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# Marginal distributions in $(2N)$ -dimensional phase space and the quantum $(N + 1)$ marginal theorem\*.

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## Abstract

We study the problem of constructing a probability density in  $2N$ -dimensional phase space which reproduces a given collection of  $n$  joint probability distributions as marginals. Only distributions authorized by quantum mechanics, i.e. depending on a (complete) commuting set of  $N$  variables, are considered. A diagrammatic or graph theoretic formulation of the problem is developed. We then exactly determine the set of “admissible” data, i.e. those types of data for which the problem always admits solutions. This is done in the case where the joint distributions originate from quantum mechanics as well as in the case where this constraint is not imposed. In particular, it is shown that a necessary (but not sufficient) condition for the existence of solutions is  $n \leq N + 1$ . When the data are admissible and the quantum constraint is not imposed, the general solution for the phase space density is determined explicitly. For admissible data of a quantum origin, the general solution is given in certain (but not all) cases. In the remaining cases, only a subset of solutions is obtained.

## I. Introduction

In quantum theory, we usually assume that probability densities for eigenvalues of two noncommuting observables cannot be measured by the same experimental set up. A related theoretical question is : does there exist a joint probability distribution of the eigenvalues of two such observables  $A$  and  $B$  which correctly reproduces the individual probabilities for  $A$  and for  $B$  as marginals (i.e., on integration over the eigenvalues of the other observable). Perhaps surprisingly, the answer to this question is yes. More generally, for a system with  $N$  configuration space variables  $q_1, q_2, \dots, q_N$ , consider complete commuting sets (CCS)  $S_1, S_2, \dots, S_n$  of observables, each  $S_i$  consisting of some coordinate and some momentum variables (where  $S_i$  and  $S_j$  must contain some mutually non-commuting observables to be considered distinct CCS). Is there a joint probability density  $\rho(q_1, \dots, q_N, p_1, \dots, p_N)$  whose marginals reproduce the quantum probability densities of the different CCS,  $S_1, S_2, \dots, S_N$ ? We shall prove that a necessary condition for this to be possible for arbitrary quantum states is  $n \leq N + 1$ ; this result is a precise no go theorem for simultaneous realization of more than  $(N + 1)$  quantum marginals.

Actually, this no go theorem also has applications to the classical arena of joint time-frequency distributions in signal processing and joint position-wave number distributions in image processing. It is therefore useful to state the problem in a general setting encompassing both classical and quantum mechanics.

Consider the general problem of reconstructing a probability density over  $\mathbb{R}^M$ , given a set of  $n$  associated joint probability distributions over subspaces of  $\mathbb{R}^M$ . In this general setting the problem can be stated as follows. Suppose first that a probability density  $\rho(y_1, \dots, y_M)$  is given and define the marginal distributions

$$\sigma_\alpha(Y_\alpha) = \int dY'_\alpha \rho(y_1, \dots, y_M) \quad (\alpha = 1, \dots, n), \quad (\text{I.1})$$

where  $Y_\alpha \cup Y'_\alpha$  is, for each  $\alpha$ , a partition of  $\{y_1, \dots, y_M\}$ . These joint distributions obey a set of compatibility conditions. Indeed, let  $Y_{\alpha\beta}$  be the set of variables that  $\sigma_\alpha(Y_\alpha)$  and  $\sigma_\beta(Y_\beta)$  have in common and introduce the partitions  $Y_\alpha = Y_{\alpha\beta} \cup Y'_{\alpha\beta}$  and  $Y_\beta = Y_{\alpha\beta} \cup Y''_{\alpha\beta}$ . Then eqs. (I.1) imply

$$\int dY'_{\alpha\beta} \sigma_\alpha(Y_{\alpha\beta}, Y'_{\alpha\beta}) = \int dY''_{\alpha\beta} \sigma_\beta(Y_{\alpha\beta}, Y''_{\alpha\beta}). \quad (\text{I.2})$$

Conversely, suppose that a set  $\{\sigma_1, \dots, \sigma_n\}$  of joint probability distributions is given, which satisfies the compatibility conditions (I.2). Is it always possible to find some probability density  $\rho$  which reproduces them as marginals, in accordance with equations (I.1)? In the affirmative, how can we “reconstruct” such  $\rho$ ’s? It turns out that eqs. (I.2) are in general only necessary conditions for the existence of a positive density  $\rho$ , and our problem is precisely to solve the questions of existence, multiplicity and explicit determination of the  $\rho$ ’s (if any).

Actually, we address these questions not in such a general setting, but in the case where  $\mathbb{R}^M = \mathbb{R}^{2N}$  is the phase space of some physical system with coordinates  $y_j$  identified with conjugate canonical variables  $\{q_j, p_j\}_{j=1}^N$  and where all distributions  $\sigma_\alpha$  depend on exactly  $N$  variables  $x_1, \dots, x_N$  restricted by the condition  $x_i = q_i$  or  $p_i$

( $i = 1, \dots, N$ ). The reason for this choice lies in the motivation of the problem within the context of quantum mechanics. Indeed, the functions  $\sigma_\alpha$  just introduced can then be understood as quantum probability distributions associated with  $n$  complete commuting sets of observables (CCS), selected among  $2^N$  possible choices (the  $2^N$  possible assignments of the variables  $x_i$ ). By “quantum probability distributions” is meant here a set of functions  $\sigma_\alpha(x_1, \dots, x_N)$  derived from a common wave function  $\langle q_1, \dots, q_N | \psi \rangle$  (in the Schrödinger representation), in accordance with the formula

$$\sigma_\alpha(x_1, \dots, x_N) = |\langle x_1, \dots, x_N | \psi \rangle|^2 \quad (\alpha = 1, \dots, n), \quad (\text{I.3})$$

or more generally, for a mixed quantum state described by the density operator  $\hat{\rho}$ ,

$$\sigma_\alpha(x_1, \dots, x_N) = \langle x_1, \dots, x_N | \hat{\rho} | x_1, \dots, x_N \rangle \quad (\alpha = 1, \dots, n). \quad (\text{I.4})$$

The problem in this physical framework is directly related to the construction of “maximally realistic quantum mechanics”, a program initiated by S.M. Roy and V. Singh in 1995 [1] and intensively pursued since then [2]-[4]. Without entering a detailed discussion of this relationship from the viewpoint of quantum physics (for which we refer the reader to [1]-[4]), let us recall the main results gained so far. In [2] it was shown that for any  $N \geq 2$ , a set of  $n = N+1$  quantum probability distributions of the special form

$$\{\sigma_1(q_1, q_2, \dots, q_N), \sigma_2(p_1, q_2, \dots, q_N), \sigma_3(p_1, p_2, q_3, \dots, q_N), \dots, \sigma_{N+1}(p_1, p_2, \dots, p_N)\} \quad (\text{I.5})$$

can be realized as a set of marginals of a common phase space probability density  $\rho(q_1, \dots, q_N, p_1, \dots, p_N)$ . Further the “no go” conjecture was made that for  $n \geq N+2$  (and for any choice of  $n$  distinct CCS), there exist sets  $\{\sigma_1, \dots, \sigma_n\}$  of quantum probability distributions which cannot be recovered as marginals of some  $\rho$ . The determination of the most general density  $\rho$  reproducing the set (I.5), as well as the status of sets of CCS-distributions different from (I.5) and not necessarily of a quantum origin (i.e. mutually compatible but not necessarily construed according to eqs. (I.4)), were left as open questions. In [4], hereafter denoted by (I), complete answers to these questions were given in the special case  $N = 2$  (see also [3] for a brief summary). In particular, the general positive solution  $\rho$  of the equations

$$\begin{cases} \int dp_1 dp_2 \rho(q_1, q_2, p_1, p_2) = \sigma_1(q_1, q_2), \\ \int dq_1 dp_2 \rho(q_1, q_2, p_1, p_2) = \sigma_2(p_1, q_2), \\ \int dq_1 dq_2 \rho(q_1, q_2, p_1, p_2) = \sigma_3(p_1, p_2), \end{cases} \quad (\text{I.6})$$

was worked out for an arbitrary set  $\{\sigma_1, \sigma_2, \sigma_3\}$  (quantum or not) and the “no go” conjecture stated above was proved for  $N = 2$ . In fact, it was shown that a necessary and sufficient condition for  $n$  ( $\leq 4$ ) arbitrarily given compatible  $\sigma_\alpha$ ’s to be marginals of a probability density in 4-dimensional phase space is  $n \leq 3$  (“Three marginal theorem”). These results were obtained by first deriving certain correlation inequalities between the  $\sigma_\alpha$ ’s from the mere existence of a positive and normalized  $\rho$ . Such inequalities, which are the analogues in phase space of the standard Bell inequalities for

spin variables [5], turn out to have an interest of their own in the context of quantum physics, as discussed in [3]-[4].

The generalization of the study performed in (I) to the case of an arbitrary number  $n$  of CCS-distributions of any species in a phase space of arbitrary dimension  $2N$ , which is precisely the aim of the present work, is not a straightforward task. It will be accomplished by means of two main tools : the Bell-like inequalities just mentioned (it turns out that no new correlation inequalities, proper to the  $2N$ -dimensional case, are needed for the present purpose) and a specific diagrammatic formulation of the problem which appears essential both for a concise exposition of our final statements and for their proof. In this way, we shall be able to treat the problem exhaustively and to give, in the general case, definite answers to the questions previously posed, in the form of a clear-cut theorem. This theorem will be stated at once in section II (Theorem 1), after having introduced a set of appropriate definitions. As a by-product, the theorem affords a proof of the “no go” conjecture for any  $N$  (Theorem 2). On the positive side, it considerably extends early results of Cohen and Zaporovanny [6] for two marginals with non intersecting sets of variables by simultaneous realizability of  $N+1$  marginals which have intersecting sets of variables as well. The rest of the paper (sections III to V) is almost entirely devoted to the (quite long!) proof of Theorem 1, and is therefore mainly technical. Our concluding comments are presented in section VI.

## II. Definitions and results

In order to give our results a precise and unambiguous form, we introduce the following definitions:

1. A **CCS-distribution** (CCS for Complete Commuting Set) in  $N$  dimensions is a probability distribution<sup>1</sup>  $\sigma(x_1, \dots, x_N)$ , with  $x_j = q_j$  or  $p_j$  for each index  $j$ .  
The CCS-distributions can occur in  $2^N$  different **types**, each **type** corresponding to one choice of the  $N$ -tuple of arguments.
2. An  **$n$ -chain** is a set  $\{\sigma_1, \dots, \sigma_n\}$  of *mutually compatible* CCS-distributions of distinct types. Here, the mutual compatibility conditions (I.2) read, for any pair  $\{\sigma_\alpha(Y_\alpha), \sigma_\beta(Y_\beta)\}$ , where  $Y_\alpha = \{y_1, \dots, y_r, Y\}$ ,  $Y_\beta = \{y'_1, \dots, y'_r, Y\}$  and  $y'_j$  is the conjugate of  $y_j$  ( $y'_j = q_j$  or  $p_j$  according as  $y_j = p_j$  or  $q_j$ ),

$$\int dy_1 \dots dy_r \sigma_\alpha(y_1, \dots, y_r, Y) = \int dy'_1 \dots dy'_r \sigma_\beta(y'_1, \dots, y'_r, Y), \quad (\text{II.1})$$

Thus, an  $n$ -chain is a possible candidate for a set of  $n$  marginals, i.e. joint probability distributions obtained from some phase space probability distribution<sup>2</sup>  $\rho(q_1, \dots, q_N, p_1, \dots, p_N)$  by integrating over some of the arguments.

