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Quantum Zeno and anti-Zeno paradoxes

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Abstract. Continuous observation of a time independent projection operator is known to prevent change of state (the quantum Zeno paradox). We discuss the recent result that generic continuous measurement of time dependent projection operators will in fact ensure change of state: an anti-Zeno paradox.

Keywords. Continuous measurements; quantum measurement theory; quantum anti-zeno paradox.

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1. Introduction

Quantum predictions for instantaneous changes of state vectors due to measurements are responsible for several paradoxes such as the Schrödinger Cat paradox, the Einstein-Podolsky–Rosen paradox, the quantum Zeno paradox and the recently discovered quantum anti-Zeno paradox. Here we discuss the quantum Zeno and anti-Zeno paradoxes which arise due to infinitely frequent measurements of time independent and time dependent projection operators respectively.

The early formulations of infinitely frequent or continuous observation are due to Von Neumann [1] and Feynman [2], who used the operator approach and the path integral approach respectively. Feynman's path integral approach was elaborated by Mensky [3] who also showed its equivalence to the phenomenological master equation approach for open quantum systems using models of system-environment coupling developed by Joos and Zeh and others [4].

Von Neumann [1] derived the remarkable result that by suitably designed continuous measurements, any pure state could be steered into any other pure state if we ignore the Hamiltonian evolution between measurements (or equivalently, for Hamiltonian equal to zero). On the other hand taking an arbitrary self-adjoint Hamiltonian into account, Misra and Sudarshan [5] asked: what is the rigorous quantum description of ideal continuous measurement of a projector E (time independent in the Schrödinger representation) over a time interval $[0, T]$? Their work led them to rigorous confirmation of a seemingly paradoxical conclusion noted earlier [6]. The conclusion 'that an unstable particle which is continuously observed to see whether it decays will never be found to decay' or that a 'watched pot never boils' [7] was christened 'Zeno's paradox in quantum theory' by Misra and Sudarshan [5]. The paradox has been theoretically scrutinized questioning the consistency of infinitely frequent measurements with time-energy and position-momentum uncertainty principles [8]. Experimental tests [9] and their different interpretations have been vigorously discussed.

In our recent letter [10] we showed that in contrast to the continuous measurement of a time independent projection operator which prevents the quantum state from changing (the quantum Zeno paradox), the generic continuous measurement of a time dependent projection operator $E_s(t)$ forces the quantum state to change with time (the quantum anti-Zeno paradox). We have emphasized that though the two effects (one inhibiting change of state and the other ensuring change of state) are physically opposite, they are mutually consistent as they refer to different experimental arrangements. We derived the anti-Zeno paradox in a very broad framework with arbitrary Hamiltonian, arbitrary density matrix states, and measurement of arbitrary but smooth time dependent projection operators. Further, Facchi *et al* [10] have discussed a special case of the quantum anti-Zeno paradox which they called ‘dynamic quantum Zeno effect’ for a spin 1/2 system guided through a closed loop in Hilbert space with a specific assumption on the time dependence of the projection operators. Kofman and Kurizki [10] noted that even for time independent measurements, when the frequency of measurements is smaller than a characteristic difference of eigenfrequencies of the system, an anti-Zeno effect results. Of course our method would yield the appropriate generalisation of their results to time dependent measurements.

Here I shall begin with a review of the quantum Zeno paradox and its intimate historical connection to the phenomenon of non-exponential decay. I then review the recent results of Balachandran and Roy [10] on continuous measurements of time dependent projection operators which lead to the much more generic quantum anti-Zeno paradox. The quantum Zeno paradox and the quantum anti-Zeno paradox demonstrate that the effect of continuous measurements on quantum states discovered by Von Neumann in the case of zero Hamiltonian, in fact hold also in the presence of arbitrary self-adjoint Hamiltonian.

