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# Continuous Time-Dependent Measurements: Quantum Anti-Zeno Paradox with Applications

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## Abstract

We derive differential equations for the modified Feynman propagator and for the density operator describing time-dependent measurements or histories continuous in time. We obtain an exact series solution and discuss its applications. Suppose the system is initially in a state with density operator $\rho(0)$ and the projection operator $E(t) = U(t)EU^\dagger(t)$ is measured continuously from $t = 0$ to $T$, where $E$ is a projector obeying $E\rho(0)E = \rho(0)$ and $U(t)$ a unitary operator obeying $U(0) = 1$ and some smoothness conditions in $t$. Then the probability of always finding $E(t) = 1$ from $t = 0$ to $T$ is unity. Generically $E(T) \neq E$ and the watched system is sure to change its state, which is the anti-Zeno paradox noted by us recently. Our results valid for projectors of arbitrary rank generalize those obtained by Anandan and Aharonov for projectors of unit rank.

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## 1. INTRODUCTION

Quantum physics specifies probabilities of ideal observations at one instant of time or of a sequence of such observations at different instants[1]. How should one describe the limit of infinitely frequent measurements or continuous observation? One of the earliest approaches to continuous quantum measurements was already suggested by Feynman[2] in his original work on the path integral. The Feynman propagator as modified by measurements is to be calculated by restricting the paths to cross (or not to cross) certain spacetime regions (where space can mean configuration space or phase space). An approximate way of doing this by incorporating Gaussian cut-offs in the phase space path integral was developed by Mensky[3] who also showed its equivalence to the phenomenological master equation approach for open quantum systems using models of system-environment coupling developed by Joos and Zeh and others[4].

On the other hand a completely different approach was initiated by Misra and Sudarshan[5] who asked: what is the rigorous quantum description of ideal continuous measurement of a projector $E$ (time-independent in the Schrödinger representation) over a time interval $[0,T]$? Their original motivation[5]: "there does not seem to be any principle, internal to quantum theory, that forbids the duration of a single measurement or the dead time between successive measurements from being arbitrarily small", led them to rigorous confirmation of a seemingly paradoxical conclusion noted earlier[6]. The conclusion "that an unstable particle which is continuously observed to see whether it decays will never be found to decay" or that a "watched pot never boils"[7] was christened "Zeno's paradox in quantum theory" by Misra and Sudarshan[5]. The paradox has been theoretically scrutinized questioning the consistency of infinitely frequent measurements with time-energy and position-momentum uncertainty principles[8]. Experimental tests[9] and their different interpretations have been rigorously discussed.

In our recent letter[10], we showed that in contrast to the continuous measurement of a time independent projection operator which prevents the quantum state from changing (the quantum Zeno paradox), the generic continuous measurement of a time-dependent projection operator $E_s(t)$ forces the quantum state to change with time (the quantum anti-Zeno paradox). We have emphasized that though the two effects (one inhibiting change of state and the other ensuring change of state) are physically opposite, they are mutually consistent as they refer to different experimental arrangements. We derived the anti-Zeno paradox in a very broad framework with arbitrary Hamiltonian, arbitrary density matrix states, and measurement of smooth time-dependent projection operators of arbitrary rank. Our results are generalisations to projectors of arbitrary rank of the earlier elegant results for rank one projectors obtained by Anandan and Aharonov and Facchi et. al.[11] who considered quantum systems guided through a closed loop in Hilbert space by measurements represented by rank one projectors. They are also generalizations to arbitrary Hamiltonians of Von Neumann's results on continuous measurements[1] in the case of zero Hamiltonian, and analogous results of Aharonov and Vardi[12]. However our results for time-dependent projectors have a completely different physical origin from those of Kofman and Kurizki[13] for time-independent measurements. They showed that when the frequency of measurements is smaller than a characteristic difference of eigenfrequencies of the system, an enhancement of decay can result.

We ask a question far more general than that of Misra and Sudarshan: what is the operator (the modified Feynman propagator) corresponding to an ideal continuous measurement of a projection operator $E_s(t)$ which has an arbitrary (but smooth) dependence on time in the Schrödinger representation? We obtain a differential equation for the operator and a series solution. We work out several applications. One of them leads us to a new watched-kettle paradox which is apparently quite the opposite of the Zeno paradox, but mathematically a far reaching generalization of it. Suppose we continuously measure from $t = 0$ to $T$ the projector $E_s(t) = U(t)EU^\dagger(t)$ where $U(t)$ is a unitary operator obeying $U(0) = 1$ and some smoothness conditions, and $E$ a projector obeying $E\rho(0)E = \rho(0)$, where $\rho(0)$ is the initial density operator. Then the probability of always finding $E_s(t) = 1$ from $t = 0$ to $T$ is unity. For the Misra-Sudarshan case, $U(t) = 1$ and we recover the usual Zeno paradox that the watched kettle does not boil. Generically $U(t)$ does not commute with $E$. Hence, for most ways of watching, the watched kettle is sure to change its state, an anti-Zeno paradox. If the system is in an eigenstate of $E$ with eigenvalue unity at $t = 0$, it will change its state with time so as to be in an eigenstate of $E_s(t)$ with eigenvalue unity at all future times.

Our computation of modified Feynman propagators corresponding to continuous measurements is in the framework of ordinary quantum mechanics. Exactly the same mathematical expressions for the propagators would arise in the 'consistent histories' or 'sum over histories' quantum mechanics of closed systems[14,15], where there is no notion of measurement. Our computations can therefore be applied also to these history-extended quantum mechanics provided that probabilities of measurement outcomes are replaced by weights of histories; the probability interpretation is restored when the probability sum rules corresponding to consistency or decoherence conditions are obeyed.

