Pramāṇa – J. Phys., Vol. 26, No. 1, January 1986, pp. L85–L91. © Printed in India.

Bosonic string theories with new boundary conditions

S M ROY and VIRENDRA SINGH

Tatal Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

MS received 25 October 1985

Abstract. We show that the classical Nambu-Goto string in D dimensions admits Poincaré invariance in d dimensions ($d \leqslant D$) if (i) $d-2$ of the transverse co-ordinates x^i are periodic and the rest quasi-periodic involving a real orthogonal matrix with $(D-d)$ $(D-d-1)/2$ free parameters, or if (ii) $d-2$ of x^i obey Neumann and the rest obey a boundary condition involving N free parameters, where $N = (D-d)^2/2$ if $D-d$ is even, and $N = [(D-d)^2-1]/2$ if $D-d$ is odd.

Keywords. Bosonic string theory; boundary condition.

PACS No. 11·10; 12·40; 11·30

String theories at present offer a hope of having a satisfactory theory of particle interactions including gravitation (Goddard *et al* 1973; Schwarz 1982; Green 1983; Brink 1984; Green and Schwarz 1984; Witten 1984; Green and Schwarz 1985). There are two known bosonic string theories (Goddard *et al* 1973; Schwarz 1982; Green 1983; Brink 1984) viz (i) closed string with periodic boundary conditions and (ii) open string with Neumann Boundary Conditions. Besides one has fermionic strings (Ramond 1971; Neveu and Schwarz 1971a, b) and heterotic strings (Gross *et al* 1985). These, however, can be embedded in the known bosonic string theories (Freund 1985; Casher *et al* 1985). It is therefore of considerable interest to investigate the possibility of new bosonic string theories. Of particular practical interest is the question, whether the absence of free parameters, a striking feature of present string theories, will persist in the new theories.

Here we report a family of new string theories based on the classical Nambu-Goto (Nambu 1970; Goto 1971; Hara 1971) action in D dimensions, but endowed with new boundary conditions. After imposing the requirement of Poincaré invariance in the "physical" d dimensions, where $d < D$, we show that we are still left with a $[(D-d)$ $(D-d-1)/2]$ parameter family of theories. The usual 'open' and 'closed' strings are thus special cases of a continuum of acceptable theories. On quantisation the usual string theories lead to restrictions on the Regge slope parameter $\alpha(0)$ and on the dimension ($D = 26$). Similar restrictions are obtained (Roy and Singh 1985) also on quantization of the new family of theories presented here.

Consider the Nambu-Goto action for a string with co-ordinate $x^\mu(\sigma, \tau)$, where $\mu = 0, 1, 2, \ldots, D-1$, $0 \leqslant \sigma \leqslant 2\pi$, and $\tau_1 \leqslant \tau \leqslant \tau_2$,

$$S = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\sigma\, L. \tag{1}$$

Here α' is a real constant of dimension $(\text{mass})^{-2}$, and

$$L = -\{(x'\cdot\dot{x})^2 - x'^2\dot{x}^2\}^{1/2}\frac{1}{2\pi\alpha'}, \tag{2}$$

$$(x')^\mu \equiv \partial x^\mu/\partial\sigma, \quad \dot{x}^\mu \equiv \partial x^\mu/\partial\tau, \tag{3}$$

and our metric is $g^{\mu\nu} = \text{diag}(1, -1, -1, \ldots)$. Being proportional to the area of the string world sheet, the action is independent of the particular choice of the parameters σ, τ used to describe that sheet. Consider deriving the equations of motion of the string from the principle of least action. For an arbitrary variation $\delta x^\mu(\sigma, \tau)$,

$$\delta S = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\sigma\, \delta x^\mu(\sigma,\tau)\left[-\frac{\partial}{\partial\sigma}\left(\frac{\partial L}{\partial x'^\mu}\right) - \frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right)\right]$$

$$+\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\left(\frac{\partial L}{\partial x'^\mu}\,\delta x^\mu\right)\Bigg|_{\sigma=0}^{2\pi} + \int_0^{2\pi}\mathrm{d}\sigma\left(\frac{\partial L}{\partial \dot{x}^\mu}\,\delta x^\mu\right)\Bigg|_{\tau=\tau_1}^{\tau=\tau_2}. \tag{4}$$