2. Ideal measurements in quantum theory

For a quantum system with a self-adjoint Hamiltonian H, an initial state vector $|\psi(0)\rangle$ evolves to a state vector $|\psi(t)\rangle$,

$$|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle. \tag{1}$$

More generally, an initial state with density operator $\rho(0)$ has the Schrödinger time evolution

$$\rho(t) = \exp(-iHt)\rho(0)\exp(iHt), \tag{2}$$

which preserves the normalization condition Tr $\rho(t) = 1$. In an ideal instantaneous measurement of a self-adjoint projection operator E, the probability of finding $E = 1$ is $\mathrm{Tr}(E\rho E)$ and on finding the value 1 for E the state collapses according to

$$\rho \to \rho' = E\rho E/\mathrm{Tr}(E\rho E). \tag{3}$$

If projectors $E_1, E_2, \ldots, E_n$ are measured at times $t_1, t_2, \ldots, t_n$ respectively, with Schrödinger evolution in between measurements, the probability $p(h)$ for the sequence of events h,

$$h: \; E_1 = 1 \text{ at } t = t_1; \; E_2 = 1 \text{ at } t = t_2; \cdots; \; E_n = 1 \text{ at } t = t_n \tag{4}$$

is [1],

$$p(h) = ||\psi_h(t')||^2, \; \psi_h(t') = K_h(t')\psi(0), \; t' > t_n. \tag{5}$$

Here $K_h(t')$ is the Feynman propagator modified by the events h,

$$K_h(t') = \exp(-iHt')A_h(t_n, t_1), \tag{6}$$

where

$$A_h(t_n, t_1) = E_H(t_n)E_H(t_{n-1}) \cdots E_H(t_1) = T \prod_{i=1}^{n} E_H(t_i), \tag{7}$$

with T denoting 'time-ordering' and the Heisenberg operators $E_H(t_i)$ are related to the Schrödinger operators by the usual relation

$$E_H(t_i) = \exp(iHt_i)E_s(t_i) \exp(-iHt_i), \; E_s(t_i) \equiv E_i. \tag{8}$$

The state vector of the system at a time t' after the events h is

$$\psi_h(t')/||\psi_h(t')||.$$

Correspondingly, if the initial state is a density operator $\rho(0)$, the probability $p(h)$ for the events h is given by

$$p(h) = \text{Tr } K_h(t')\rho(0)K_h^\dagger(t') = \text{Tr } A_h(t_n, t_1) \; \rho(0)A_h^\dagger(t_n, t_1), \tag{9}$$

and the state at $t' > t_n$ is

$$K_h(t')\rho(0)K_h^\dagger(t')/\text{Tr } (K_h(t')\rho(0)K_h^\dagger(t')).$$

3. Non-exponential decay

In spite of the apparent ubiquitousness of the exponential decay law, it can be shown that the basic principles of quantum mechanics imply that the exponential law of decay of an unstable particle must break down both at very short and at very long times. I shall closely follow a presentation due to Martin [11].

Consider an unstable particle B decaying into particles $C + D$, ... etc.

$$B \to C + D, \ldots. \tag{10}$$

Suppose that the total Hamiltonian H has a lower bound M,

$$H \geq M, \tag{11}$$

and the initial state is denoted by $|B\rangle$,

$$|\psi(t=0)\rangle = |B\rangle. \tag{12}$$

Then, at time t,

$$|\psi(t)\rangle = \mathrm{e}^{-iHt}|B\rangle. \tag{13}$$

The probability amplitude of finding the undecayed state $|B\rangle$ at time t is

$$A(t) = \langle B|\mathrm{e}^{-iHt}|B\rangle; \tag{14}$$

and the probability of finding $|B\rangle$ at time t is,

$$|A(t)|^2 = \langle\psi(t)|E|\psi(t)\rangle, \tag{15}$$

where E denotes the projector

$$E = |B\rangle\langle B|. \tag{16}$$

Let $\{|\lambda, r\rangle\}$ denote a complete set of eigenvectors of H,

$$H|\lambda, r\rangle = (M+\lambda)|\lambda, r\rangle, \quad \lambda \geq 0, \tag{17}$$

where r is a degeneracy index and the eigenvalues $(M+\lambda)$ of H are $\geq M$ by assumption. We have

$$\mathbf{1} = \int_0^\infty \mathrm{d}\lambda E_\lambda, \quad E_\lambda \equiv \sum_r |\lambda, r\rangle\langle\lambda, r|. \tag{18}$$