## 2. BASIC PRINCIPLES

For a quantum system with a self-adjoint Hamiltonian $H$, an initial state vector $|\psi(0)\rangle$ evolves to a state vector $|\psi(t)\rangle$,

$$|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle. \tag{2.1}$$

More generally, an initial state with density operator $\rho(0)$ has the Schrödinger time evolution

$$\rho(t) = \exp(-iHt)\rho(0)\exp(iHt), \tag{2.2}$$

which preserves the normalization condition Tr $\rho(t) = 1$. In an ideal instantaneous measurement of a self-adjoint projection operator $E$, the probability of finding $E = 1$ is $\mathrm{Tr}(E\rho E)$

and on finding the value 1 for $E$, the state collapses according to

$$\rho \to \rho' = E\rho E/\mathrm{Tr}(E\rho E). \tag{2.3}$$

If projectors $E_1, E_2, \cdots, E_n$ are measured at times $t_1, t_2, \cdots, t_n$ respectively, with Schrödinger evolution in between measurements, the probability $p(h)$ for the sequence of events $h$,

$$h:\ E_1 = 1 \text{ at } t = t_1;\ E_2 = 1 \text{ at } t = t_2; \cdots;\ E_n = 1 \text{ at } t = t_n \tag{2.4}$$

is[1]

$$p(h) = ||\psi_h(t')||^2,\ \psi_h(t') = K_h(t')\psi(0),\ t' > t_n. \tag{2.5}$$

Here $K_h(t')$ is the Feynman propagator modified by the events $h$,

$$K_h(t') = \exp(-iHt')A_h(t_n, t_1) \tag{2.6}$$

where,

$$A_h(t_n, t_1) = E_H(t_n)E_H(t_{n-1})\cdots E_H(t_1) = T\prod_{i=1}^{n} E_H(t_i), \tag{2.7}$$

with $T$ denoting 'time-ordering' and the Heisenberg operators $E_H(t_i)$ are related to the Schrödinger operators by the usual relation

$$E_H(t_i) = \exp(iHt_i)E_s(t_i)\exp(-iHt_i),\ E_s(t_i) \equiv E_i. \tag{2.8}$$

The state vector of the system at a time $t'$ after the events $h$ is

$$\psi_h(t')/||\psi_h(t')||. \tag{2.9}$$

(We shall omit the ket symbol except when confusion can arise thereby). Correspondingly, if the initial state is a density operator $\rho(0)$, the probability $p(h)$ for the events $h$ is given by

$$p(h) = \mathrm{Tr}\ K_h(t')\rho(0)K_h^\dagger(t') = \mathrm{Tr}\ A_h(t_n, t_1)\ \rho(0)A_h^\dagger(t_n, t_1), \tag{2.10}$$

and the state at $t' > t_n$ is

$$K_h(t')\rho(0)K_h^\dagger(t')/\mathrm{Tr}\ (K_h(t')\rho(0)K_h^\dagger(t')). \tag{2.11}$$

In the history extended quantum mechanics of closed systems[14,15], exactly the same expression (2.10) for $p(h)$ is adopted, with $h$ denoting the history (2.4) without any mention of measurements, and $p(h)$ being the weight of the history. The weight $p(h)$ is rechristened as probability when certain consistency conditions are obeyed.

## 3. REPEATED MEASUREMENTS WITH ZERO HAMILTONIAN

We recall first von Neumann's[1] fundamental work on the change of state due to measurements alone, ignoring the Hamiltonian evolution between measurements. A state vector $|\phi\rangle$ has the density operator $\rho_\phi$ obeying

$$\rho_\phi = |\phi\rangle\langle\phi|, \quad \rho_\phi^2 = \rho_\phi. \tag{3.1}$$

Given any other pure state $|\psi\rangle$, von Neumann constructed a beautiful demonstration that repetition of a sufficiently large number of suitable measurements will change $\rho_\phi$ to an ensemble whose density operator differs from $\rho_\psi$ by an arbitrarily small amount. Now since a good definition of entropy $S(\rho)$ of a state $\rho$ must (by hypothesis) have the property that measurements only increase it, we need that $S(\rho_\psi) - S(\rho_\phi) \geq 0$. Interchanging the roles of $\psi$ and $\phi$, we obtain $S(\rho_\phi) - S(\rho_\psi) \geq 0$. Therefore,

$$S(\rho_\phi) = S(\rho_\psi) \tag{3.2}$$

for any two pure states $\phi, \psi$. This led von Neumann to define the entropy corresponding to an arbitrary density operator $\rho$ as

$$S = -\mathrm{Tr}\ \rho\ \ell n\ \rho, \tag{3.3}$$

a complete set of which is zero for any pure state and positive for any mixture state. The von Neumann entropy now plays a fundamental role in providing a quantitative measure of decoherence, for example in quantum information processing.

We give von Neumann's demonstration of changing an initial $\rho_\phi$ into $\rho_\psi$ by infinitely repeated measurements in the case of $\phi$ and $\psi$ being orthogonal states. (This is enough. If they are not orthogonal we can find a state $\chi$ orthogonal to both $\phi$ and $\psi$, change from $\rho_\phi$ to $\rho_\chi$, and then from $\rho_\chi$ to $\rho_\psi$).

If an observable $R$ with a complete set of nondegenerate orthonormal eigenvectors $|\phi_n\rangle$ is measured on a state with density operator $\rho$, the states with density operators $|\phi_n\rangle\langle\phi_n|$ are obtained with probabilities $\langle\phi_n|\rho|\phi_n\rangle$. A mixed state $\rho'$ results.

$$\rho \xrightarrow{R} \rho' = \sum_n E_n \rho E_n, \;\; E_n = |\phi_n\rangle\langle\phi_n|. \tag{3.4}$$

This result will be used repeatedly to steer $\rho_\phi$ into $\rho_\psi$. Let $k$ be a positive integer and $|\psi^{(\nu)}\rangle$, with $\nu = 0, 1, \cdots, k$, be a set of normalized states ($\| \, |\psi^{(\nu)}\rangle \, \| = 1$) which interpolate between $|\phi\rangle = |\psi^{(0)}\rangle$ and $|\psi\rangle = |\psi^{(k)}\rangle$, e.g.

$$|\psi^{(\nu)}\rangle = \cos\left(\frac{\pi\nu}{2k}\right)|\phi\rangle + \sin\left(\frac{\pi\nu}{2k}\right)|\psi\rangle. \tag{3.5}$$