The type of an  $n$ -chain is defined by the types of its elements.

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<sup>1</sup>Here and in the following, probability distributions are understood as positive normalized measures, with an absolutely continuous part and (possibly) Dirac measures.

<sup>2</sup>We shall use sometimes the notation  $\rho(\vec{q}, \vec{p})$ .

3. An  $n$ -chain is **admissible** if there exists at least one phase space probability distribution  $\rho$  reproducing all the CCS distributions of the  $n$ -chain, namely such that

$$\sigma_\alpha(x_1, \dots, x_N) = \int dx'_1 \dots dx'_N \rho(q_1, \dots, q_N, p_1, \dots, p_N), \quad (\alpha = 1, \dots, n) \quad (\text{II.2})$$

where  $x'_i$  is the conjugate of  $x_i$ . Eqs. (II.2) imply (II.1). Using the notation  $Z_\alpha = \{x_1, \dots, x_N\}$ ,  $Z'_\alpha = \{x'_1, \dots, x'_N\}$ , and  $d^N Z'_\alpha = dx'_1 \dots dx'_N$ , (II.2) can be rewritten as

$$\sigma_\alpha(Z_\alpha) = \int d^N Z'_\alpha \rho(\vec{q}, \vec{p}), \quad (\alpha = 1, \dots, n)$$

4. An  $n$ -chain is a **quantum  $n$ -chain** if there exists at least one quantum state described by the density operator  $\hat{\rho}$  such that eqs. (I.4) hold.

Note that in that case the compatibility conditions (II.1) are automatically satisfied.

5. Two CCS-distributions  $\sigma_\alpha$  and  $\sigma_\beta$  are **contiguous** if they differ by the assignment of only one variable  $x_i$  (to  $q_i$  and  $p_i$ ), namely

$$\sigma_\alpha(x_1, \dots, x_{i-1}, q_i, x_{i+1}, \dots, x_N) \quad \text{and} \quad \sigma_\beta(x_1, \dots, x_{i-1}, p_i, x_{i+1}, \dots, x_N).$$

We call  $i$  the **index** of the pair  $\{\sigma_\alpha, \sigma_\beta\}$ .

6. To each  $n$ -chain we associate a **graph** which is constructed as follows:

- to every CCS-distribution of the chain, associate a **vertex** characterized by the collection of the variables of this distribution,
- connect two vertices by a **link** if they correspond to contiguous CCS-distributions. We call **index** of the link the index of the pair of contiguous CCS-distributions, and we say that the two vertices are contiguous.

As usual, there are connected and disconnected graphs, tree graphs and graphs with loops. A graph  $G$  completely determines the type of the associated  $n$ -chain, so that we can speak of a **chain of type  $G$** .

7. An  $n$ -chain and its associated graph are said to be **proper** if no two links have the same index.

Since there are at most  $N$  possible indices, a proper graph has at most  $N$  links, and thus, if it is connected, at most  $(N + 1)$  vertices. Furthermore, a graph with a loop cannot be proper. Therefore, a *connected proper graph is necessarily a tree graph with at most  $(N + 1)$  vertices*.

8. A graph  $G$  is **fully admissible** if all  $n$ -chains of type  $G$  are admissible.

A graph  $G$  is **quantum admissible** if all quantum  $n$ -chains of type  $G$  are admissible.

Full admissibility entails quantum admissibility.

A graph  $G$  is **non admissible** if it is not quantum (and a fortiori not fully) admissible.

9. Let a non connected graph  $G$  be subgraph of a connected graph  $G_c$ . We call **insertions** the vertices of  $G_c$  which are not vertices of  $G$ .  $G_c$  is called  **$G$ -simple** if all its insertions have only two legs.

In the following, in order to shorten the writing, particularly when drawing graphs, we often replace the variables  $q_j$  and  $p_j$  by the indices  $j$  and  $j'$  respectively. For example, to a CCS-distribution  $\sigma(q_1, p_2, q_3)$  we associate the vertex  $\boxed{12'3}$  in a rectangle box. As for the inserted vertices, we use instead round boxes, e.g.  $(12'3)$ .

We are now in a position to give a complete characterization of the graphs, or equivalently of the types of  $n$ -chains, according to their (full, quantum or non) admissibility.

### Theorem 1

1. If a graph  $G$  is proper and connected, then it is fully admissible.
2. If  $G$  is proper but non connected, then
  - a. if  $G$  is subgraph of a proper connected graph  $G_c$ , then
    - i. if  $G_c$  is  $G$ -simple,  $G$  is fully admissible,
    - ii. if  $G_c$  is not  $G$ -simple,  $G$  is quantum, but not fully, admissible.
  - b. if  $G$  is not subgraph of a proper connected graph  $G_c$ , then it is non admissible.
3. If  $G$  is non proper, then it is non admissible.

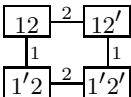
This theorem is complemented by the explicit construction of all the phase space distributions  $\rho$  reproducing a given chain of type  $G$ , a construction which is completed only in the case of full admissibility, that is to say when  $G$  is either connected and proper, or subgraph of a connected, proper and  $G$ -simple graph  $G_c$  (see section IV-B). In the case of quantum admissibility, when the connected graph  $G_c$  is proper but not  $G$ -simple, the situation is not as favorable and the general expression of  $\rho$  is not known (see section V-B-2).

Remarks:

1. Given a graph  $G$ , a proper  $G_c$ , when there is one (case 2.a), is in general not unique. It is a consequence of the theorem that either all the proper  $G_c$ 's are  $G$ -simple, or none of them is.

Notice that this is a pure graph theoretic statement.

2. For  $N = 2$ , the main result of section IV of (I), which was derived from Bell-like correlation inequalities, is the following : both in the classical and quantum cases, there exist 4-chains  $\sigma_1(q_1, q_2), \sigma_2(p_1, q_2), \sigma_3(q_1, p_2)$  and  $\sigma_4(p_1, p_2)$  which cannot be reproduced as marginals of any probability distribution  $\rho(\vec{q}, \vec{p})$ . In the language of the present paper, this can be rephrased as :

The non proper graph  is non admissible.

This is the simplest case of part 3 of the above theorem, and actually it is the clue of its proof.

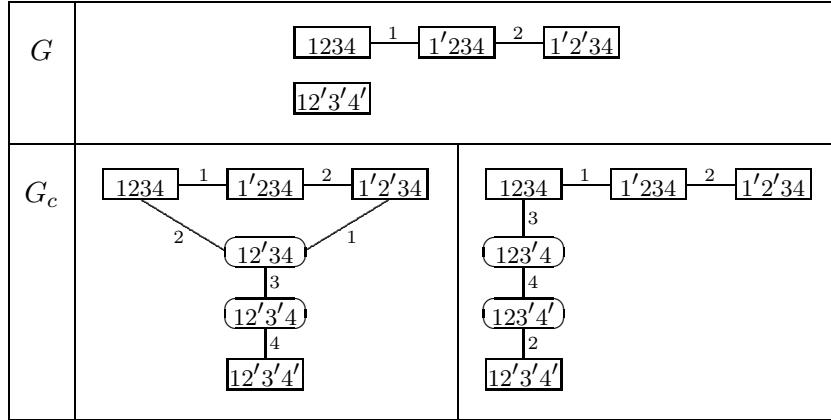


FIGURE 1: A non connected proper graph  $G$  which is not subgraph of any proper connected graph  $G_c$  :  $G$  is non admissible. Two possible (non proper)  $G_c$ 's are shown : one with a loop, the other one a tree graph.

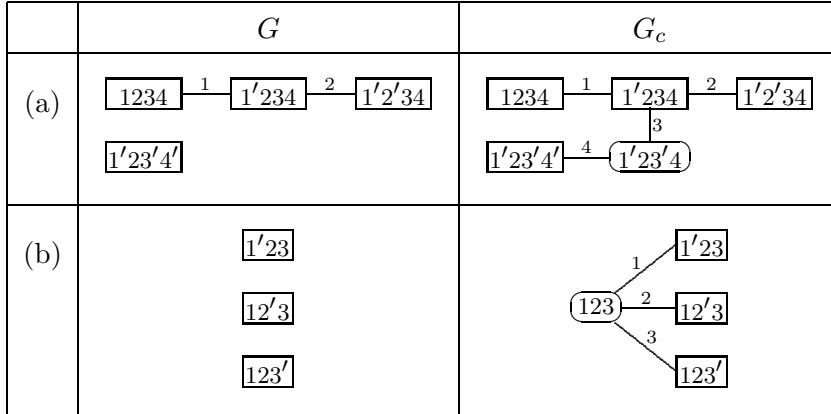


FIGURE 2: (a) a non connected proper graph  $G$  which is subgraph of a proper connected  $G$ -simple graph  $G_c$  :  $G$  is fully admissible. (b) a non connected proper graph  $G$  which is subgraph of a proper connected non  $G$ -simple graph  $G_c$  :  $G$  is quantum but not fully admissible.

An immediate corollary of Theorem 1 is

**Theorem 2 (N + 1 Marginal Theorem)**

*A necessary condition for all quantum  $n$ -chains of a given type to be admissible is  $n \leq N + 1$ .*

It suffices to use parts 2.b and 3 of Theorem 1 and to note that  $G$  cannot have more vertices than  $G_c$ , and that a proper  $G_c$  has at most  $N + 1$  vertices (see the

remark after the above definition 7 of proper  $n$ -chains and graphs).

Notice that, in contradistinction with the *three marginal theorem* of (I), the above theorem gives only a necessary condition. This is because only proper connected graphs  $G_c$  are involved when  $N = 2$  and  $n \leq N + 1$ , whereas non proper connected  $G_c$ 's do appear as soon as  $N \geq 3$ . Of course, for proper connected graphs  $G$  for which  $n$  must be  $\leq N + 1$ , part 1 of Theorem 1 guarantees admissibility for  $N \geq 3$  also.

Theorem 1 will be proved in sections III to V. In order to help understanding its content, we give in Figs. 1 and 2 examples of the various cases encountered.

### III. Non proper graphs

Consider a non proper graph  $G$ . By definition, there exist at least two links with the same index (say 1) connecting a first pair of vertices  $(V, V')$  and a second pair  $(W, W')$ . Then there necessarily exists a second index (say 2) such that the variables  $x_1$  and  $x_2$  have in the four vertices  $V, V', W$  and  $W'$  the assignments as shown in Fig. 3.



FIGURE 3: *Critical quartet in a non proper graph.*

Quite generally, in a graph  $G$  (proper or not), a set of four vertices where a pair of variables takes the four possible assignments will be called a *critical quartet*.