Using this resolution of the identity we have

$$\begin{aligned} A(t) &= \langle B|\mathrm{e}^{-iHt}\int_0^\infty \mathrm{d}\lambda E_\lambda|B\rangle \\ &= \mathrm{e}^{-iMt}\langle B|\int_0^\infty \mathrm{d}\lambda \mathrm{e}^{-i\lambda t}E_\lambda|B\rangle \\ &= \mathrm{e}^{-iMt}a(t), \end{aligned} \tag{19}$$

where

$$a(t) = \int_{-\infty}^\infty \mathrm{d}\lambda \mathrm{e}^{-i\lambda t}\omega(\lambda), \tag{20}$$

with

$$\omega(\lambda) = \sum_r |\langle B|\lambda, r\rangle|^2 \geq 0, \quad \text{for } \lambda \geq 0, \tag{21}$$

and

$$\omega(\lambda) = 0 \ \text{ for } \ \lambda < 0. \tag{22}$$

We shall now prove the impossibility of exponential decay for $t \to \infty$.

Consider the analytic continuation of $\mathrm{Re}\, a(t)$ to complex t-plane,

$$r(t) = \frac{a(t) + a^{\star}(t^{\star})}{2} = \int_0^{\infty} \mathrm{d}\lambda \cos(\lambda t)\omega(\lambda) \tag{23}$$

which implies

$$\omega(\lambda) = \frac{1}{2\pi} \int_0^{\infty} \mathrm{d}t \cos(\lambda t) r(t). \tag{24}$$

Exponential decay would imply that $|A(t)|$ and consequently $r(t)$ must decay exponentially for $t \to \infty$,

$$|r(t)| \leq C \exp(-\gamma t). \tag{25}$$

This implies that the cosine Fourier representation of $\omega(\lambda)$ given above can be continued analytically into the strip $|\mathrm{Im}\ \lambda| < \gamma$. This is impossible since we know that $\omega(\lambda) = 0$ for $\lambda < 0$. Hence the hypothesis of exponential decay at long times must be false. What we have used is essentially the Paley–Wiener theorem.

Khalfin [6] noted that exponential decay cannot hold for short times either. Denoting

$$\langle B|H|B\rangle = \bar{H}, \tag{26}$$

we have

$$\begin{aligned} A(t) &= \mathrm{e}^{-i\bar{H}t} \langle B| \mathrm{e}^{-i(H-\bar{H})t} |B\rangle \\ &= \mathrm{e}^{-i\bar{H}t} \left[1 - \frac{t^2}{2!} \langle B|(H-\bar{H})^2|B\rangle + \cdots \right]. \end{aligned} \tag{27}$$

Assuming that the series on the right-hand side has a finite radius of convergence we have

$$|A(t)|^2 = 1 + O|t^2), \text{ for } t \to 0, \tag{28}$$

instead of what exponential decay requires,

$$\mathrm{e}^{-\Gamma t} \to 1 - \Gamma t \text{ for } t \to 0. \tag{29}$$

The nonexponential behaviour (28) is intimately connected to the quantum-Zeno paradox.

4. Quantum Zeno paradox

Infinitely frequent (or continuous) observation of the same observable prevents change of state [5]. This elementary consequence of the quantum measurement postulates has been variously described: ‘watched unstable particle does not decay’, ‘watched clock does not move’, ‘watched kettle does not boil’ etc. Its paradoxical nature is sometimes thought of as an ‘example of taking quantum measurement postulates seriously and not liking the results’. Its experimental tests [9] have not yet settled questions of interpretation.

Let us give here an elementary proof of the quantum Zeno paradox. Apply the Khalfin argument repeatedly. Starting with an initial state $|B\rangle$, and measuring the projector $E = |B\rangle\langle B|$ repeatedly, at times T/n, $2T/n$, ..., T, the probability of finding $E = 1$ in each of these measurements is

$$\begin{aligned} |\langle B|\mathrm{e}^{-iHT/n}|B\rangle|^{2n} &= \left|\mathrm{e}^{-i\bar{H}T/n}\left[1-\frac{T^2}{2n^2}\langle B|(H-\bar{H})^2|B\rangle+\cdots\right]\right|^{2n} \\ &= 1-\frac{T^2}{n}\langle B|(H-\bar{H})^2|B\rangle+\cdots \\ &\stackrel{n\to\infty}{\longrightarrow} 1, \end{aligned} \tag{30}$$

provided that $\langle B|\mathrm{e}^{-iH\tau}|B\rangle$ is analytic at $\tau = 0$ (see Chiu *et al* [5]).