To $|\psi^{(\nu)}\rangle \equiv |\psi_1^{(\nu)}\rangle$, adjoin a set of orthonormal vectors $|\psi_2^{(\nu)}\rangle$, $|\psi_3^{(\nu)}\rangle$, $\cdots$ to obtain a complete orthonormal set of eigenvectors of an observable $R^{(\nu)}$ with the respective eigenvalues $\lambda_1^{(\nu)}, \lambda_2^{(\nu)}, \cdots$ which are all different. Starting with the initial density operator $\rho^{(0)} = |\phi\rangle\langle\phi|$, successively measure $R^{(1)}, R^{(2)}, \cdots, R^{(k)}$ to obtain a final density operator $\rho^{(k)}$:

$$\rho^0 \xrightarrow{R^{(1)}} \rho_1 \xrightarrow{R^{(2)}} \rho_2 \cdots \xrightarrow{R^{(k)}} \rho^{(k)}. \tag{3.6}$$

Here $\rho^{(\nu)}$ is obtained from $\rho^{(\nu-1)}$ after measurement of $R^{(\nu)}$:

$$\rho^{(\nu-1)} \xrightarrow{R^{(\nu)}} \rho^{(\nu)} = \sum_n E_n^{(\nu)} \rho^{(\nu-1)} E_n^{\nu}, \tag{3.7}$$

where

$$E_n^{(\nu)} = |\psi_n^{(\nu)}\rangle\langle\psi_n^{(\nu)}|. \tag{3.8}$$

The crucial step in proving that $\rho^{(k)} \to \rho_\psi$ for $k \to \infty$ will be a lower bound on

$$\langle\psi|\rho^{(k)}|\psi\rangle = \sum_n \langle\psi^{(k)}|E_n^{(k)}\rho^{(k-1)}E_n^{(k)}|\psi^{(k)}\rangle = \langle\psi^{(k)}|\rho^{(k-1)}|\psi^{(k)}\rangle. \tag{3.9}$$

A lower bound can be obtained by repeated application of

$$\langle\psi^{(\nu+1)}|\rho^{(\nu)}|\psi^{(\nu+1)}\rangle = \sum_n \langle\psi^{(\nu+1)}|E_n^{(\nu)}\rho^{(\nu-1)}E_n^{(\nu)}|\psi^{(\nu+1)}\rangle$$
$$\geq |\langle\psi^{(\nu+1)}|\psi^{(\nu)}\rangle|^2\langle\psi^{(\nu)}|\rho^{(\nu-1)}|\psi^{(\nu)}\rangle, \tag{3.10}$$

together with

$$\langle\psi^{(\nu+1)}|\psi^{(\nu)}\rangle = \cos\left(\frac{\pi}{2k}\right), \ \langle\psi^{(1)}|\rho^{(0)}|\psi^{(1)}\rangle = \cos^2\left(\frac{\pi}{2k}\right). \tag{3.11}$$

Hence,

$$\langle\psi|\rho^{(k)}|\psi\rangle \geq \left[\cos\left(\frac{\pi}{2k}\right)\right]^{2k} \underset{k\to\infty}{\longrightarrow} 1. \tag{3.12}$$

Since Tr $\rho^{(k)} = 1$ and $\rho^{(k)}$ is a nonnegative operator, we have

$\rho^{(k)}_{nn} \underset{k\to\infty}{\longrightarrow} \delta_{n1}$ (no sum over $n$),

and also, for $m \neq n$,

$|\rho^{(k)}_{mn}|^2 \leq (\rho^k)_{mm}(\rho^k)_{nn} \underset{k\to\infty}{\longrightarrow} 0.$

Hence,

$$\rho^{(k)} \underset{k\to\infty}{\longrightarrow} |\psi\rangle\langle\psi|. \tag{3.13}$$

This completes von Neumann's demonstration.

## 4. CONTINUOUS MEASUREMENTS WITH ARBITRARY HAMILTONIAN

The purpose now is to obtain an exact operator expression for the modified Feynman propagator $K_h(t')$ due to infinitely frequent measurements in some earlier interval of time allowing for arbitrary Hamiltonian evolution. We assume that the projection operators $E_s(t_i)$ measured at time $t_i$ are values at $t_i$ of a projection valued function $E_s(t)$. We make also the technical assumption that the corresponding Heisenberg operator $E_H(t)$ is weakly analytic. We therefore seek to calculate

$$K_h(t') = \exp(-iHt')A_h(t, t_1), \tag{4.1}$$

where

$$A_h(t,t_1) = \lim_{n\to\infty} T \prod_{i=1}^{n} E_H(t_1 + (t-t_1)(i-1)/(n-1)) \tag{4.2}$$

which is the $n \to \infty$ limit of Eq. (2.7) with a specific choice of the $t_i$. Let us also introduce the projectors $\bar{E}_i$ which are the orthogonal complements of the projectors $E_i$,

$$\bar{E}_i = 1 - E_i \tag{4.3}$$

and a sequence of events $\bar{h}$ complementary to the sequence $h$,

$$\bar{h} : \bar{E}_1 = 1 \text{ at } t = t_1;\; \bar{E}_2 = 1 \text{ at } t = t_2, \cdots, \bar{E}_n = 1 \text{ at } t = t_n. \tag{4.4}$$

Corresponding to Eqs. (2.6), (2.7), (4.1), (4.2), we have equations with $E \to \bar{E}$, $h \to \bar{h}$. Thus,

$$K_{\bar{h}}(t') = \exp(-iHt')A_{\bar{h}}(t,t_1), \tag{4.5}$$

$$A_{\bar{h}}(t,t_1) = \lim_{n\to\infty} T \prod_{i=1}^{n} \bar{E}_H(t_1 + (t-t_1)(i-1)/(n-1)). \tag{4.6}$$