We now prove

**Lemma 1** *A graph  $G$  containing a critical quartet is non admissible.*

Given an  $n$ -chain  $\{\sigma_\alpha\}_{\alpha=1,\dots,n}$  of type  $G$  in  $N$  dimensions and a partition of the set of indices  $\{1, \dots, N\} = J \cup K$ , let us introduce the distributions

$$\tilde{\sigma}_\alpha(X_J) = \int \prod_{k \in K} dx_k \sigma_\alpha(X_J, X_K).$$

Some of these  $\tilde{\sigma}_\alpha$ 's may coincide. We call  $J$ -reduced  $n'$ -chain the maximal set of  $n'$  distinct  $\tilde{\sigma}_\alpha$ 's ( $n' \leq n$ ). Obviously, a necessary condition for the  $n$ -chain to be admissible is that the associated  $J$ -reduced  $n'$ -chain be admissible. Now, consider a quantum  $n$ -chain constructed with a factorized wave function of the form

$$\Psi(q_1, \dots, q_N) = \Psi_1(Q_J) \Psi_2(Q_K). \quad (\text{III.1})$$

Choosing then  $J = \{1, 2\}$ , to be the indices of a critical quartet, our results in (I) (see Remark 2 after Theorem 1) immediately imply the non-admissibility of the  $J$ -reduced 4-chain. Hence, the non-admissibility of the  $n$ -chain  $\sigma_\alpha$  itself, and Lemma 1 follows.

This establishes part 3 of Theorem 1, namely that a non proper graph  $G$  is non admissible.

## IV. Connected proper graphs

Let  $G$  be a connected proper graph. We establish part 1 of Theorem 1 by associating to any chain of type  $G$  a particular phase space distribution  $\rho_0$  reproducing this chain. The explicit construction of such a  $\rho_0$  is described in section IV-A. In section IV-B, we derive the expression of the most general phase space distribution reproducing the given chain.

### IV-A. Particular solution

Let  $\mathcal{C}_n = \{\sigma_1, \dots, \sigma_n\}$  be an  $n$ -chain of type  $G$ . Since  $G$  is a proper tree graph with  $(n-1)$  links, there are exactly  $(N-n+1)$  variables which have the same assignments in all the distributions  $\sigma_\alpha$ . After a possible renumbering of the indices of the  $x_i$ 's, we can therefore assume that in  $\sigma_\alpha(x_1, \dots, x_{n-1}, x_n, \dots, x_N)$  the assignment of each of the variables  $x_n, \dots, x_N$  is independent of  $\alpha$ . These variables, which will play a purely passive role, are henceforth denoted collectively by  $T$ , whereas  $T'$  will stand for the set of conjugate variables  $\{x'_n, \dots, x'_N\}$ .

The solution  $\rho_0(q_1, \dots, q_N, p_1, \dots, p_N)$  of eqs. (II.2) is constructed as the product of “vertex functions”, “propagators” and an arbitrary positive function of  $T'$ . The former elements are defined by the following “Feynman rules”:

- 1) to each vertex  $\boxed{x_1, \dots, x_N}$  of  $G$ , we associate the **vertex function**  $\sigma_\alpha(x_1, \dots, x_N)$  of the chain  $\mathcal{C}_n$ ,
- 2) for each link  $l_i$  of  $G$  carrying the index  $i$ , by using the compatibility condition (II.1) for the pair  $\{\sigma_{\alpha_i}, \sigma_{\beta_i}\}$  of contiguous CCS-distributions attached to this link, we define the integrated distribution

$$\begin{aligned} \sigma_{\alpha_i \beta_i}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) &= \int dq_i \sigma_{\alpha_i}(x_1, \dots, x_{i-1}, q_i, x_{i+1}, \dots, x_N), \\ &= \int dp_i \sigma_{\beta_i}(x_1, \dots, x_{i-1}, p_i, x_{i+1}, \dots, x_N). \end{aligned} \tag{IV.1}$$

Then, to the link  $l_i$  we associate the **propagator**

$$\varpi_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \equiv \begin{cases} \frac{1}{\sigma_{\alpha_i \beta_i}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)} \\ \quad \text{if } (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in \mathcal{S}_{\alpha_i \beta_i}, \\ 0 \quad \text{otherwise,} \end{cases} \tag{IV.2}$$

where  $\mathcal{S}_{\alpha_i \beta_i}$  is the (essential) support of  $\sigma_{\alpha_i \beta_i}$ .

The support properties of the  $\sigma_\alpha$ 's,  $\sigma_{\alpha\beta}$ 's and  $\rho_0$ , and the relations between them (due to compatibility and positivity) are not innocent in the forthcoming considerations, and we should pay attention to them. However, doing so leads to cumbersome technicalities which are in fact straightforward generalizations of those developed in the rigorous proof of Theorem 1 in (I) for the case  $N = 2$ . Thus, in this section IV, we shall ignore such inessential complications.

The function  $\rho_0$  is then written as

$$\rho_0 = \left( \prod_{\alpha=1}^n \sigma_\alpha \prod_{i=1}^{n-1} \varpi_i \right) \zeta, \quad (\text{IV.3})$$

where  $\zeta(T')$  is an arbitrary non negative function in  $L^1(\mathbb{R}^{N-n+1}, d^{N-n+1}T')$  with normalization

$$\int d^{N-n+1}T' \zeta(T') = 1. \quad (\text{IV.4})$$

That the expression (IV.3) for  $\rho_0$  solves the equations (II.2) results from the following property:

Let  $\hat{V} = [x_1, \dots, x_N]$  be a one-leg vertex of the proper tree graph  $G$  and  $i$  be the index of the link  $\hat{l}$  attached to it. Let  $\hat{\sigma}(x_1, \dots, x_N)$  be the corresponding element of the chain  $\mathcal{C}_n$ . Then

$$\begin{aligned} \int dq_i \rho_0^{(n)}(\vec{q}, \vec{p}) &= \rho_0^{(n-1)}(q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_N, p_1, \dots, p_N) \\ &\quad \text{if } x_i = q_i, \\ \int dp_i \rho_0^{(n)}(\vec{q}, \vec{p}) &= \rho_0^{(n-1)}(q_1, \dots, q_N, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_N) \\ &\quad \text{if } x_i = p_i, \end{aligned} \quad (\text{IV.5})$$

where  $\rho_0^{(n)}$  is the above defined  $\rho_0$  associated to the  $n$ -chain  $\mathcal{C}_n$ , and  $\rho_0^{(n-1)}$  is the  $\rho_0$  similarly associated to the  $(n-1)$ -chain  $\mathcal{C}_{n-1}$  obtained from  $\mathcal{C}_n$  by removing the CCS-distribution  $\hat{\sigma}$ . Notice that the reduced chain  $\mathcal{C}_{n-1}$  corresponds to the reduced graph  $G^{(n-1)}$  obtained from  $G$  by removing the one-leg vertex  $\hat{V}$  and the link  $\hat{l}$ . Hence  $\rho_0^{(n-1)} = \rho_0 / (\hat{\sigma} \varpi_i)$ .

Equation (IV.5) holds because: *i*) in  $\rho_0^{(n)}$  the variable  $x_i$  appears only in the factor  $\hat{\sigma}$ , *ii*) the propagator of the link  $\hat{l}$  is precisely the inverse of the integral of  $\hat{\sigma}$  over  $x_i$ .

Now, given any  $\sigma_\alpha$  in the chain  $\mathcal{C}_n$ , corresponding to the vertex  $V_\alpha$  of  $G$ , one can start the reduction  $G^{(n)} \rightarrow G^{(n-1)}$  of the tree graph  $G^{(n)} \equiv G$  at some arbitrarily chosen one-leg vertex  $\hat{V} \neq V_\alpha$ , and repeat it  $(n-1)$  times in such a way that one is left with the graph  $G^{(1)}$  consisting solely of the vertex  $V_\alpha$ . To this “peeling process”  $G^{(n)} \rightarrow G^{(n-1)} \rightarrow \dots \rightarrow G^{(1)}$  of tree graphs is naturally associated, via eqs. (IV.5), a reduction  $\rho_0 \equiv \rho_0^{(n)} \rightarrow \rho_0^{(n-1)} \rightarrow \dots \rightarrow \rho_0^{(1)}$  of functions  $\rho_0^{(m)}$ , which eventually produces

$$\rho_0^{(1)}(x_1, \dots, x_{n-1}, T, T') = \int dx'_1 \dots dx'_{n-1} \rho_0 = \sigma_\alpha(x_1, \dots, x_{n-1}, T) \zeta(T'), \quad (\text{IV.6})$$

and hence, by integrating over  $T'$  :

$$\int dx'_1 \dots dx'_N \rho_0 = \sigma_\alpha(x_1, \dots, x_N). \quad (\text{IV.7})$$

Notice that, although the order of the repeated integrations over the  $x'_i$ ’s is imposed by the steps of the peeling process, this order becomes obviously irrelevant in the above equation: the equations (II.2) are valid for  $\rho = \rho_0$  indeed.

Finally, to illustrate our Feynman rules, let us take as an example the graph  $G^{(5)}$  of Fig. 4. The distribution  $\rho_0$  associated to any 5-chain of type  $G^{(5)}$  is

$$\begin{aligned} \rho_0(\vec{q}, \vec{p}) &= \sigma_1(q_1 q_2 q_3 q_4) \frac{1}{\sigma_{12}(q_2 q_3 q_4)} \sigma_2(p_1 q_2 q_3 q_4) \frac{1}{\sigma_{23}(p_1 q_3 q_4)} \sigma_3(p_1 p_2 q_3 q_4) \\ &\quad \frac{1}{\sigma_{24}(p_1 q_2 q_4)} \sigma_4(p_1 q_2 p_3 q_4) \frac{1}{\sigma_{45}(p_1 q_2 p_3)} \sigma_5(p_1 q_2 p_3 p_4). \end{aligned}$$

Here and in the sequel, we keep writing the propagators as  $1/\sigma_{\alpha\beta}$ , although they are strictly given by eq. (IV.2).

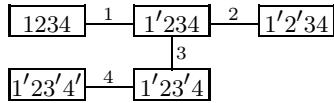


FIGURE 4: a proper connected graph with  $N=4$  and  $n=5$ .

#### IV-B. General solution

Let us define the  $n$  following positive measures, each one associated to a particular component  $\sigma_\alpha$  of the  $n$ -chain  $\mathcal{C}_n$

$$d\mu_\alpha = \begin{cases} \frac{\rho_0(\vec{q}, \vec{p})}{\sigma_\alpha(Z_\alpha)} d^N Z'_\alpha & \text{if } Z_\alpha = (x_1, \dots, x_N) \in \mathcal{S}_\alpha, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{IV.8})$$

where  $\mathcal{S}_\alpha$  denotes the (essential) support of  $\sigma_\alpha$ . Due to eq. (IV.7) these  $(x_1, \dots, x_N)$ -dependent measures<sup>3</sup> are normalized for all  $(x_1, \dots, x_N) \in \mathcal{S}_\alpha$ .