5. Quantum anti-Zeno paradox

We now discuss the results of Balachandran and Roy [10] on continuous measurements of time dependent projectors. Consider infinitely frequent measurements of the projection operators $E_s(t_i)$ which are values at times t_i of a projection valued function $E_s(t)$. We make the technical assumption that the corresponding Heisenberg operator $E_H(t)$ is weakly analytic. We seek to calculate the modified Feynman propagator

$$K_h(t') = \exp(-iHt')A_h(t,t_1), \tag{31}$$

where

$$A_h(t,t_1) = \lim_{n\to\infty} T\prod_{i=1}^{n} E_H(t_1+(t-t_1)(i-1)/(n-1)) \tag{32}$$

which is the $n\to\infty$ limit of eq. (7) with a specific choice of the t_i. Let us also introduce the projectors $\bar{E}_i = 1 - E_i$ which are the orthogonal complements of the projectors E_i, and a sequence of events $\bar{h}$ complementary to the sequence h,

$$\bar{h} : \bar{E}_1 = 1 \text{ at } t = t_1;\ \bar{E}_2 = 1 \text{ at } t = t_2, \ldots, \bar{E}_n = 1 \text{ at } t = t_n. \tag{33}$$

Corresponding to eqs (6), (7), (31), (32), we have equations with $E \to \bar{E}$, $h \to \bar{h}$. The special interest in $K_{\bar{h}}(t')$ is that it is closely related to the propagator

$$K_{h'}(t') \equiv \exp(-iHt') - K_{\bar{h}}(t') = \exp(-iHt')[1 - A_{\bar{h}}(t,t_1)],\ h' \equiv \underset{i}{U} E_i, \tag{34}$$

which represents the modified Feynman propagator corresponding to the union of the events E_i, i.e. to at least one of the events $E_s(t_i) = 1$ occurring, with t_i lying between t_1 and t. Our object is to obtain exact operator expressions for the propagators K_h, $K_{\bar{h}}$ which have been defined above by formal infinite products.

We see from eq. (31) that $A_h(t,t_1)(A_{\bar{h}}(t,t_1))$ represents the modification of the Feynman propagator due to the continuous measurement corresponding to the sequence of events $h(\bar{h})$. Consider first the operators $A_h(t_i,t_1),\ A_{\bar{h}}(t_i,t_1)$ before taking the $n\to\infty$ limit, and note that

$$A_h(t_i,t_1)=E_H(t_i)A_h(t_{i-1},t_1),\ A_{\bar{h}}(t_i,t_1)=\bar{E_H}(t_i)A_{\bar{h}}(t_{i-1},t_1). \tag{35}$$

The relation $\bar{E}_H^2=\bar{E}_H$ implies $A_{\bar{h}}(t_{i-1},t_1)=\bar{E}_H(t_{i-1})A_{\bar{h}}(t_{i-1},t_1)$. We thus have

$$A_{\bar{h}}(t_i,t_1)-A_{\bar{h}}(t_{i-1},t_1)=(\bar{E_H}(t_i)-\bar{E_H}(t_{i-1}))A_{\bar{h}}(t_{i-1},t_1), \tag{36}$$

and a similar relation for A_h. Dividing by $t_i-t_{i-1}=\delta t$, taking the limit $n\to\infty$ (i.e., $\delta t\to 0$) and assuming that $E_H(t)$ is weakly analytic at $t=0$ we obtain the differential equations,

$$\frac{\mathrm{d}A_{\bar{h}}(t,t_1)}{\mathrm{d}t}=\frac{\mathrm{d}\bar{E}_H(t)}{\mathrm{d}t}A_{\bar{h}}(t_-,t_1),\ \frac{\mathrm{d}A_h(t,t_1)}{\mathrm{d}t}=\frac{\mathrm{d}E_H(t)}{\mathrm{d}t}A_h(t_-,t_1), \tag{37}$$

where the arguments t_- on the right-hand sides indicate that in case of any ambiguity in defining the operator products the arguments have to be taken as $t-\epsilon$ with $\epsilon\to 0$ from positive values and

$$\frac{\mathrm{d}E_H(t)}{\mathrm{d}t}=i[H,E_H(t)]+\exp(iHt)\frac{\mathrm{d}E_s(t)}{\mathrm{d}t}\exp(-iHt). \tag{38}$$