The special interest in $K_{\bar{h}}(t')$ is that it is closely related to the propagator

$$K_{h'}(t') \equiv \exp(-iHt') - K_{\bar{h}}(t') = \exp(-iHt')[1 - A_{\bar{h}}(t,t_1)],\; h' \equiv \underset{i}{U} E_i, \tag{4.7}$$

which represents the modified Feynman propagator corresponding to the union of the events $E_i$, i.e. to at least one of the events $E_s(t_i) = 1$ occurring, with $t_i$ lying between $t_1$ and $t$. Though the $E_H(t_i)$ are in general not position projectors, we represent them in Fig. 1 by space regions and hence we represent $A_h(t,t_1)$ which is a product of the $E_H(t_i)$ at various $t_i$ by a spacetime region. This enables us to visualize the propagator $K_{h'}$ as corresponding to Feynman paths which intersect the spacetime region at least once, the propagator $K_{\bar{h}}$ as corresponding to paths which do not intersect the spacetime region at all and the propagator $K_h$ as corresponding to paths which stay inside the region $A_h(t,t_1)$ for all times between $t_1$ and $t$. Our object is to obtain exact operator expressions for the propagators $K_h$, $K_{\bar{h}}$ which are defined by equations (4.1), (4.5) with $A_h(t,t_1)$ and $A_{\bar{h}}(t,t_1)$ being given by the formal infinite products in Eqs. (4.2) and (4.6). The operator results we obtain will also provide evaluations of the path integral formulae for the propagators in history-extended quantum mechanics[14,15].

## 5. DIFFERENTIAL EQUATION AND SERIES SOLUTION FOR OPERATORS REPRESENTING CONTINUOUS MEASUREMENT

We see from Eqs. (4.1) and (4.5) that the modifications of the Feynman propagator due to the sequences of events $h$ and $\bar{h}$ consist respectively in multiplication by the operators $A_h(t,t_1)$ and $A_{\bar{h}}(t,t_1)$. Thus $A_h(t,t_1)(A_{\bar{h}}(t,t_1))$ represents the continuous measurement corresponding to the sequence of events $h(\bar{h})$. Consider first the operators $A_h(t_i,t_1),\ A_{\bar{h}}(t_i,t_1)$ before taking the $n\to\infty$ limit, and note the crucial identities

$$\bar{E_H}(t_i)A_h(t_i,t_1)=0,\ E_H(t_i)A_{\bar{h}}(t_i,t_1)=0 \tag{5.1}$$

which follow from $\bar{E}E=E\bar{E}=0$ for any projection operator $E$. Note also that

$$A_h(t_i,t_1)=E_H(t_i)A_h(t_{i-1},t_1),\ A_{\bar{h}}(t_i,t_1)=\bar{E_H}(t_i)A_{\bar{h}}(t_{i-1},t_1). \tag{5.2}$$

The relation $\left(\bar{E}_H(t_{i-1})\right)^2=\bar{E}_H(t_{i-1})$ implies $A_{\bar{h}}(t_{i-1},t_1)=\bar{E}_H(t_{i-1})A_{\bar{h}}(t_{i-1},t_1)$. Hence,

$$A_{\bar{h}}(t_i,t_1)-A_{\bar{h}}(t_{i-1},t_1)=(\bar{E_H}(t_i)-\bar{E_H}(t_{i-1}))A_{\bar{h}}(t_{i-1},t_1). \tag{5.3}$$

Dividing by $t_i-t_{i-1}=\delta t$, taking the limit $n\to\infty$ (i.e., $\delta t\to 0$) and assuming that $E_H(t)$ is weakly analytic at $t=0$ we obtain the differential eqn.

$$\frac{dA_{\bar{h}}(t,t_1)}{dt}=\frac{d\bar{E}_H(t)}{dt}A_{\bar{h}}(t_-,t_1) \tag{5.4}$$

where the argument $t_-$ on the right-hand side indicates that in case of any ambiguity in defining the operator product on the right, the argument of $A_{\bar{h}}$ has to be taken as $t-\epsilon$ with $\epsilon\to 0$ from positive values. We obtain similarly,

$$\frac{dA_h(t,t_1)}{dt}=\frac{dE_H(t)}{dt}A_h(t_-,t_1), \tag{5.5}$$

with

$$\frac{dE_H(t)}{dt}=i[H,E_H(t)]+\exp(iHt)\frac{dE_s(t)}{dt}\exp(-iHt). \tag{5.6}$$

Further $A_{\bar{h}}(t,t_1),A_h(t,t_1)$ must obey the initial conditions

$$A_{\bar{h}}(t_1,t_1)=\bar{E_H}(t_1),\ A_h(t_1,t_1)=E_H(t_1). \tag{5.7}$$

The measurement differential equations (5.4) and (5.5) are reminiscent of Schrödinger equation for the time evolution operator except for the fact that the operators $d\bar{E}_H/dt$, $dE_H/dt$ are hermitean whereas in Schrödinger theory the antihermitean operator $H/i$ would occur. Using the initial conditions (5.7), we obtain the explicit solutions,

$$A_{\bar{h}}(t,t_1) = T\exp\left(\int_{t_1}^{t} dt' \frac{d\bar{E}_H(t')}{dt'}\right)\bar{E}_H(t_1), \tag{5.8}$$

$$A_h(t,t_1) = T\exp\left(\int_{t_1}^{t} dt' \frac{dE_H(t')}{dt'}\right)E_H(t_1), \tag{5.9}$$

where the time-ordered exponential in (5.8) for example has the series expansion

$$T\exp\left(\int_{t_1}^{t} dt' \frac{d\bar{E}_H(t')}{dt'}\right) = 1 + \sum_{n=1}^{\infty}\int_{t_1}^{t} dt'_1 \int_{t_1}^{t'_1} dt'_2 \cdots \int_{t_1}^{t'_{n-1}} dt'_n T\prod_{i=1}^{n}\frac{d\bar{E}_H(t'_i)}{dt'_i}. \tag{5.10}$$

We assume that the time-ordered operator products appearing on the right-hand side exist at least as distributions. The distributional character occurs naturally for operators with continuous spectrum even when the $\bar{E}_H(t)$ (or $E_H(t)$) at different times commute, and implies that the series on the right-hand side must be taken as the definition of the exponential on the left-hand side; we may not do the integral of $d\bar{E}_H(t')/dt'$ on the left-hand side. (This will be clarified in examples.) Multiplying the expressions (5.8) and (5.9) for $A_{\bar{h}}(t,t_1)$ and $A_h(t,t_1)$ on the left by $\exp(-iHt')$ then completes the evaluation of the modified Feynman propagators $K_{\bar{h}}(t')$ and $K_h(t)$.