It is convenient to write the general solution  $\rho$  we are looking for in the form

$$\rho = \rho_0(1 + \lambda h). \quad (\text{IV.9})$$

Here, the function  $h(\vec{q}, \vec{p})$  will be chosen as to ensure eq. (II.2), which results in the linear equations

$$\int d^N Z'_\alpha \rho_0(\vec{q}, \vec{p}) h(\vec{q}, \vec{p}) = 0, \quad (\alpha = 1, \dots, n), \quad (\text{IV.10})$$

whereas the real constant  $\lambda$  is a normalisation factor which will be useful to control the positivity of  $\rho$ . Thanks to definitions (IV.8), eqs. (IV.10) are equivalent to

$$\int d\mu_\alpha h = 0 \quad (\alpha = 1, \dots, n). \quad (\text{IV.11})$$

We then observe that, for any  $\alpha$  and any function  $g$  in  $L^1(\mathbb{R}^{2N}, \rho_0 d^N q d^N p)$

$$\int d\mu_\alpha \left( \int d\mu_\alpha g \right) = \int d\mu_\alpha g, \quad (\text{IV.12})$$

---

<sup>3</sup>Notice also that the  $d\mu_\alpha$ 's are continuous linear mappings of  $L^1(\mathbb{R}^{2N}, \rho_0 d^N q d^N p)$  into itself.

since  $\int d\mu_\alpha g$  does not depend any longer on the integration variables  $(x'_1, \dots, x'_N)$  and  $d\mu_\alpha$  is normalized. That is, the linear operators  $P_\alpha : L^1(\mathbb{R}^{2N}, \rho_0 d^N q d^N p) \rightarrow L^1(\mathbb{R}^{2N}, \rho_0 d^N q d^N p)$  defined by  $P_\alpha g = \int d\mu_\alpha g$  are projectors,

$$P_\alpha^2 = P_\alpha \quad (\alpha = 1, \dots, n). \quad (\text{IV.13})$$

The set  $\{P_\alpha\}$  enjoys certain algebraic properties which are crucial for the construction of the general solution  $h$  of eqs. (IV.11) :

**Lemma 2**

a) *The projectors  $P_\alpha$  and  $P_\beta$  associated with any pair  $\{V_\alpha, V_\beta\}$  of contiguous vertices commute*

$$[P_\alpha, P_\beta] = 0. \quad (\text{IV.14})$$

b) *If  $V_\alpha$ ,  $V_\beta$  and  $V_\gamma$ , are three vertices of the connected proper (tree) graph  $G$  such that  $V_\alpha$  belongs to the (unique) path connecting  $V_\beta$  to  $V_\gamma$ , and is contiguous to at least one of these two vertices, then*

$$P_\gamma P_\alpha P_\beta = P_\gamma P_\beta. \quad (\text{IV.15})$$

The proof is given in Appendix A. We stress that the contiguity of  $V_\alpha$  with  $V_\beta$  or  $V_\gamma$  is essential for the validity of eq. (IV.15).

Let us now introduce the central object of our construction, namely the operator

$$\Pi = 1 - \sum_{\alpha=1}^n P_\alpha + \sum_{i=1}^{n-1} P_{\alpha_i} P_{\beta_i}, \quad (\text{IV.16})$$

where  $P_{\alpha_i}$  and  $P_{\beta_i}$  denote the operators  $P$  associated with the two (contiguous) vertices  $V_{\alpha_i}$  and  $V_{\beta_i}$  attached to the link with index  $i$ . Thanks to Lemma 2, it is readily shown that  $\Pi$  is annihilated by all the projectors  $P_\alpha$  :

$$P_\gamma \Pi = 0 \quad (\gamma = 1, \dots, n). \quad (\text{IV.17})$$

Indeed :

$$\begin{aligned} P_\gamma \Pi &= P_\gamma - \sum_{\alpha=1}^n P_\gamma P_\alpha + \sum_{i=1}^{n-1} P_\gamma P_{\alpha_i} P_{\beta_i}, \\ &= -P_\gamma \sum_{\substack{\alpha=1 \\ \alpha \neq \gamma}}^n P_\alpha + \sum_{i=1}^{n-1} P_\gamma P_{\alpha_i} P_{\beta_i}. \end{aligned} \quad (\text{IV.18})$$

But, according to eqs. (IV.15) :

$$P_\gamma P_{\alpha_i} P_{\beta_i} = P_\gamma P_{\delta_i}, \quad (\text{IV.19})$$

where  $\delta_i = \beta_i$  (resp.  $\alpha_i$ ) if  $V_{\alpha_i}$  (resp.  $V_{\beta_i}$ ) belongs to the path connecting  $V_\gamma$  to  $V_{\beta_i}$  (resp.  $V_{\alpha_i}$ ). Hence

$$\sum_{i=1}^{n-1} P_\gamma P_{\alpha_i} P_{\beta_i} = P_\gamma \sum_{\substack{\alpha=1 \\ \alpha \neq \gamma}}^n P_\alpha, \quad (\text{IV.20})$$

which entails eq. (IV.17) (the property  $\Pi P_\gamma = 0$  ( $\gamma = 1, \dots, n$ ), which also holds as a consequence of eq. (IV.15), will not be used here). Note that eq. (IV.20) would not be valid if the graph  $G$  were not connected.

Furthermore,  $\Pi$  is itself a projector :

$$\Pi^2 = \Pi, \quad (\text{IV.21})$$

as immediately deduced from

$$\Pi^2 = (1 - \sum_{\alpha=1}^n P_\alpha + \sum_{i=1}^{n-1} P_{\alpha_i} P_{\beta_i}) \Pi$$

and eq. (IV.17). This operator allows us to write down at once the general solution of eqs. (IV.11), i.e.

$$P_\alpha h = 0 \quad (\alpha = 1, \dots, n), \quad (\text{IV.22})$$

as

$$h = \Pi f, \quad (\text{IV.23})$$

where  $f$  is an arbitrary function in  $L^1(\mathbb{R}^{2N}, d^N q d^N p)$ . That eq. (IV.23) implies eqs. (IV.22) is trivial due to eq. (IV.17). Conversely, any function  $h$  satisfying eqs. (IV.22) assumes the form (IV.23) : since then  $h = \Pi h$ , it suffices to take  $f = h$ .

We now have to give the representation formula for  $h$  resulting from eqs. (IV.23) and (IV.16) an explicit form in terms of the data of the problem, namely the elements of the chain  $\mathcal{C}_n$ . For this purpose, it is necessary to use appropriate notations. First, we denote by  $Z_\alpha$  the collection of arguments of the vertex function  $\sigma_\alpha$ , and  $Z'_\alpha$  the collection of the conjugate arguments (a notation already used in the definition (IV.8)). Then

$$(P_\alpha f)(Z_\alpha) = \frac{1}{\sigma_\alpha(Z_\alpha)} \int d^N Z'_\alpha \rho_0(\vec{q}, \vec{p}) f(\vec{q}, \vec{p}). \quad (\text{IV.24})$$

Second, we denote by  $\sigma_{\alpha_i}(X_i, x_i)$  and  $\sigma_{\beta_i}(X_i, x'_i)$  the vertex functions of the vertices  $V_{\alpha_i}$  and  $V_{\beta_i}$ , where  $X_i = \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N\}$ . Accordingly, we write  $\rho_0(X_i, X'_i, x_i, x'_i)$  for  $\rho_0(\vec{q}, \vec{p})$ , where  $X'_i = \{x'_1, \dots, x'_{i-1}, x'_{i+1}, \dots, x'_N\}$ , and so on. With these notations

$$(P_{\alpha_i} P_{\beta_i} f)(X_i) = \int d^{N-1} X'_i dx'_i \frac{\rho_0(X_i, X'_i, x_i, x'_i)}{\sigma_{\alpha_i}(X_i, x_i)} (P_{\beta_i} f)(X_i, x'_i), \quad (\text{IV.25})$$

and

$$(P_{\beta_i} f)(X_i, x'_i) = \int d^{N-1} X'_i dx_i \frac{\rho_0(X_i, X'_i, x_i, x'_i)}{\sigma_{\beta_i}(X_i, x'_i)} f(X_i, X'_i, x_i, x'_i). \quad (\text{IV.26})$$

In the r.h.s. of eq. (IV.25), one observes that the integral  $\int d^{N-1} X'_i \rho_0$  can be performed explicitly by means of the “peeling process” described in section IV-A :

$$\int d^{N-1} X'_i \rho_0(X_i, X'_i, x_i, x'_i) = \frac{\sigma_{\alpha_i}(X_i, x_i) \sigma_{\beta_i}(X_i, x'_i)}{\sigma_{\alpha_i \beta_i}(X_i)}. \quad (\text{IV.27})$$

This equation obtains by stopping the peeling process at the reduced graph made of the two vertices  $V_{\alpha_i}$ ,  $V_{\beta_i}$  and the link between them. Here appears the propagator  $1/\sigma_{\alpha_i\beta_i}$  with

$$\sigma_{\alpha_i\beta_i}(X_i) = \int dx_i \sigma_{\alpha_i}(X_i, x_i) = \int dx'_i \sigma_{\beta_i}(X_i, x'_i). \quad (\text{IV.28})$$

Then, inserting eqs. (IV.26) and (IV.27) in eq. (IV.25), one gets, after simplifications,

$$(P_{\alpha_i} P_{\beta_i} f)(X_i) = \int d^{N-1} X'_i dx_i dx'_i \frac{\rho_0(X_i, X'_i, x_i, x'_i)}{\sigma_{\alpha_i\beta_i}(X_i)} f(X_i, X'_i, x_i, x'_i). \quad (\text{IV.29})$$

Finally, collecting eqs. (IV.23), (IV.16), (IV.24) and (IV.29), we obtain the expression of the function  $h$  we were looking for

$$\begin{aligned} h(\vec{q}, \vec{p}) &= f(\vec{q}, \vec{p}) - \sum_{\alpha=1}^n \frac{1}{\sigma_{\alpha}(Z_{\alpha})} \int d^N Z'_{\alpha} \rho_0(\vec{q}, \vec{p}) f(\vec{q}, \vec{p}) \\ &+ \sum_{i=1}^{n-1} \frac{1}{\sigma_{\alpha_i\beta_i}(X_i)} \int d^{N-1} X'_i dx_i dx'_i \rho_0(\vec{q}, \vec{p}) f(\vec{q}, \vec{p}). \end{aligned} \quad (\text{IV.30})$$

Equations (IV.9) and (IV.30) provide us with the general solution  $\rho$  of the linear system (II.2). It remains to enforce the positivity of this solution. Let us denote by  $m_+$  (resp.  $-m_-$ ) the (essential) supremum (resp. infimum) of  $h$ . Because of eq. (IV.10),  $m_+$  and  $m_-$  are strictly positive when  $h$  does not vanish identically. Then, from eq. (IV.9), the condition  $\rho \geq 0$  is equivalent to the condition on the parameter  $\lambda$

$$-\frac{1}{m_+} \leq \lambda \leq \frac{1}{m_-}. \quad (\text{IV.31})$$

We stress that the allowed interval  $[-\frac{1}{m_+}, \frac{1}{m_-}]$  is not zero as soon as the range of the arbitrary function  $f$  is (essentially) bounded. Indeed, assuming that  $A \leq f \leq B$  (almost) everywhere, one finds from eq. (IV.30) that  $m_{\pm} \leq n(B - A)$ .