Further $A_{\bar{h}}(t,t_1),A_h(t,t_1)$ must obey the initial conditions

$$A_{\bar{h}}(t_1,t_1)=\bar{E_H}(t_1),\ A_h(t_1,t_1)=E_H(t_1). \tag{39}$$

The measurement differential equations (37) are reminiscent of Schrödinger equation for the time evolution operator except for the fact that the operators $\mathrm{d}\,\bar{E}_H/\mathrm{d}t$, $\mathrm{d}E_H/\mathrm{d}t$ are hermitian whereas in Schrödinger theory the antihermitian operator H/i would occur. Using the initial conditions we obtain the explicit solutions,

$$A_h(t,t_1)=T\exp\left(\int_{t_1}^{t}\mathrm{d}t'\frac{\mathrm{d}E_H(t')}{\mathrm{d}t'}\right)E_H(t_1), \tag{40}$$

and a similar equation with $h\to\bar{h}$, $E_h\to\bar{E}_h$, where the time ordered exponentials have the series expansion

$$T\exp\left(\int_{t_1}^{t}\mathrm{d}t'\frac{\mathrm{d}E_H(t')}{\mathrm{d}t'}\right)=1+\sum_{n=1}^{\infty}\int_{t_1}^{t}\mathrm{d}t'_1\int_{t_1}^{t'_1}\mathrm{d}t'_2\cdots\int_{t_1}^{t'_{n-1}}\mathrm{d}t'_nT\prod_{i=1}^{n}\frac{\mathrm{d}E_H(t'_i)}{\mathrm{d}t'_i}. \tag{41}$$

In general the time-ordered operator products appearing on the right-hand side are distributions and the series on the right-hand side must be taken as the definition of the exponential on the left-hand side; we may not do the integral of $\mathrm{d}E_H(t')/\mathrm{d}t'$ on the left-hand side. Multiplying the expressions for $A_{\bar{h}}(t,t_1)$ and $A_h(t,t_1)$ on the left by $\exp(-iHt')$ then completes the evaluation of the modified Feynman propagators $K_{\bar{h}}(t')$ and $K_h(t)$. These equations will enable us to derive both the Zeno paradox and the anti-Zeno paradox.

The Zeno paradox: Let the initial state be $|\psi_0\rangle$ and let the projection operator $|\psi_0\rangle\langle\psi_0|$ be measured at times $t_1, t_2, \ldots, t_n$ with $t_j - t_{j-1} = (t_n - t_1)/(n-1)$ and $t_n = t$, and let $n \to \infty$. Then, the definition (7) yields

$$A_h(t, t_1) = \lim_{n\to\infty} \mathrm{e}^{iHt}|\psi_0\rangle\langle\psi_0|\exp(-iH(t-t_1)/(n-1))|\psi_0\rangle^{n-1}\langle\psi_0|\mathrm{e}^{-iHt_1}$$
$$= \exp(i(H-\bar{H})t)|\psi_0\rangle\langle\psi_0|\exp(-i(H-\bar{H})t_1), \quad (42)$$

where $\bar{H}$ denotes $\langle\psi_0|H|\psi_0\rangle$ and we assume that $\langle\psi_0|\exp(-iH\tau)|\psi_0\rangle$ is analytic at $\tau = 0$. Our differential equation also yields exactly this solution for $A_h(t, t_1)$. Taking $t_1 = 0$, we deduce that the probability $p(h)$ of finding the system in the initial state at all times up to t is given by

$$p(h) = ||K_h(t)|\psi_0\rangle||^2 = ||\bar{\mathrm{e}}^{i\bar{H}t}|\psi_0\rangle||^2 = 1, \quad (43)$$

which is the Zeno paradox. (The result can also be generalized to the case of an initial state described by a density operator, and the measured projection operator being of arbitrary rank but leaving the initial state unaltered, see below.)