## 6. EXAMPLES

(i) *Operator With Continuous Spectrum Commuting with Hamiltonian*

For a one-dimensional free particle, $H = p^2/(2m)$, consider measuring

$$E_s(t') = \int_{\lambda_L(t')}^{\lambda_R(t')} dp|p><p| \tag{6.1}$$

continuously for $t'\epsilon[t_1,t]$. Since $E_s(t')$ commutes with $H$, $E_H(t') = E_s(t')$, and

$$\bar{E}_H(t') = \bar{E}_L(t') + \bar{E}_R(t'), \tag{6.2}$$

where

$$\bar{E}_L(t') = \int_{-\infty}^{\lambda_L(t')} dp|p><p|, \;\; \bar{E}_R(t') = \int_{\lambda_R(t')}^{\infty} dp|p><p|. \tag{6.3}$$

We assume that $\lambda_L(t') < \lambda_R(t'')$ for all $t', t'' \epsilon [t_1, t]$, and $< p|q >= \delta(p-q)$. Hence, $\bar{E}_L(t')\bar{E}_R(t'') = 0$ and Eq. (5.10) yields

$$A_{\bar{h}}(t, t_1) = A_L(t, t_1) + A_R(t, t_1), \tag{6.4}$$

where

$$A_L(t, t_1) = \left[1 + \sum_{n=1}^{\infty} \int_{t>t'_n>t'_{n-1}\cdots t'_1>t_1} dt'_1 \; dt'_2 \cdots dt'_n \; T \prod_{i=1}^{n} \frac{d\bar{E}_L(t'_i)}{dt'_i}\right] \bar{E}_L(t_1) \tag{6.5}$$

and $A_R$ is given by a similar expression with $L \to R$. The orthogonality relations between states $|p>$ imply that the integrand is a product of $\delta$-functions,

$$T \prod_{i=1}^{n} \frac{d\bar{E}_L(t'_i)}{dt'_i} \bar{E}_L(t_1) = \int_{-\infty}^{\lambda_L(t_1)} dp|p><p| \prod_{i=1}^{n} \dot{\lambda}_L(t'_i)\delta(\lambda_L(t'_i) - p). \tag{6.6}$$

The integrals over $t'_1, \cdots, t'_n$ are now easily done. The $\delta$-functions vanish for $p < \min \lambda_L$, where $\min \lambda_L$ denotes the minimum value of $\lambda_L(t')$ for $t' \epsilon [t_1, t]$. Hence,

$$A_L(t, t_1) = \bar{E}_L(t_1) + \int_{\min \lambda_L}^{\lambda_L(t_1)} dp|p><p| \sum_{n=1}^{N_p} \sum_{\{t'_1, \cdots t'_n\}} sgn\left(\prod_{i=1}^{n} \dot{\lambda}_L(t'_i)\right), \tag{6.7}$$

where $N_p$ is the number of values of $t'$ in the interval $[t_1, t]$ for which $\lambda_L(t') = p$, and for each $n$ we sum over all $n$-tuples $\{t'_1, \cdots t'_n\}$ such that $\lambda_L(t'_1) = \cdots = \lambda_L(t'_n) = p$ with $t > t'_n > t'_{n-1} \cdots t'_1 > t_1$. Hence

$$A_L(t, t_1) = \int_{-\infty}^{\min \lambda_L} dp|p><p| + \int_{\min \lambda_L}^{\lambda_L(t_1)} dp|p><p| \prod_{i=1}^{N_p} \left(1 + sgn\left(\dot{\lambda}_L(t'_i)\right)\right). \tag{6.8}$$

Note that for $N_p = 1$, $\dot{\lambda}_L(t'_1) < 0$, and that for $N_p \geq 2$, $\dot{\lambda}_L(t'_i)$ must have opposite signs for consecutive integers $i$. Hence,

$$\prod_{i=1}^{N_p} \left(1 + sgn(\dot{\lambda}_L(t'_i))\right) = 0, \quad \text{for } N_p \geq 1. \tag{6.9}$$

An entirely similar evaluation gives $A_R(t, t_1)$. Finally, we get

$$A_{\bar{h}}(t, t_1) = \int_{-\infty}^{\min \lambda_L} dp|p><p| + \int_{\max \lambda_R}^{\infty} dp|p><p|, \tag{6.10}$$

where $\max \lambda_R$ is the maximum value of $\lambda_R(t')$ for $t' \epsilon [t_1, t]$. Of course this answer is correct, and it can easily be deduced directly from the product of projectors in Eq. (4.6). But we have obtained here a non-trivial test of the contribution of terms of arbitrary order in the expansion of the time-ordered exponential in Eq. (5.8).

(ii) *Continuous Measurement of Spin Component along Time-Varying Direction* $\vec{n}(t)$

For a spin 1/2 particle with Hamiltonian $H = -(1/2)\sigma_y \alpha$, let the projector

$$E_s(t) = \frac{1 + \vec{\sigma} \cdot \vec{n}(t)}{2}, \tag{6.11}$$

be measured continuously, where

$$\vec{n}(t) = (\sin\theta(t), 0, \cos\theta(t)) \tag{6.12}$$

with $\theta(0) = 0$. Defining $\epsilon(t) = \theta(t) + \alpha t$, we deduce that

$$E_H(t) = \exp\left[-\frac{i}{2}\sigma_y \epsilon(t)\right] \frac{1 + \sigma_z}{2} \exp\left[\frac{i}{2}\sigma_y \epsilon(t)\right], \tag{6.13}$$

and that the first five terms in the expansion of the time-ordered exponential in Eq. (5.8) are (for $t_1 = 0$) given by

$$\begin{aligned} T\exp\left(\int_0^t dt'\ d\bar{E}_H(t')/dt'\right) &= 1 - \frac{1}{2}\left[\sigma_z(\cos\epsilon - 1) + \sigma_x \sin\epsilon\right] \\ &+\frac{1}{4}\left[1 - \cos\epsilon - i\sigma_y(\epsilon - \sin\epsilon)\right] - \frac{1}{8}\Big[\sigma_z\{\epsilon\sin\epsilon + 2\cos\epsilon - 2\} \\ &+\sigma_x\{2\sin\epsilon - \epsilon(\cos\epsilon + 1)\}\Big] - \frac{1}{16}\Big[\frac{1}{2}\epsilon^2 + \epsilon\sin\epsilon + 3(\cos\epsilon - 1) \\ &+ i\sigma_y\{2\epsilon + \epsilon\cos\epsilon - 3\sin\epsilon\}\Big] + 0(\epsilon^5). \end{aligned} \tag{6.14}$$