Equations (IV.9), (IV.30) and (IV.31) for  $\rho$  constitute the generalization of the results in section V of (I) (see eqs. (V.8) to (V.10) there) to phase spaces of arbitrary dimension, in one case of full admissibility (connected proper graphs).

## V. Non connected proper graphs

Throughout this section, devoted to the proof of part 2 of Theorem 1, a vertex (or insertion) with only two legs will be called *simple vertex*.

### V-A. Non proper $G_c$

Our purpose here is to establish part 2.b of Theorem 1. To this end we first construct a particular connected graph  $G_c$  such that the proper graph  $G$  is subgraph of  $G_c$ . By hypothesis  $G_c$  is not proper. We then show that this implies the existence of at least one critical quartet in the graph  $G$ . According to Lemma 1, the statement 2.b immediately follows.

Let  $G = \bigcup_k G_k$  be the decomposition of a proper graph  $G$  into connected components  $G_k$ 's. Each  $G_k$  is a proper graph, hence a tree. We connectify  $G$  recursively according to the following scheme. Assume we have already connectified the components  $G_1, \dots, G_k$  into a connected *diagram*  $\Gamma_k$ . We define  $\Gamma_{k+1}$  as follows. Let us call *segment* a linear chain of inserted simple vertices and links, and define its length by the number of its links. We choose one of the shortest segments which connect  $\Gamma_k$  to the component  $G_r$  ( $r = k+1, \dots$ ). Among the  $G_r$ 's we select one, say  $G_{k+1}$ , which minimizes the length of the attached segment. We call  $\Sigma_k$  this segment. The diagram  $\Gamma_{k+1}$  is defined by  $\Gamma_{k+1} = \Gamma_k \cup G_{k+1} \cup \Sigma_k$ .

Note that

- 1) in the above construction, two contiguous vertices of a  $\Gamma$  are not necessarily linked, so that the *diagrams*  $\Gamma$  are not always *graphs* as defined in section II. The advantage of this construction is that the  $\Gamma_k$ 's are trees.
- 2) the segment  $\Sigma_k$  is attached to  $G_{k+1}$  through a vertex of  $G$  and attached to  $\Gamma_k$  through either a vertex of  $G$  or an inserted vertex of  $\Gamma_k$  (as represented in Fig. 5). In the latter case, this inserted vertex becomes non simple (if it was simple before).

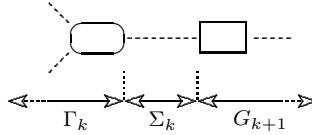


FIGURE 5: the diagram  $\Gamma_{k+1}$

- 3) there is in general some arbitrariness in the construction of the  $\Gamma_k$ 's. First, the recursive process has to be initialized by the choice of one component as  $\Gamma_1$ . Next, in the subsequent steps of the process, there is a possible arbitrariness in the choice of  $G_{k+1}$  and its attached segment.

Once this connectification process is completed, we end up with a connected tree diagram  $\Gamma_c$  which contains all the components of  $G$ . If  $\Gamma_c$  is not a graph, we obtain a graph  $G_c$  by adding links between all pairs of contiguous vertices which are still not linked in  $\Gamma_c$ .

$G_c$  and  $\Gamma_c$  may coincide or not. In the latter case, it is important to notice that  $G_c$  and  $\Gamma_c$  are still either both proper or both non proper. This follows from the fact that i) when going from  $\Gamma_c$  to  $G_c$ , a loop of  $G_c$  is created each time one adds a link, ii) a loop contains at least two pairs of links carrying the same index.

Let us define the diagrams  $\Omega_k$  by

$$\Omega_k = \Gamma_k \cup G_{k+1} \cup G_{k+2} \cup \dots$$

which satisfy the inclusion relations

$$\Omega_1 \equiv G \subset \Omega_2 \subset \dots \subset \Omega_c \equiv \Gamma_c \subset G_c.$$

Now, by hypothesis  $G_c$ , and thus also  $\Omega_c = \Gamma_c$ , are non proper, whereas  $\Omega_1 = G$  is proper. This implies the existence of an integer  $k$  such that  $\Omega_k \subset \Omega_{k+1}$  with  $\Omega_k$

proper and  $\Omega_{k+1}$  non proper. From the observation that

$$\Omega_{k+1} = \Omega_k \cup \Sigma_k,$$

we deduce that there exists at least one index, say 1, which is carried by just two links, one  $l_1$  in  $\Sigma_k$  and a second one  $l'_1$  in  $\Omega_k$ . The link  $l'_1$  may appear either in  $\Gamma_k$  or in the components  $G_{k+1}, \dots$  of  $G$ , which leads us to distinguish three cases :

- a)  $l'_1 \subset G_{k+1}$ ,
- b)  $l'_1 \subset \Gamma_k$ ,
- c)  $l'_1 \subset G_{k+2}$  or  $G_{k+3}$  or ...

We now proceed with a few remarks which will be useful in the forthcoming argument, although not always explicitly referred to thereby :

- 1) All the end points (one leg vertices) of the  $\Gamma_k$ 's belong to  $G$ .
- 2) Any link of a  $\Gamma_k$  belongs to at least one linear chain with end vertices belonging to  $G$ .
- 3) On a segment, the indices of the links can be reordered at our convenience. This should be kept in mind when constructing the  $\Gamma_k$ 's.
- 4) Two links carrying the same index cannot be attached to a common vertex.

As a consequence of these last two remarks, since all the  $\Sigma_k$ 's are shortest connecting chains,

- 5) all segments  $\Sigma_k$  are proper, and
- 6) on a connected tree  $\Gamma_k$ , the (unique) path joining two links carrying the same index contains either two vertices of  $G$ , or one vertex of  $G$  and one inserted non simple vertex, or two inserted non simple vertices.

### Case a)

Let  $V$  be any vertex of  $\Gamma_k$  belonging to  $G$ , and  $V'$  be the vertex of  $G_{k+1}$  where the segment  $\Sigma_k$  attaches. Figure 6 exhibits the relevant part of  $\Omega_{k+1}$ , namely the linear chain joining the vertices  $V$  and  $V'$ , and the linear chain from  $V'$  to the link  $l'_1$  in  $G_{k+1}$ . Moreover, in accordance with Remark 3), we have attached the link  $l_1$  to  $V'$ . We have also called 2 the index of the other link attached to  $V'$ .

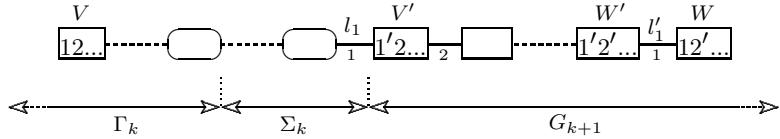


FIGURE 6: Case a)

According to Remarks 3) and 4), there is no link with index 2 in  $\Sigma_k$ . Furthermore, since  $\Gamma_k \cup G_{k+1}$  is proper, no new link of index 1 or 2 can appear in the linear chain

connecting  $V$  to  $W$ . As a consequence, the vertices  $V$ ,  $V'$ ,  $W$  and  $W'$  constitute a critical quartet.

**Case b)**

The relevant part of  $\Omega_{k+1}$  is displayed in Fig. 7. Here,  $V$  is a vertex of  $\Gamma_k$  belonging to  $G$ , such that the (unique) path joining it to  $\Sigma_k$  contains the link  $l'_1$ .

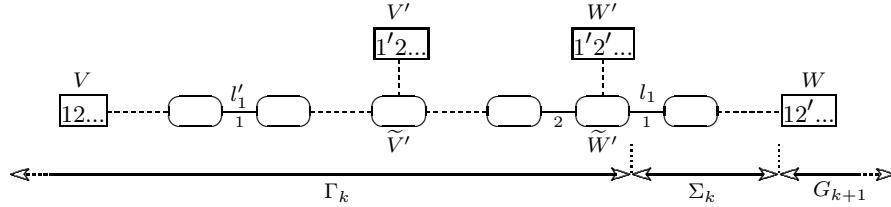


FIGURE 7: Case b)

The existence of the non simple vertices  $\tilde{V}'$  and  $\tilde{W}'$  results from Remark 6). They may possibly coincide with respectively the vertices  $V'$  and  $W'$  of  $G$ . As previously, no new index 1 or 2 can appear in the chain displayed in Fig. 7, which implies that the vertices  $V$ ,  $V'$ ,  $W$  and  $W'$  constitute a critical quartet.

**Case c)**

In that case, the relevant part of  $\Omega_{k+1}$  is made of two disconnected parts, as displayed in Fig. 8. As previously, the non simple inserted vertex  $\tilde{V}$  may possibly coincide with  $V$ .

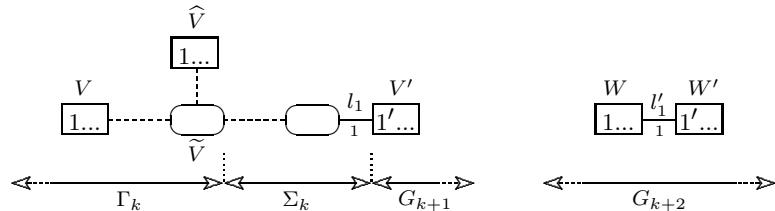


FIGURE 8: Case c)

Let us denote by  $I$  the set of indices appearing (each only once) in the links between  $V$  and  $\tilde{V}$ , by  $J$  the set of indices appearing in the segment  $\Sigma_k$  but the index 1, and by  $K$  all the remaining indices. We can assume  $I \cap J = \emptyset$  (otherwise the configuration would also enter the case b)). We further split the sets  $I$ ,  $J$  and  $K$  as  $I = I_1 \cup I_2$ ,  $J = J_1 \cup J_2$ ,  $K = K_1 \cup K_2$ . Here,  $I_1$  and  $I_2$  are introduced to separate the variables which have the same or different assignments in the vertices  $V$  on the one hand and in the vertices  $W$  and  $W'$  on the other hand, and similarly for the splitting of  $J$  and  $K$ . Then  $\{1, I_1, I_2, J_1, J_2, K_1, K_2\}$  is a partition of  $\{1, 2, \dots, N\}$ , and the assignments of the variables in the vertices  $V$ ,  $V'$ ,  $W$  and  $W'$  of  $G$ , and  $\tilde{V}$

take the form

$$\left\{ \begin{array}{lcl} V & = & \{1 \mathcal{I}_1 \mathcal{I}_2 \mathcal{J}_1 \mathcal{J}_2 \mathcal{K}_1 \mathcal{K}_2\} \\ \tilde{V} & = & \{1 \mathcal{I}'_1 \mathcal{I}'_2 \mathcal{J}_1 \mathcal{J}_2 \mathcal{K}_1 \mathcal{K}_2\} \\ V' & = & \{1' \mathcal{I}'_1 \mathcal{I}'_2 \mathcal{J}'_1 \mathcal{J}'_2 \mathcal{K}_1 \mathcal{K}_2\} \\ W & = & \{1 \mathcal{I}_1 \mathcal{I}'_2 \mathcal{J}_1 \mathcal{J}'_2 \mathcal{K}_1 \mathcal{K}'_2\} \\ W' & = & \{1' \mathcal{I}_1 \mathcal{I}'_2 \mathcal{J}_1 \mathcal{J}'_2 \mathcal{K}_1 \mathcal{K}'_2\} \end{array} \right. \quad (\text{V.1})$$