The anti-Zeno paradox: The above result may suggest that continuous observation inhibits change of state. Now we prove a far more general result which shows that a generic continuous observation actually ensures change of state. Suppose that the initial state is described by a density operator $\rho(0)$, and we measure the projection operator

$$E_s(t') = U(t')EU^\dagger(t') \quad (44)$$

continuously for $t'\epsilon[0, t]$. Here E is an arbitrary projection operator (which need not even be of finite rank) which leaves the initial state unaltered,

$$E\rho(0)E = \rho(0), \quad (45)$$

and $U(t')$ is a unitary operator which coincides with the identity operator at $t' = 0$,

$$U^\dagger(t')U(t') = U(t')U^\dagger(t') = 1, U(0) = 1. \quad (46)$$

The Heisenberg operator $E_H(t')$ is then

$$E_H(t') = V(t')EV^\dagger(t'), \;\; V(t') = \mathrm{e}^{iHt'}U(t'). \quad (47)$$

Clearly $V(t')$ is also a unitary operator. The definition (7) yields, for $t_1 \geq 0$,

$$A_h(t_n, t_1) = V(t_n)\left(T\prod_{i=1}^{n-1} X(t_i)\right)V^\dagger(t_i), \; n \geq 2 \quad (48)$$

where

$$X(t_i) \equiv EV^\dagger(t_{i+1})V(t_i)E, \quad (49)$$

and $A_h(t_1, t_1) = V(t_1)EV^\dagger(t_1)$. Denoting

$$Y(t_j) = T\prod_{i=1}^{j-1} X(t_i),\ j \geq 2, \tag{50}$$

$Y(t_1) = E$ and noting that $EY(t_{j-1}) = Y(t_{j-1})$, we have

$$Y(t_j) - Y(t_{j-1}) = E(V^\dagger(t_j)V(t_{j-1}) - 1)EY(t_{j-1}). \tag{51}$$

Taking $t_{j-1} = t',\ t_j = t' + \delta t,\ n \to \infty$, we have $\delta t = 0(1/n)$, and

$$E(V^\dagger(t' + \delta t)V(t') - 1)E = \delta t E\frac{\mathrm{d}V^\dagger(t')}{\mathrm{d}t'}V(t')E + 0(\delta t)^2. \tag{52}$$

To derive that the last term on the right-hand side is $0(\delta t)^2$ in the weak sense (i.e., for matrix elements between any two arbitrary state vectors in the Hilbert space), we make the smoothness assumption that $E(V^\dagger(t'+\tau)V(t')-1)E$ is analytic in τ at $\tau = 0$ in the weak sense. (It may be seen that this reduces to analyticity of $\langle\psi_0|\exp(-iH\tau)|\psi_0\rangle$ in the usual Zeno case). Hence the $n \to \infty$ limit yields

$$A_h(t, t_1) = V(t)Y(t)V^\dagger(t_1), \tag{53}$$

where

$$\frac{\mathrm{d}Y(t')}{\mathrm{d}t'} = E\frac{\mathrm{d}V^\dagger(t')}{\mathrm{d}t'}V(t')EY(t'). \tag{54}$$

Solving the differential equation we obtain,

$$A_h(t, t_1) = V(t)T\exp\left(\int_{t_1}^{t}\mathrm{d}t' E\frac{\mathrm{d}V^\dagger(t')}{\mathrm{d}t'}V(t')E\right)EV^\dagger(t_1). \tag{55}$$

It is satisfying to note that this expression indeed solves our basic differential equation (37) as can be verified very easily by direct substitution.

The most crucial point for deriving the anti-Zeno paradox is that the operator

$$T\exp\left(\int_{t_1}^{t}\mathrm{d}t' E\frac{\mathrm{d}V^\dagger(t')}{\mathrm{d}t'}V(t')E\right) \equiv W(t, t_1)$$

is unitary, because $(\mathrm{d}V^\dagger(t')/\mathrm{d}t')V(t')$ is anti-hermitian as a simple consequence of the unitarity of $V(t')$. Taking $t_1 = 0$, eq. (9) gives the probability of finding $E_s(t') = 1$ for all t' from $t' = 0$ to t as

$$p(h) = \mathrm{Tr}\left(V(t)W(t,0)EV^\dagger(0)\rho(0)V(0)EW^\dagger(t,0)V^\dagger(t)\right) = \mathrm{Tr}\rho(0) = 1, \tag{56}$$

where we have used $V(0) = 1$, $E\rho(0)E = \rho(0)$, the unitarity of $V(t)$ and the unitarity of $W(t, 0)$. This completes the demonstration of the anti-Zeno paradox: continuous observation of $E_s(t) = U(t)EU^\dagger(t)$ with $U(t) \neq 1$ ensures that the initial state must change with time such that the probability of finding $E_s(t) = 1$ at all times during the duration of the measurement is unity.

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