The condition $\delta S = 0$ then yields the usual Euler-Lagrange equations. If the variations are subjected to $\delta x^\mu(\sigma, \tau_1) = \delta x^\mu(\sigma, \tau_2) = 0$, and to boundary conditions at $\sigma = 0, 2\pi$ such that

$$\frac{\partial L}{\partial x'^\mu}\,\delta x^\mu(\sigma,\tau)\Bigg|_{\sigma=0}^{2\pi} = 0. \tag{5}$$

To elucidate the nature of these boundary conditions it is convenient to choose σ, τ to obtain an orthonormal *transverse gauge*:

$$x'\cdot\dot{x} = 0,\; x'^2 + \dot{x}^2 = 0,\; x^+ \equiv \frac{x^0 + x^1}{\sqrt{2}} \equiv q^+ + p^+\tau. \tag{6}$$

Then $x^- \equiv (x^0 - x^1)/\sqrt{2}$ can be solved for in terms of the transverse $x^i (i = 2, 3, \ldots, D-1)$ and one integration constant using

$$\dot{x}^- = \frac{(\dot{x}^{\mathrm{Tr}})^2 + (x'^{\mathrm{Tr}})^2}{2p^+},$$

$$x'^- = \frac{\dot{x}^{\mathrm{Tr}}\cdot x'^{\mathrm{Tr}}}{p^+}, \quad x^{\mathrm{Tr}} \equiv (x^2, x^3, \ldots x^{D-1}). \tag{7}$$

To separate the first d dimensions in which we wish Poincaré invariance from the remaining $(D-d)$ it will be convenient to use the notation

$$x^A = (x^2, \ldots, x^{d-1}),\; x^B = (x^d, \ldots, x^{D-1}). \tag{8}$$

The Euler-Lagrange equations now become

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right)x(\sigma,\tau) = 0, \tag{9}$$

and the boundary conditions (5) becomes

$$x'^{\mathrm{Tr}}\cdot\delta x^{\mathrm{Tr}}(\sigma,\tau)\Bigg|_{\sigma=0}^{2\pi} = 0. \tag{10}$$

We shall show that equations (10) are obeyed not only for the usually discussed boundary conditions of open strings ($x'^{\mathrm{Tr}} = 0$ at $\sigma = 0, 2\pi$) and closed strings ($x^{\mathrm{Tr}}(2\pi, \tau) - x^{\mathrm{Tr}}(0, \tau) = x'^{\mathrm{Tr}}(2\pi, \tau) - x'^{\mathrm{Tr}}(0, \tau) = 0$), but also for a much larger class of boundary conditions. The first step is to realize that (10) may be rewritten as

$$(\psi + \phi)^T(\delta\psi - \delta\phi) = 0, \tag{11}$$

with the superscript T denoting transpose, and ψ and ϕ denoting the $2(D-2)$ dimensional column vectors.

$$\psi = \begin{Bmatrix} (x'(0,\tau) + x(0,\tau))^A \\ (x'(2\pi,\tau) - x(2\pi,\tau))^A \\ (x'(0,\tau) + x(0,\tau))^B \\ (x'(2\pi,\tau) - x(2\pi,\tau))^B \end{Bmatrix}, \phi = \begin{Bmatrix} (x'(0,\tau) - x(0,\tau))^A \\ (x'(2\pi,\tau) + x(2\pi,\tau))^A \\ (x'(0,\tau) - x(0,\tau))^B \\ (x'(2\pi,\tau) + x(2\pi,\tau))^B \end{Bmatrix}. \tag{12}$$

The Euler Lagrange equation (9) are linear in x. To have super-position principle we also seek linear homogeneous boundary conditions of the form $A_1\psi + A_2\phi = 0$ where A_1, A_2 are $2(D-2) \times 2(D-2)$ dimensional matrices. A_1 and A_2 are to be found such that for any solution $x(\sigma, \tau)$ of the equations of motion and the boundary conditions $A_1\psi + A_2\phi = 0$, any variation $\delta x(\sigma, \tau)$ subject to $A_1\delta\psi + A_2\delta\phi = 0$ obeys (11). Clearly, the variation $\delta x(\sigma, \tau) = \lambda x(\sigma, \tau)$ where λ is a constant obeys $A_1\delta\psi + A_2\delta\phi = \lambda(A_1\psi + A_2\phi) = 0$, and hence we require

$$(\psi + \phi)^T(\psi - \phi) = 0, \tag{13}$$

i.e., $$\psi^T\psi = \phi^T\phi, \text{ and } \phi^T\psi - \psi^T\phi = 0. \tag{14}$$