In these formulas,  $\mathcal{I}_1, \mathcal{I}_2, \dots$  represent collections of variables  $q$  and  $p$ . In accordance with our convention (see sect.II),  $\mathcal{I}_1$  is written as a set of indices, namely those of  $I_1$  but each one being primed or not. As for  $\mathcal{I}'_1$ , it is written as a set of the same indices, non primed (resp. primed) if primed (resp. non primed) in  $\mathcal{I}_1$ . Similarly for the other sets  $\mathcal{I}_2, \mathcal{I}'_2, \dots$

Let us define the distance  $d(U, U')$  between two vertices  $U$  and  $U'$  as the number of variables with different assignments in  $U$  and  $U'$ . Inspecting eq. (V.1), one readily obtains

$$\left\{ \begin{array}{lcl} d(\tilde{V}, V') & = & 1 + j_1 + j_2, \\ d(\tilde{V}, W) & = & i_1 + j_2 + k_2, \end{array} \right. \quad (\text{V.2})$$

where  $j_1 = \text{card } J_1$ , and so on. Since  $\Sigma_k$  is one of the shortest segments connecting  $\Gamma_k$  to one of the components  $G_{k+1}, G_{k+2}, \dots$ , one must have  $d(\tilde{V}, V') \leq d(\tilde{V}, W)$ , which entails

$$i_1 + k_2 \geq 1.$$

This means that the sets  $I_1$  and  $K_2$  cannot be both empty. If  $K_2 \neq \emptyset$ , we choose the index 2 in  $K_2$ , so that the vertices  $V, V', W$  and  $W'$  constitute a critical quartet. If  $K_2 = \emptyset$ ,  $I_1$  is not empty,  $\tilde{V}$  does not coincide with  $V$ , and thus  $\tilde{V}$  is a non simple inserted vertex, necessarily linked in  $\Gamma_k$  to a vertex  $\hat{V}$  of  $G$  as displayed in Fig. 8. Choosing now the index 2 in  $I_1$ , one finds that the vertices  $\hat{V}, V', W$  and  $W'$  constitute a critical quartet.

## V-B. Proper $G_c$

When  $G$  is subgraph of a proper connected graph  $G_c$ , the latter is a tree graph which can be decomposed into the connected components  $G_i$  of  $G$  and connecting segments  $\Sigma_k$ , even when  $G_c$  does not coincide with the specific graph  $G_c$  constructed in the previous section V-A. One needs to distinguish two cases:

- a) all the segments  $\Sigma_k$ 's are disjoint. Then all insertions are simple and  $G_c$  is  $G$ -simple.
- b) at least two segments have one common vertex. This vertex is not a simple insertion and thus  $G_c$  is not  $G$ -simple.

### V-B-1. $G$ -simple $G_c$

Here, as in section IV, we establish part 2.a.i of Theorem 1 by associating to any chain  $\mathcal{C}$  of type  $G$  a particular phase space distribution  $\rho_0$  reproducing this chain. We also

give the general form of the  $\rho$ 's reproducing the chain.

The construction of  $\rho_0$  proceeds through the “Feynman rules” of section IV-A, complemented with propagators associated with the segments of  $G_c$ . Let  $\Sigma$  be such a segment connecting the vertices  $V_\alpha$  and  $V_\beta$  of  $G$ , and let  $r$  be its length. Let  $\sigma_\alpha(X, Y)$  and  $\sigma_\beta(X, Y')$  be the corresponding vertex functions of  $\mathcal{C}$ , where  $X$  (resp.  $Y$ ) denote the set of variables which have the same (resp. a different) assignment in  $\sigma_\alpha$  and  $\sigma_\beta$ . The compatibility of  $\sigma_\alpha$  and  $\sigma_\beta$  allows us to define

$$\sigma_{\alpha\beta}(X) \equiv \int d^r Y \sigma_\alpha(X, Y) = \int d^r Y' \sigma_\beta(X, Y') . \quad (\text{V.3})$$

For different segments  $\Sigma_l$  labelled by the index  $l$ , we use the notation  $\sigma_{\alpha_l\beta_l}(X_l)$ .

To the segment  $\Sigma_l$  we now associate the propagator

$$\varpi_l(X_l) = \begin{cases} \frac{1}{\sigma_{\alpha_l\beta_l}(X_l)} & \text{if } X_l \in \mathcal{S}_{\alpha_l\beta_l}, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{V.4})$$

where  $\mathcal{S}_{\alpha_l\beta_l}$  is the support of  $\sigma_{\alpha_l\beta_l}$ . This amounts to consider  $\Sigma_l$  as a new kind of link (in the graph  $G_c$ ) which we call *composite link*. Then the function  $\rho_0$  reads

$$\rho_0(\vec{q}, \vec{p}) = \left( \prod_{\alpha=1}^n \sigma_\alpha(Z_\alpha) \right) \left( \prod_{l=1}^{n-1} \varpi_l(X_l) \right) \zeta(T') , \quad (\text{V.5})$$

where the second product in the r.h.s. is performed on all links, namely the links of  $G$  and the composite links of  $G_c$ . Note that, since  $G_c$  is a connected tree, the total number of links is  $(n-1)$ . As for the function  $\zeta(T')$ , which is arbitrary but non negative and normalized, it takes care of the “passive” variables  $T = \{x_n, x_{n+1}, \dots, x_N\}$  and their conjugate  $T'$ , as in eq. (IV.3).

The proof that the  $\rho_0$  of eq. (V.5) solves the equations (II.2) relies on the “peeling process” described in section IV-A. Here, this process has to be extended to the case where the one-leg vertex  $\hat{V}$  introduced there is attached to a composite link. Let  $\hat{\sigma}(X, Y)$  be the vertex function of  $\mathcal{C}$  associated to  $\hat{V}$ , where  $X$  (resp.  $Y$ ) is the set of variables whose assignment does not change (resp. changes) through the composite link. Then eqs. (IV.5) become

$$\int d^r Y \rho_0^{(m)}(\vec{q}, \vec{p}) = \rho_0^{(m-1)}(X, X', Y') , \quad (\text{V.6})$$

and the rest of the proof is completely similar to that given in section IV-A.

The determination of the general solution  $\rho$  of eqs. (II.2) is also carried out along the lines followed in section IV-B, by using this time the extended peeling process. The definitions of the measures  $d\mu_\alpha$  and of the projectors  $P_\alpha$  (which now involve the function  $\rho_0$  of eq. (V.5)) are unchanged. The Lemma 2 still holds, though with an extended acceptance of “contiguity” : two vertices  $V_{\alpha_l}$  and  $V_{\beta_l}$  of  $G$  connected by a composite link  $\Sigma_l$  of  $G_c$  are declared contiguous. Actually, only minor changes are needed to generalize the proof given in Appendix A (essentially the substitution  $x_i \rightarrow Y$ ). The operator  $\Pi$  is now defined as

$$\Pi = 1 - \sum_{\alpha=1}^n P_\alpha + \sum_{l=1}^{n-1} P_{\alpha_l} P_{\beta_l} , \quad (\text{V.7})$$

where the last sum in the r.h.s. is a sum over all links of  $G_c$ , composite or not. Its properties (IV.17) and (IV.21) remain true and, with  $\rho$  written as in eq. (IV.9), one finds that the general solution for  $h$  is given again by eq. (IV.23). A change then occurs in eq. (IV.27) when the two vertices  $V_{\alpha_i}$  and  $V_{\beta_i}$  there become two vertices  $V_{\alpha_l}$  and  $V_{\beta_l}$  connected by a composite link. In this case eq. (IV.27) becomes

$$\int d^{N-r_l} X'_l \rho_0(X_l, X'_l, Y_l, Y'_l) = \frac{\sigma_{\alpha_l}(X_l, Y_l) \sigma_{\beta_l}(X_l, Y'_l)}{\sigma_{\alpha_l \beta_l}(X_l)}. \quad (\text{V.8})$$

One ends up with the following expression of  $h$ , generalizing the representation formula (IV.30)

$$\begin{aligned} h(\vec{q}, \vec{p}) &= f(\vec{q}, \vec{p}) - \sum_{\alpha=1}^n \frac{1}{\sigma_\alpha(Z_\alpha)} \int d^N Z'_\alpha \rho_0(\vec{q}, \vec{p}) f(\vec{q}, \vec{p}) \\ &+ \sum_{l=1}^{n-1} \frac{1}{\sigma_{\alpha_l \beta_l}(X_l)} \int d^{N-r_l} X'_l d^{r_l} Y_l d^{r_l} Y'_l \rho_0(\vec{q}, \vec{p}) f(\vec{q}, \vec{p}). \end{aligned} \quad (\text{V.9})$$

We remind the reader that, in this formula:

- i)  $f$  is an arbitrary function in  $L^1(\mathbb{R}^{2N}, \rho_0 d^N q d^N p)$ ;
- ii) the first sum in the r.h.s. is over all vertices  $V_\alpha$  of  $G$ ;  $Z_\alpha$  denotes the collection of arguments of the vertex function  $\sigma_\alpha$  and  $Z'_\alpha$  the collection of conjugate arguments;
- iii) the second sum is over all links (of length  $r_l$ ), that is the simple links of  $G$  and the composite links of  $G_c$ ; the definition of the functions  $\sigma_{\alpha_l \beta_l}$  occurring in the sum, as well as the meaning of the collections of variables  $X_l$ ,  $X'_l$ ,  $Y_l$  and  $Y'_l$ , are provided by eq. (V.3), which reduces to eq. (IV.28) in the case of a simple link of index  $i$ .

### V-B-2. Non $G$ -simple $G_c$

It remains to prove part 2.a.ii of Theorem 1.

Consider first an arbitrary *quantum* chain  $\mathcal{C}$  of type  $G$ . The CCS-distributions of  $\mathcal{C}$  are then expressed in terms of some density operator  $\hat{\rho}$ , in accordance with eqs. (I.4). But, in this case, a set of CCS-distributions associated with the insertions of  $G_c$  can also be computed through these equations. One thus obtains an extended chain  $\mathcal{C}_c$  of compatible CCS-distributions associated with all the vertices of  $G_c$ . Since  $G_c$  is proper, the chain  $\mathcal{C}_c$  is admissible (irrespective of the fact that  $G_c$  is not  $G$ -simple), which implies the admissibility of  $\mathcal{C}$ . Hence the quantum admissibility of the graph  $G$ .

To prove that  $G$  is not fully admissible, we show that there are (non quantum) chains of type  $G$  which are not admissible.

By assumption  $G_c$  contains at least one insertion  $V$  with  $k \geq 3$  legs, say the vertex  $(12 \dots N)$  with legs carrying the indices  $j = 1, 2, \dots, k$ . These legs connect  $V$  to  $k$  subgraphs  $G_c^{(j)}$  of  $G_c$  which are proper connected trees, but mutually disconnected (otherwise  $G_c$  would contain loops). By removing all the insertions, together with

their legs, from each of the  $G_c^{(j)}$ 's, we obtain  $k$  subgraphs  $G^{(j)}$  of  $G$  which are proper (not necessarily connected) trees. The vertices  $V_l^{(j)}$  of each  $G^{(j)}$  are of the form  $\boxed{1, 2, \dots, j-1, j', j+1, \dots, k, \mathcal{J}_l^{(j)}}$ , where  $\mathcal{J}_l^{(j)}$  represents a set  $\{k+1, k+2, \dots, N\}$  of indices, each one being primed or not.