Note that for $t \to 0$, $\epsilon(t) \equiv \epsilon$ is of order $t$ and that the successive square brackets are of orders $\epsilon, \epsilon^2, \epsilon^3, \epsilon^4$ respectively for $\epsilon \to 0$. An analogous result has been obtained by Facchi et. al.[11] for a specific time dependence of $\epsilon(t)$. Eq. (5.8) then gives $A_{\bar{h}}$ and the formula (2.10) the probability $p(\bar{h})$ which can be tested experimentally.

## 7. QUANTUM ANTI-ZENO PARADOX

We recall first the usual Zeno paradox. Let the initial state be $|\psi_0>$ and let the projection operator $|\psi_0><\psi_0|$ be measured at times $t_1, t_2, \cdots t_n$ with $t_j - t_{j-1} = (t_n - t_1)/(n-1)$ and $t_n = t$, and let $n \to \infty$. Then, the definition (2.7) yields,

$$A_h(t, t_1) = \lim_{n\to\infty} e^{iHt}|\psi_0><\psi_0| \exp(-iH(t-t_1)/(n-1))|\psi_0>^{n-1}<\psi_0|e^{-iHt_1}$$

$$= \exp(i(H-\bar{H})t)|\psi_0><\psi_0| \exp(-i(H-\bar{H})t_1), \tag{7.1}$$

where $\bar{H}$ denotes $<\psi_0|H|\psi_0>$ and we assume that[13] $<\psi_0|\exp(-iH\tau)|\psi_0>$ is analytic at $\tau = 0$. Our differential equation also yields exactly this solution for $A_h(t, t_1)$. Taking $t_1 = 0$, we deduce that the probability $p(h)$ of finding the system in the initial state at all times upto $t$ is given by

$$p(h) = ||K_h(t)|\psi_0>||^2 = ||\bar{e}^{i\bar{H}t}|\psi_0>||^2 = 1, \tag{7.2}$$

which is the Zeno paradox. (The result can also be generalized to the case of an initial state described by a density operator, and a measured projection operator of arbitrary rank leaving the initial state unaltered, see below.)

### Anti-Zeno Paradox

The above result may suggest that continuous observation inhibits change of state. Now we prove a far more general result which shows that a generic continuous observation actually ensures change of state. Suppose that the initial state is described by a density operator $\rho(0)$, and we measure the projection operator

$$E_s(t') = U(t')EU^\dagger(t') \tag{7.3}$$

continuously for $t'\epsilon[0, t]$. Here $E$ is an arbitrary projection operator (which need not even be of finite rank) which leaves the initial state unaltered,

$$E\rho(0)E = \rho(0), \tag{7.4}$$

and $U(t')$ is a unitary operator which coincides with the identity operator at $t' = 0$,

$$U^\dagger(t')U(t') = U(t')U^\dagger(t') = 1, U(0) = 1. \tag{7.5}$$

The Heisenberg operator $E_H(t')$ is then

$$E_H(t') = V(t')EV^\dagger(t'), \quad V(t') = e^{iHt'}U(t'). \tag{7.6}$$

Clearly $V(t')$ is also a unitary operator. The definition (2.7) yields, for $t_1 \geq 0$,

$$A_h(t_n, t_1) = V(t_n)(T \prod_{i=1}^{n-1} X(t_i))V^\dagger(t_i), \ n \geq 2 \tag{7.7}$$

where

$$X(t_i) \equiv EV^\dagger(t_{i+1})V(t_i)E, \tag{7.8}$$

and $A_h(t_1, t_1) = V(t_1)EV^\dagger(t_1)$. Denoting

$$Y(t_j) = T \prod_{i=1}^{j-1} X(t_i), \ j \geq 2; \ Y(t_1) = E, \tag{7.9}$$

and noting that $EY(t_{j-1}) = Y(t_{j-1})$, we have

$$Y(t_j) - Y(t_{j-1}) = E(V^\dagger(t_j)V(t_{j-1}) - 1)EY(t_{j-1}). \tag{7.10}$$

Taking $t_{j-1} = t', \ t_j = t' + \delta t, \ n \to \infty$, we have $\delta t = 0(1/n)$, and

$$E(V^\dagger(t' + \delta t)V(t') - 1)E = \delta t E \frac{dV^\dagger(t')}{dt'} V(t')E + 0(\delta t)^2. \tag{7.11}$$

To derive that the last term on the right-hand side is $0(\delta t)^2$ in the weak sense (i.e., for matrix elements between any two arbitrary state vectors in the Hilbert space), we make the smoothness assumption that $E(V^\dagger(t' + \tau)V(t') - 1)E$ is analytic in $\tau$ at $\tau = 0$ in the weak sense. (It may be seen that this reduces to analyticity of $< \psi_0 | \exp(-iH\tau) | \psi_0 >$ in the usual Zeno case[16]). Hence the $n \to \infty$ limit yields,

$$A_h(t, t_1) = V(t)Y(t)V^\dagger(t_1), \tag{7.12}$$

where

$$\frac{dY(t')}{dt'} = E \frac{dV^\dagger(t')}{dt'} V(t')EY(t'). \tag{7.13}$$

Solving the differential equation, we obtain,

$$A_h(t,t_1) = V(t)T\exp(\int_{t_1}^{t} dt' E\frac{dV^\dagger(t')}{dt'}V(t')E)EV^\dagger(t_1). \tag{7.14}$$

It is satisfying to note that this expression indeed solves our basic differential equation (5.5) as can be verified very easily by direct substitution.

