec{Q}''_e)$ and calculate the corresponding induced charges $(\Delta\vec{Q}''_m, \Delta\vec{Q}''_e)$ using eq.(6.14). Finally, using the equation for charge conservation,

$$(\vec{Q}'_m + \Delta\vec{Q}'_m, \vec{Q}'_e + \Delta\vec{Q}'_e) = (\vec{Q}''_m + \Delta\vec{Q}''_m, \vec{Q}''_e + \Delta\vec{Q}''_e) \tag{6.16}$$

we get

$$\begin{aligned}\vec{Q}''_e =&(-\tilde{c}\lambda_1'^{(0)} + \tilde{a})\vec{Q}'_e - \tilde{c}\lambda_2'^{(0)} M^{(0)} L\vec{Q}'_m \\ \vec{Q}''_m =&(-\tilde{c}\lambda_1'^{(0)} + \tilde{a})\vec{Q}'_m + \tilde{c}\lambda_2'^{(0)} M^{(0)} L\vec{Q}'_e\end{aligned} \tag{6.17}$$

Using eq.(6.12), and that $\lambda'^{(0)}$ is the SL(2,Z) transform of $\lambda^{(0)}$ by the element $\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ we recover eqs.(5.5), (5.6).

Although this provides a rederivation of the results of the last section, this derivation makes it clear how the charge lost by the particle is deposited on the string. As the particle goes farther away from the wall, the charge spreads over wider region of the wall, thereby decreasing the charge density induced on the wall. When the distance of the particle from the wall is much larger than the string size, the total charge induced on the wall becomes negligible, showing that all the induced charge flows back to the boundary of the wall, i.e. the string.

7. Scattering of Dual Strings

We shall now use the results derived above to study the scattering of dual strings. As we saw earlier, a dual string is characterized by a group element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$, or, equivalently, the group element $\tilde{g}$ defined in eq.(3.3).⋆

⋆ Note that $\tilde{g}$ remains invariant under a change $g \to gT$. Since it is the element $\tilde{g}$ that characterizes inequivalent strings, we see that the elements g and gT describe the same string. As we shall see, all our results will be invariant under the transformation $g \to gT$.

We shall first find the allowed spectrum of $\vec{Q}_e$ and $\vec{Q}_m$, or equivalently of $\vec{\alpha}$ and $\vec{\beta}$ defined through eq.(5.1), for a dual string characterized by a given element g. We start with the observation that for an ordinary string $(\vec{\alpha}, \vec{\beta}) = (\vec{l}, 0)$ where $\vec{l} \in P$, since these states do not carry any magnetic charge. From this the allowed values of $\vec{\alpha}$ and $\vec{\beta}$ for the dual string can be found by SL(2,Z) transformation, and are given by,

$$\begin{pmatrix} \vec{\alpha} \\ -\vec{\beta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \vec{l} \\ 0 \end{pmatrix} = \begin{pmatrix} a\vec{l} \\ c\vec{l} \end{pmatrix} \tag{7.1}$$

This shows that the magnetic and electric charges of a dual string characterized by a specific SL(2,Z) element g are related. We shall denote the state of a dual string by the quantum numbers $(g, \vec{l}, \ldots)$ where $\ldots$ denote other quantum numbers which are not of interest for our analysis.

We shall now consider two such strings, characterized by the quantum numbers $(g_1, \vec{l}_1)$ and $(g_2, \vec{l}_2)$ respectively, and consider a scattering where string 1 passes through string 2 without touching it. We shall assume that string 2 is a long string, where string 1 is small in size. The first point to note is that after string 1 passes through string 2, it is characterized by a new group element

$$g_1' = \tilde{g}_2^{-1} g_1 \tag{7.2}$$

Proof: Before scattering, if we go around string 1, the new field configuration ϕ' is related to the original field configuration ϕ through the relation $\phi' = g_1\phi$. When string 1 crosses the fictitious wall D during the process of scattering with the string 2, we use a different coordinate system $\psi = \tilde{g}_2^{-1}\phi$. In this new coordinate system, the relation $\phi' = g_1\phi$ may be expressed as,

$$\psi' = \tilde{g}_2^{-1}\tilde{g}_1\tilde{g}_2\psi \tag{7.3}$$

Thus $\tilde{g}_1' = \tilde{g}_2^{-1}\tilde{g}_1\tilde{g}_2$. Using the relations $\tilde{g}_1 = g_1 T g_1^{-1}$ and $\tilde{g}_1' = g_1' T g_1'^{-1}$ we get eq.(7.2) up to a transformation of the form $g_1' \to g_1' T$.

Using eqs.(5.5), (7.1) and (7.2), we see that the electric and magnetic charge quantum numbers $\vec{\alpha}'_1$ and $\vec{\beta}'_1$ of string 1 after scattering are given by,

$$\begin{pmatrix} \vec{\alpha}'_1 \\ -\vec{\beta}'_1 \end{pmatrix} = \tilde{g}_2^{-1} \begin{pmatrix} \vec{\alpha}_1 \\ -\vec{\beta}_1 \end{pmatrix} = \tilde{g}_2^{-1} g_1 \begin{pmatrix} \vec{l}_1 \\ 0 \end{pmatrix} = g'_1 \begin{pmatrix} \vec{l}_1 \\ 0 \end{pmatrix} \tag{7.4}$$

showing that,

$$\vec{l}'_1 = \vec{l}_1 \tag{7.5}$$

Eqs.(7.2) and (7.5) determine the quantum numbers of the string 1 after scattering. Let us now study the quantum numbers of string 2 after scattering. First of all, note that during this scattering process g_2 and $\tilde{g}_2$ remains unchanged, i.e.

$$g'_2 = g_2 \tag{7.6}$$

Conservation of electric and magnetic charge implies that the charges lost by string 1 must be deposited on string 2. This gives,

$$\begin{pmatrix} \vec{\alpha}'_2 \\ -\vec{\beta}'_2 \end{pmatrix} = \begin{pmatrix} \vec{\alpha}_2 \\ -\vec{\beta}_2 \end{pmatrix} + \begin{pmatrix} \vec{\alpha}_1 \\ -\vec{\beta}_1 \end{pmatrix} - \begin{pmatrix} \vec{\alpha}'_1 \\ -\vec{\beta}'_1 \end{pmatrix} \tag{7.7}$$

Using eqs.(7.1), (7.4) and (3.3) we get,

$$\begin{pmatrix} \vec{\alpha}'_2 \\ -\vec{\beta}'_2 \end{pmatrix} = g_2 \begin{pmatrix} \vec{l}'_2 \\ 0 \end{pmatrix} \tag{7.8}$$

where,

$$\vec{l}'_2 = \vec{l}_2 + (a_2 c_1 - a_1 c_2) \vec{l}_1 \tag{7.9}$$

Eqs.(7.2), (7.5), (7.6) and (7.9) describe the final result of scattering when string 1 passes through string 2.

8. Summary

To summarize, in this paper we have studied the classical scattering of dual strings (strings carrying electric and magnetic charges) and have shown that there is a definite change of quantum numbers of the string as a result of the scattering. The changes in the quantum numbers are determined by the quantum numbers of the original strings, and depend on which string passed through the other during the scattering.

The picture of magnetically charged string states that we have used is valid for sufficiently long string states, but is not useful for description of point like string states. Description of such states are likely to be found in 't Hooft - Polyakov like monopole solutions in string theory [21].

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