We now use the following lemma, the proof of which is given in Appendix B.

### Lemma 3

*In a  $2k$ -dimensional phase space with  $k \geq 3$ , there exist  $k$ -chains of compatible distributions  $\{\tau_1(p_1, q_2, q_3, \dots, q_k), \tau_2(q_1, p_2, q_3, \dots, q_k), \dots, \tau_k(q_1, q_2, \dots, q_{k-1}, p_k)\}$  which are not admissible.*

Let us construct a chain  $\mathcal{C}$  of type  $G$  by assigning to each vertex  $V_l^{(j)}$  of  $G^{(j)}$  ( $j = 1, \dots, k$ ) the CCS-distributions

$$\sigma_l^{(j)}(q_1, \dots, q_{j-1}, p_j, q_{j+1}, \dots, q_k, X_l^{(j)}) = \tau_j(q_1, \dots, q_{j-1}, p_j, q_{j+1}, \dots, q_k) \tilde{\sigma}_l^{(j)}(X_l^{(j)}), \quad (\text{V.10})$$

where  $X_l^{(j)}$  denotes the set of variables  $x_i$  corresponding to  $\mathcal{J}_l^{(j)}$  and the  $\tilde{\sigma}_l^{(j)}$ 's are arbitrary probability distributions depending on these variables, only subjected to the apposite compatibility conditions<sup>4</sup>. The elements of the chain  $\mathcal{C} = \{\sigma_l^{(j)}\}_{j=1, \dots, k; l}$  are evidently compatible CCS-distributions. Let us pretend that these distributions are marginals of some phase space density  $\rho$ . Then, by defining the reduced density

$$\tilde{\rho}(q_1, \dots, q_k, p_1, \dots, p_k) = \int dq_{k+1} dp_{k+1} \dots dq_N dp_N \rho(\vec{q}, \vec{p}) \quad (\text{V.11})$$

in a  $2k$ -dimensional phase space, one finds that

$$\begin{aligned} \tau_j &= \int d^{N-k} X_l^{(j)} \sigma_l^{(j)} \quad (\text{any } l), \\ &= \int dp_1 \dots dp_{j-1} dq_j dp_{j+1} \dots dp_k \tilde{\rho} \quad (j = 1, \dots, k). \end{aligned} \quad (\text{V.12})$$

This would mean that the reduced  $k$ -chain  $\tilde{\mathcal{C}} = \{\tau_j\}_{j=1, \dots, k}$  is always admissible, in contradistinction with Lemma 3. We conclude that there exist chains of type  $G$  which are not admissible.

The two statements in part 2.a.ii of Theorem 1 are now established and the proof of this theorem is complete.

Finally, we would like to obtain an explicit expression of all the phase space densities  $\rho$  solving eqs. (II.2) for a given quantum chain  $\mathcal{C}$  of type  $G$ , by following again the method of section IV. However, serious complications crop up in the final step of the procedure.

First, a particular solution  $\rho_0$  is obtained by applying the formula (IV.3) to the extended chain  $\mathcal{C}_c$ . Alternatively, one can determine other particular solutions  $\rho'_0$  by using, instead of  $\mathcal{C}_c$ , the chains  $\mathcal{C}'_c$  obtained from  $\mathcal{C}_c$  by removing all or some of

<sup>4</sup>That such  $\tilde{\sigma}_l^{(j)}$ 's always exist is easy to see, e.g. by choosing completely factorized forms for them.

the *simple* insertions of  $G_c$  and applying the procedure of section V-B-1 involving composite links and their associated propagators. Whatever  $\rho_0$  is chosen, we keep writing the general solution in the form  $\rho = \rho_0(1 + \lambda h)$ , as in eq. (IV.9).

Then a change appears in the determination of the function  $h$ , because one does not have to require the density  $\rho$  to reproduce all the CCS-distributions of the chain  $\mathcal{C}_c$  (or  $\mathcal{C}'_c$ ), but only those of the given chain  $\mathcal{C}$ . This means that  $h$  should satisfy the eqs. (IV.22), where the index  $\alpha$  now refers to the only elements of the *initial chain*  $\mathcal{C}$ . As a consequence, the form (IV.16) of the appropriate operator  $\Pi$  (to be used in eq. (IV.23)) is no longer valid, since the properties (IV.17) and (IV.21) hold only if the underlying graph is *connected*. Notice that a similar difficulty already appeared when dealing with  $G$ -simple graphs  $G_c$  in the previous subsection. There, it was overcome by introducing composite links which eventually allowed us to remove the insertions of  $G_c$ . Unfortunately, no such device presents itself for non  $G$ -simple  $G_c$ 's, and constructing the “good” projector  $\Pi$  in this case seems to be quite a difficult problem, which we leave unsolved here.

Of course, the projector  $\Pi_c$  associated with the chain  $\mathcal{C}_c$  already provides us with a large class of solutions, but certainly not all the solutions.

## VI. Conclusions

We have investigated the extent to which it is possible to reproduce a given set of joint probability distributions  $\sigma_\alpha(x_1, x_2, \dots, x_N)$  with  $x_i = q_i$  or  $p_i$ , in arbitrary number  $n$  and with arbitrary position-momentum assignments of the  $x_i$ 's, as marginals of some probability density  $\rho(\vec{q}, \vec{p})$  in  $2N$ -dimensional phase space. We have been able to give a complete characterization of those sets which can always be reproduced by a  $\rho(\vec{q}, \vec{p})$  (admissible sets), irrespective of the functional form of the  $\sigma_\alpha$ 's provided they are compatible, and both for quantum probability distributions  $\sigma_\alpha$  and for more general (classical) ones. This has been achieved by introducing a specific, powerful diagrammatic method and by relying on previous results [3]-[4] obtained in the case  $N = 2$  by means of Bell-like inequalities in phase space.

When both classical and quantum sets  $\{\sigma_\alpha\}_{\alpha=1, \dots, n}$  are admissible, we have constructed the general solution  $\rho(\vec{q}, \vec{p})$  of the problem. When only quantum sets are admissible, we have the explicit expression of a large class of solutions, which however is not exhaustive. Concerning the dynamical aspect which is completely ignored in this paper, our results in the quantum case motivate the construction of realistic quantum mechanics reproducing  $(N + 1)$  marginals at all times  $t$  and thus considerably improving on the de Broglie-Bohm mechanics [7], which reproduces only one  $\sigma_\alpha$  (the position probability distribution  $\sigma(\vec{q}, t)$ ).

On the other hand, all cases of non admissibility have been identified. For quantum  $\sigma_\alpha$ 's again, this may be viewed as a general contextuality theorem of the Gleason-Kochen-Specker type [8], which also extends a previous result of this type due to Martin and Roy [9]. At the same time, this provides a proof of a long-standing conjecture, the “ $(N + 1)$  marginal theorem”.

From the mathematical standpoint, the parts of our main theorem (Theorem 1)

pertaining to the quantum case are essentially new statements concerning multi-dimensional Fourier transforms in  $L^2(\mathbb{R}^{2N}, d^N q d^N p)$ . These statements vastly extend the results of Cohen and Zaparovanny [6] for two non-intersecting marginals to the case of  $N+1$  marginals containing overlapping variables. Thus, they can be expected to open new applications in classical signal and image processing [10]. From the physical point of view, our results completely settle, at a formal level, the question of “maximal reality” raised and already investigated in special cases [1]-[4]. Their possible relevance for related fundamental problems of quantum theory (in particular for helping towards a clarification of the still controversial problem of measurement) remains to be explored.

## Appendix A. Proof of Lemma 2

### A. a) Commutation relation (IV.14)

Consider two contiguous vertices  $V_\alpha$  and  $V_\beta$  of  $G$  connected by a link  $l_i$  with index  $i$ , and denote by  $\sigma_\alpha(x_i, X, T)$  and  $\sigma_\beta(x'_i, X, T)$  the corresponding distributions of the chain  $\mathcal{C}_n$ , where  $X = \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1}\}$ . By removing the link  $l_i$ , the proper tree graph  $G$  is broken in two connected components  $G_\alpha$  and  $G_\beta$  :  $G = G_\alpha \cup l_i \cup G_\beta$ . To this splitting clearly corresponds a partition  $\{X_\alpha, X_\beta\}$  of the variables  $X$  such that, among  $X$  and their conjugate  $X'$ , all the variables  $\{X_\alpha, X'_\alpha, X_\beta\}$  and only them appear in the vertices of  $G_\alpha$ , whereas all the variables  $\{X_\alpha, X_\beta, X'_\beta\}$  and only them appear in the vertices of  $G_\beta$ . In parallel, the particular solution  $\rho_0$  given in eq. (IV.3) factorizes as

$$\rho_0 = \frac{\rho_\alpha \rho_\beta}{\sigma_{\alpha\beta}} \zeta, \quad (\text{A.1})$$

where the function  $\rho_\alpha$  (the “ $\rho_0/\zeta$ ” of the subchain of  $\mathcal{C}_n$  of type  $G_\alpha$ ) depends only on  $(x_i, X_\alpha, X'_\alpha, X_\beta)$ , the function  $\rho_\beta$  depends only on  $(x'_i, X_\alpha, X_\beta, X'_\beta)$ , and

$$\sigma_{\alpha\beta}(X, T) = \int dx_i \sigma_\alpha(x_i, X, T) = \int dx'_i \sigma_\beta(x'_i, X, T).$$

The measures  $d\mu_\alpha$  and  $d\mu_\beta$  as defined by eq. (IV.8) now take the form :

$$\begin{cases} d\mu_\alpha &= \frac{\rho_\alpha}{\sigma_\alpha} dX'_\alpha \frac{1}{\sigma_{\alpha\beta}} \rho_\beta dX'_\beta dx'_i \zeta dT', \\ d\mu_\beta &= \rho_\alpha dX'_\alpha \frac{1}{\sigma_{\alpha\beta}} \frac{\rho_\beta}{\sigma_\beta} dX'_\beta dx_i \zeta dT'. \end{cases} \quad (\text{A.2})$$

Hence, for any  $g(\vec{q}, \vec{p}) \in L^1(\mathbb{R}^{2N}, \rho_0 d^N q d^N p)$  :

$$P_\alpha P_\beta g = \frac{1}{\sigma_{\alpha\beta}} \int dx'_i \int dX'_\alpha \frac{\rho_\alpha}{\sigma_\alpha} \int dX'_\beta \rho_\beta \int dT' \zeta (P_\beta g).$$

The integrations over  $X'_\alpha$ ,  $X'_\beta$  and  $T'$  can be performed explicitly since  $P_\beta g$  does not depend on these variables. Noticing that

$$\int dX'_\alpha \rho_\alpha = \sigma_\alpha, \quad \int dX'_\beta \rho_\beta = \sigma_\beta, \quad (\text{A.3})$$

and taking account of eq. (IV.4), we get

$$P_\alpha P_\beta g = \frac{1}{\sigma_{\alpha\beta}} \int dx'_i \sigma_\beta (P_\beta g), \quad (\text{A.4})$$

$$= \frac{1}{\sigma_{\alpha\beta}} \int dx'_i \sigma_\beta \frac{1}{\sigma_{\alpha\beta}} \int dx_i dX'_\alpha dX'_\beta dT' \rho_\alpha \frac{\rho_\beta}{\sigma_\beta} \zeta g. \quad (\text{A.5})$$

Since  $\sigma_\beta$  does not depend on  $x_i$ ,  $X'_\alpha$  and  $X'_\beta$ , the factors  $\sigma_\beta$  and  $1/\sigma_\beta$  in eq. (A.5) cancel each other. This gives

$$P_\alpha P_\beta g = \frac{1}{(\sigma_{\alpha\beta})^2} \int dx_i dx'_i dX'_\alpha dX'_\beta dT' \rho_\alpha \rho_\beta \zeta g. \quad (\text{A.6})$$

The r.h.s. of this equation is symmetric in  $\alpha \leftrightarrow \beta$ , which establishes eq. (IV.14).