The most crucial point for deriving the anti-Zeno paradox is that the operator

$$T\exp(\int_{t_1}^{t} dt' E\frac{dV^\dagger(t')}{dt'}V(t')E) \equiv W(t,t_1)$$

is unitary, because $(dV^\dagger(t')/dt')V(t')$ is anti-hermitian as a simple consequence of the unitarity of $V(t')$. Taking $t_1 = 0$, Eq. (2.10) gives the probability of finding $E_s(t') = 1$ for all $t'$ from $t' = 0$ to $t$ as

$$p(h) = \mathrm{Tr}\left(V(t)W(t,0)EV^\dagger(0)\rho(0)V(0)EW^\dagger(t,0)V^\dagger(t)\right) = \mathrm{Tr}\rho(0) = 1, \tag{7.15}$$

where we have used $V(0) = 1$, $E\rho(0)E = \rho(0)$, the unitarity of $V(t)$ and the unitarity of $W(t,0)$. This completes the demonstration of the anti-Zeno paradox: continuous observation of $E_s(t) = U(t)EU^\dagger(t)$ with $U(t) \neq 1$ ensures that the initial state must change with time such that the probability of finding $E_s(t) = 1$ at all times during the duration of the measurement is unity.

This remarkable result means that during continuous observation the quantum state (whether pure or represented by a density matrix) has an effectively unitary evolution! Explicitly, for initially pure states

$$\psi_h(t) = K_h(t)\psi(0), ||\psi_h(t)|| = 1, \tag{7.16}$$

and for initial density matrix states,

$$\rho_h(t) = K_h(t)\rho(0)K_h^\dagger(t),\ \mathrm{Tr}\rho_h(t) = 1. \tag{7.17}$$

Our explicit expressions for $K_h(t)$ yield,

$$E_s(t)\psi_h(t) = \psi_h(t) \tag{7.18}$$

and the "effective" unitary evolution[17]

$$i\frac{\partial\psi_h(t)}{\partial t} = \left\{E_s(t)HE_s(t) + i\left[\frac{dE_s(t)}{dt}, E_s(t)\right]\right\}\psi_h(t), \tag{7.19}$$

the operator in the parenthesis on the right-hand side being Hermitean.

## 8. MEASUREMENTS REPRESENTED BY PROJECTORS OF FINITE RANK AND COMPARISON WITH PREVIOUS WORK

In order to compare with previous work[11] and also to bring out the simplicity of our explicit formulae consider projectors $E$ (and therefore $E_s(t)$) of finite rank.

Rank one. If $E$ is of rank one, then

$$E = |\psi(0)\rangle\langle\psi(0)|,\ E_s(t) = |\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)|, \tag{8.1}$$

where

$$|\tilde{\psi}(t)\rangle = U(t)|\psi(0)\rangle.$$

Our formulae yield, taking $t_1 = 0$,

$$K_h(t) = U(t)|\psi(0)\rangle\langle\psi(0)| \exp\left(i\int_0^t dt'\phi(t')\right), \tag{8.2}$$

and

$$|\psi_h(t)\rangle = |\tilde{\psi}(t)\rangle \exp\left(i\int_0^t dt'\phi(t')\right), \tag{8.3}$$

where

$$\phi(t') = \langle\tilde{\psi}(t')|\left(i\frac{\partial}{\partial t'} - H\right)|\tilde{\psi}(t')\rangle, \tag{8.4}$$

which is exactly the result obtained by Anandan and Aharonov[11].

Rank n. If $E$ is of rank $n$, then

$$E = \sum_{\alpha=1}^{n}|\alpha\rangle\langle\alpha|,\ E_s(t) = \sum_{\alpha=1}^{n}|\tilde{\psi}_\alpha(t)\rangle\langle\tilde{\psi}_\alpha(t)|, \tag{8.5}$$

where,

$$|\tilde{\psi}_\alpha(t)\rangle = U(t)|\alpha\rangle. \tag{8.6}$$

We find that

$$|\psi_h(t)\rangle = U(t)T\exp\left(\int_0^t dt' \sum_{\alpha,\beta=1}^{n} |\alpha\rangle f_{\alpha\beta}(t')\langle\beta|\right)|\psi(0)\rangle \tag{8.7}$$

where $f_{\alpha\beta}(t')$ is the anti-Hermitian matrix,

$$\begin{aligned} f_{\alpha\beta}(t') &= i\langle\alpha|U^\dagger(t')\left(i\frac{\partial}{\partial t'} - H\right)U(t')|\beta\rangle \\ &= i\langle\tilde{\psi}_\alpha(t')|\left(i\frac{\partial}{\partial t'} - H\right)|\tilde{\psi}_\beta(t')\rangle. \end{aligned} \tag{8.8}$$

Note that the time-ordering instruction is now essential as the matrices $f(t')$, and $f(t'')$ with $t' \neq t''$ do not commute. Eq. (8.7) is thus a non-trivial generalisation of the Anandan-Aharonov result (8.3).

## 9. MATHEMATICAL REMARKS

The great generality of the present results with respect to the ordinary Zeno paradox[5] derives from the fact that the unitary operator $V(t)$ need not even obey the semigroup law[5] $V(t)V(s) = V(t+s)$ which played a crucial role in the Misra-Sudarshan proof. Further, the following remarks about the set of pairs $(E,\rho)$ [with $\rho$ a density operator] fulfilling $E\rho E = \rho$ can be made. The first is that as $E$ and $\rho$ are self-adjoint, this condition is equivalent to either of the requirements $E\rho = \rho$, or $\rho E = \rho$. They mean just that $\rho$ is zero on the range of $(1-E)$. The properties of the pairs $(E,\rho)$ in a finite-dimensional quantum theory are simple. In that case, the density operators, being a convex set, are connected and contractible while the connected components of projectors $E$ consist of all the projectors of the same rank. Thus for fixed rank $n$ of projectors, the allowed pairs $(E,\rho)$ form a connected space with the structure of a fibre bundle, with projectors forming the base and a fibre being a convex set. This bundle is trivial, the fibres being contractible. If the quantum Hilbert space $\mathcal{H}_{n+k}$ is of dimension $n+k$, its unitary group $U(n+k) = \{U\}$ acts on $(E,\rho)$ by conjugation: $E \to UEU^{-1}$, $\rho \to U\rho U^{-1}$. This action is an automorphism of the bundle. Since any two projectors of the same rank are unitarily related, it is also transitive on the base. The nature of the base follows from this remark. The stability group of $E$ is $U(n)\times U(k)$ where

$U(n)$ and $U(k)$ act as identities on the range of $(1 - E)$ and $E$ respectively. Thus the base, as is well-known, is the Grassmannian[18] $G_{n,k}(C) = U(n+k)/[U(n) \times U(k)]$. When we pass to quantum physics in infinite dimensions, the space of connected projectors are determined by orbits of infinite-dimensional unitary groups, and, in addition, a projector can itself be of infinite rank. In this manner, general applications of our results will involve infinite-dimensional Grassmannians (on which there are excellent reviews[19]).

### 10. CONCLUSION

It should be stressed that within standard quantum mechanics and its measurement postulates, both the usual Zeno paradox and the anti-Zeno paradox derived here are theorems. The two paradoxes appear 'paradoxical' and 'mutually contradictory' only when we forget Bohr's insistence that quantum results depend not only on the quantum state, but also on the entire disposition of the experimental apparatus. Indeed the apparatus to measure $E$ and $U(t)EU^\dagger(t)$ are different. It would be interesting to analyse how these results appear in a quantum theory of closed systems (including the apparatus) in which there is no notion of measurements. It will also be interesting to devise experimental tests of the anti-Zeno effect along lines used to test the ordinary Zeno effect[9].

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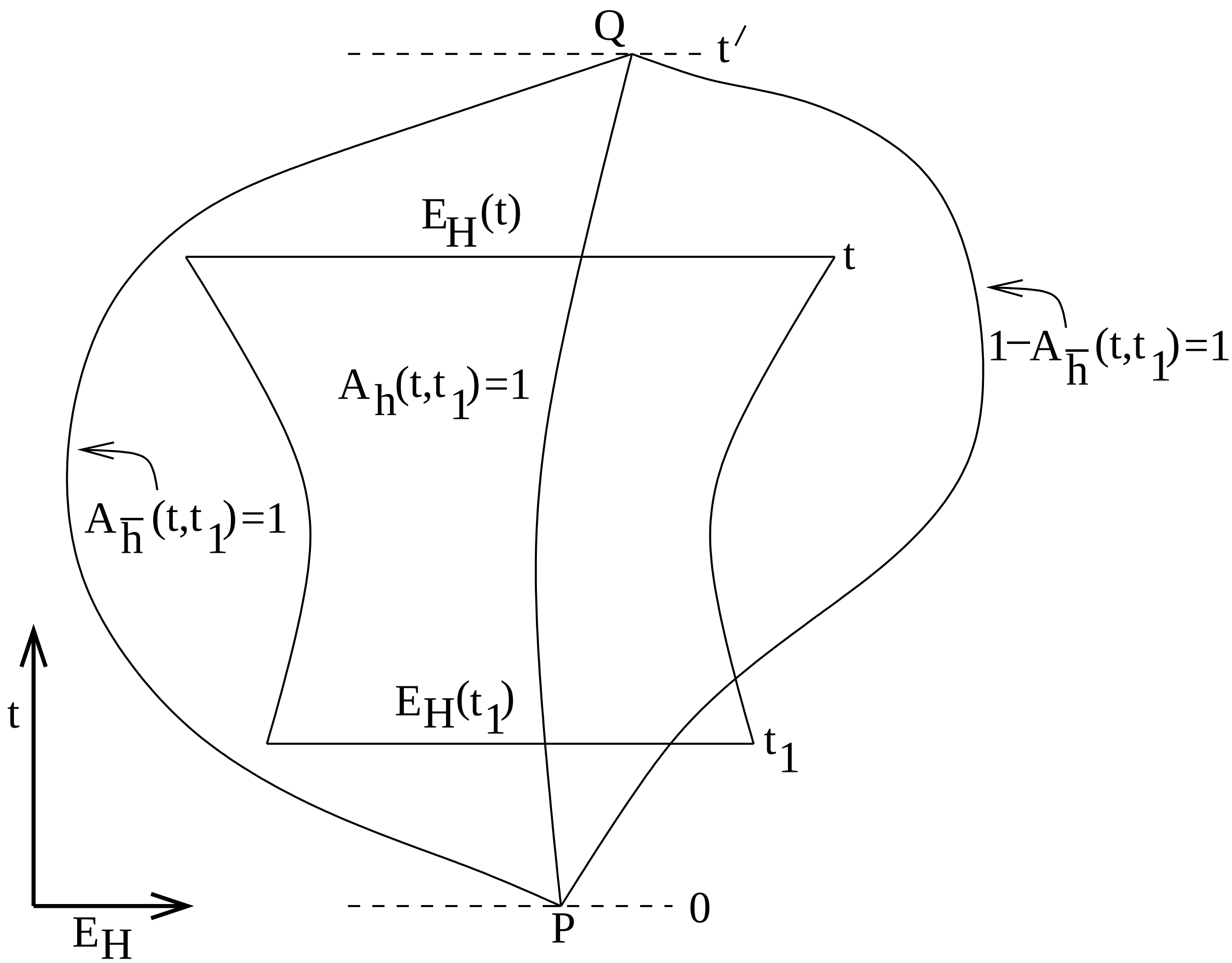


Fig. 1. We may visualize the product of projectors $A_h(t,t_1)$ by a spacetime region if we represent $E_H(t)$ by a space region (though $E_H(t)$ need not be a position projector). In the path integral approach sum over paths which intersect the spacetime region at least once $(1 - A_{\bar{h}}(t,t_1) = 1)$ yield the propagator $K_{h'}(t')$, paths which stay inside the region for times between $t_1$ and $t$, $(A_h(t,t_1) = 1)$ yield the propagator $K_h(t')$, and paths which do not intersect the spacetime region at all $(A_{\bar{h}}(t,t_1) = 1)$ yield $K_{\bar{h}}(t')$.