### A. b) Relation (IV.15)

Let  $V_\alpha$ ,  $V_\beta$  and  $V_\gamma$  be now three vertices of  $G$  such that  $V_\alpha$  belongs to the path connecting  $V_\gamma$  to  $V_\beta$  and is contiguous to  $V_\beta$ . Consider again the connected subgraphs  $G_\alpha$  and  $G_\beta$  defined in A.a) above, together with the partition  $\{X_\alpha, X_\beta\}$  of the variables  $X$ , and distinguish in  $G_\alpha$  the linear subgraph  $G_{\alpha\gamma}$  made of the vertices  $V_\alpha$ ,  $V_\gamma$  and the path connecting them. Denote by  $I_{\alpha_1}$  the set of indices of the links of  $G_{\alpha\gamma}$  and by  $I_{\alpha_2}$  the set of indices of the remaining links in  $G_\alpha$ . To this splitting corresponds a further partition  $\{X_{\alpha_1}, X_{\alpha_2}\}$  of the variables  $X_\alpha$ , as indicated in Fig. 9 :

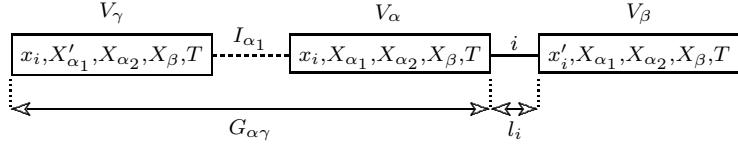


FIGURE 9: The path between  $V_\gamma$  and  $V_\beta$  in  $G$ .

With a factorization of the measure  $d\mu_\gamma$  analogous to those of eq. (A.2), we can write

$$P_\gamma P_\alpha P_\beta g = \int dX_{\alpha_1} dX'_{\alpha_2} dX'_\beta dx'_i dT' \frac{\rho_\alpha}{\sigma_\gamma} \frac{1}{\sigma_{\alpha\beta}} \rho_\beta \zeta (P_\alpha P_\beta g). \quad (\text{A.7})$$

Here, we can perform explicitly the integrations over  $x'_i$ ,  $X'_\beta$ ,  $X'_{\alpha_2}$  and  $T'$ , for  $P_\alpha P_\beta g$  does not depend on these variables. First :

$$\int dx'_i dX'_\beta \rho_\beta = \int dx'_i \sigma_\beta = \sigma_{\alpha\beta}. \quad (\text{A.8})$$

The left equality in eq. (A.8) results, as in eq. (A.3), from the “peeling process” (described in section IV.A) corresponding to the reduction of the graph  $G_\beta$  to the vertex  $V_\beta$ . Similarly, the (partial) peeling process corresponding to the reduction  $G_\alpha \rightarrow G_{\alpha\gamma}$  yields

$$\int dX'_{\alpha_2} \rho_\alpha = \rho_{\alpha\gamma}, \quad (\text{A.9})$$

where  $\rho_{\alpha\gamma}$  is the “ $\rho_0/\zeta$ ” of the subchain of type  $G_{\alpha\gamma}$ . Thanks to eqs. (A.8), (A.9) and (IV.7), equation (A.7) boils down to

$$P_\gamma P_\alpha P_\beta g = \int dX_{\alpha_1} \frac{\rho_{\alpha\gamma}}{\sigma_\gamma} (P_\alpha P_\beta g)$$

or, by inserting the expression (A.4) of  $P_\alpha P_\beta g$  :

$$P_\gamma P_\alpha P_\beta g = \int dX_{\alpha_1} \frac{\rho_{\alpha\gamma}}{\sigma_\gamma} \int dx'_i \frac{\sigma_\beta}{\sigma_{\alpha\beta}} (P_\beta g). \quad (\text{A.10})$$

On the other hand :

$$P_\gamma P_\beta g = \int dX_{\alpha_1} dX'_{\alpha_2} dX'_\beta dx'_i dT' \frac{\rho_0}{\sigma_\gamma} \zeta (P_\beta g), \quad (\text{A.11})$$

where the integrations over  $X'_\beta$ ,  $X'_{\alpha_2}$  and  $T'$  can be performed explicitly since  $P_\beta g$  does not depend on these variables. The integration over  $X'_\beta$  first produces, through

$r=2$

the partial peeling process corresponding to  $G \rightarrow (G_\alpha \cup l_i \cup V_\beta)$  :

$$\int dX'_\beta \rho_0 = \rho_\alpha \frac{1}{\sigma_{\alpha\beta}} \sigma_\beta. \quad (\text{A.12})$$

Then eqs. (A.9) and (IV.4) are used again for the integrations over  $X'_{\alpha_2}$  and  $T'$  respectively. Altogether, this reduces the expression (A.11) to the r.h.s. of eq. (A.10). Therefore  $P_\gamma P_\alpha P_\beta g = P_\gamma P_\beta g$ , which establishes eq. (IV.15) in the case where  $V_\alpha$  is contiguous to  $V_\beta$ .

The proof of eq. (IV.15) in the case where  $V_\alpha$  is contiguous to  $V_\gamma$  is completely similar.

q.e.d.

## Appendix B. Proof of Lemma 3

We construct a particular  $k$ -chain of compatible distributions  $\tau_j$  and we prove that there is no positive phase space density  $\rho$  reproducing these distributions as marginals. We take  $\tau_j$  of the form:

$$\tau_j(q_1, \dots, q_{j-1}, p_j, q_{j+1}, \dots, q_k) = \gamma_j(p_j) \bar{\tau}_j(q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_k) \quad (j = 1, \dots, k)$$

where  $\{\bar{\tau}_1, \dots, \bar{\tau}_k\}$  is a  $k$ -chain of compatible, reduced distributions and the  $\gamma_j(p_j)$ 's are arbitrary, normalized one variable distributions.

Let us look for a phase space density  $\rho$  reproducing the  $\tau_j$ 's. The  $\bar{\tau}_j$ 's are given in terms of the configuration space density

$$\bar{\rho}(q_1, \dots, q_k) \equiv \int d^k p \rho(q_1, \dots, q_k, p_1, \dots, p_k) \quad (\text{B.1})$$

as

$$\bar{\tau}_j(q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_k) = \int dq_j \bar{\rho}(q_1, \dots, q_k). \quad (\text{B.2})$$

We now choose the  $\bar{\tau}_j$ 's as follows:

$$\left\{ \begin{array}{lcl} \bar{\tau}_1(q_2, \dots, q_k) & = & \prod_{r=2}^k T_r - \prod_{r=2}^k U_r, \\ \bar{\tau}_j(q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_k) & = & \prod_{\substack{r=1 \\ r \neq j}}^k T_r + \prod_{\substack{r=1 \\ r \neq j}}^k U_r, \quad (j = 2, \dots, k), \end{array} \right. \quad (\text{B.3})$$

where

$$\left\{ \begin{array}{lcl} T_r & = & \frac{1}{2} [\delta(q_r - 1) + \delta(q_r + 1)], \\ U_r & = & \frac{1}{2} [\delta(q_r - 1) - \delta(q_r + 1)], \end{array} \right. \quad (r = 1, \dots, k).$$

The  $\bar{\tau}_j$ 's, which appear as sums of  $2^{k-2}$  monomials of the form

$$\frac{1}{2^{k-2}} \prod_{\substack{r=1 \\ r \neq j}}^k \delta(q_r - \varepsilon_r) \quad \varepsilon_r = \pm 1,$$

are obviously positive and normalized. Furthermore:

$$\int dq_i \bar{\tau}_j(q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_k) = \prod_{\substack{r=1 \\ r \neq i, j}}^k T_r \quad (i \neq j).$$

The r.h.s. of this equation is symmetric in  $i \leftrightarrow j$ , which entails the compatibility of the  $\bar{\tau}_j$ 's.

Clearly, the most general *positive*  $\bar{\rho}$  obeying equations (B.2) is the sum of  $2^k$  terms proportional to  $\prod_{r=1}^k \delta(q_r - \varepsilon_r)$ . Equivalently,  $\bar{\rho}$  can be written as a homogeneous polynomial  $P(\{T_r\}, \{U_r\})$  of degree  $k$  which, for each index  $r$ , is linear in  $T_r$  and  $U_r$ . Then, since  $\int dq_j T_j = 1$  and  $\int dq_j U_j = 0$ , we can express  $\int dq_j \bar{\rho}$  as  $\partial P / \partial T_j$ , so that eqs. (B.2) and (B.3) yield:

$$\begin{cases} \frac{\partial P}{\partial T_1} = \prod_{r=1}^k T_r - \prod_{\substack{r=1 \\ r \neq j}}^k U_r, \\ \frac{\partial P}{\partial T_j} = \prod_{\substack{r=1 \\ r \neq j}}^k T_r + \prod_{\substack{r=1 \\ r \neq j}}^k U_r, \quad (j = 2, \dots, k). \end{cases}$$

The general solution of these equations is:

$$P = \prod_{r=1}^k T_r - T_1 \prod_{\substack{r=1 \\ r \neq j}}^k U_r + \sum_{j=2}^k T_j \prod_{\substack{r=1 \\ r \neq j}}^k U_r + \lambda \prod_{r=1}^k U_r, \quad (\text{B.4})$$

where  $\lambda$  is an arbitrary real parameter.

Now, whatever the value of  $\lambda$  is,  $P$ , and thus  $\rho$ , are not positive. To show this, it is sufficient to look at the coefficients of the two monomials

$$\delta(q_1 + 1) \prod_{r=2}^k \delta(q_r - 1) \quad \text{and} \quad \delta(q_1 - 1) \delta(q_2 + 1) \delta(q_3 + 1) \prod_{r=4}^k \delta(q_r - 1)$$

which appear in eq. (B.4) if  $k \geq 3$ . One finds  $-(k - 1 + \lambda)/2^k$  and  $(k - 5 + \lambda)/2^k$  respectively, the sum of which is independent of  $\lambda$  and negative.

q.e.d.

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