Hence the boundary conditions must be of the form

$$\psi = U\phi, \quad U^T = U^{-1} = U, \tag{15}$$

where U is a real $2(D-2) \times 2(D-2)$ dimensional matrix. Conversely, any arbitrary variation δx respecting the boundary conditions (15) is directly seen to obey (11). We thus have

Theorem 1. In the orthonormal transverse gauge, the Nambu-Goto action is stationary under variations subject to $\delta x(\sigma, \tau_1) = \delta x(\sigma, \tau_2) = 0$ if $x(\sigma, \tau)$ obeys the equations of motion $(\partial^2/\partial\tau^2 - \partial^2/\partial\sigma^2)x = 0$ and the boundary conditions $\psi = U\phi$ where U is a symmetric, real orthogonal matrix.

It is trivial to check that the usual boundary conditions are of this form e.g. open strings correspond to $U = -1$.

We now show that the requirement of relativistic (Poincaré) invariance in the first d dimensions ($2 \leqslant d \leqslant D$) can be used to restrict the free parameters of U. To include physical Poincaré invariance it is desirable to have $d \geqslant 4$. This restriction is however not insisted upon in the present work.

Poincaré invariance. Let $x^\mu(\sigma, \tau)$ be one solution of the equations of motion and the boundary conditions in the transverse gauge. To impose Poincaré invariance, we require (Goddard *et al* 1973; Schwarz 1982; Green 1983; Brink 1984) that the new function $y^\mu(\sigma, \tau)$ given by

$$y^\mu(\sigma, \tau) = x^\mu(\tilde{\sigma}, \tilde{\tau}) + a^\mu + \omega^{\mu\nu}x_\nu(\sigma, \tau), \tag{16}$$

is another such solution, provided that a^μ and $\omega^{\mu\nu}(= -\omega^{\nu\mu})$ are infinitesimal translation and Lorentz transformation parameters, and $\tilde{\sigma}-\sigma$, $\tilde{\tau}-\tau$ are infinitesimal reparametrization transformations which ensure that $y(\sigma, \tau)$ is also in the transverse gauge (6), with

$$y^{+}=q^{+}+a^{+}+(p^{+}+\omega_{v}^{+}p^{v})\tau. \tag{17}$$

Here

$$p^{v}\equiv\frac{1}{2\pi}\int_{0}^{2\pi}\mathrm{d}\sigma\,\dot{x}^{v}(\sigma,\tau). \tag{18}$$

Further, we assume that a^{μ} and $\omega^{\mu\nu}$ are zero for $\mu, \nu > d$ because we are interested in Poincaré invariance in d dimensions only. We then have

$$\tilde{\tau}-\tau=-\omega_{v}^{+}(x^{v}(\sigma,\tau)-p^{v}\tau)/p^{+}, \tag{19}$$

$$\tilde{\sigma}-\sigma=-(\omega_{v}^{+}/p^{+})\left[\int_{0}^{\tau}\mathrm{d}\tau' x'^{v}(0,\tau')+\int_{0}^{\sigma}\mathrm{d}\sigma'(\dot{x}^{v}(\sigma',\tau)-p^{v})\right], \tag{20}$$

and

$$y^{\mu}(\sigma,\tau)=x^{\mu}(\sigma,\tau)+a^{\mu}+\omega^{\mu\nu}x_{\nu}(\sigma,\tau)+x'^{\mu}(\sigma,\tau)(\tilde{\sigma}-\sigma)$$
$$+\dot{x}^{\mu}(\sigma,\tau)(\tilde{\tau}-\tau). \tag{21}$$

By the construction of the $\tilde{\sigma}$, $\tilde{\tau}$ it is ensured that

$$(\partial^{2}/\partial\tau^{2}-\partial^{2}/\partial\sigma^{2})x^{\mu}(\tilde{\sigma},\tilde{\tau})=0,$$

and hence $y^{\mu}(\sigma, \tau)$ obey the correct equations of motion. The only non-trivial thing to impose is that y^{μ} obeys the same boundary conditions as x^{μ}. We do this in several steps.

Step 1. Translational invariance in the first $d-2$ transverse dimensions requires that a^{Tr} must obey the boundary conditions (15), i.e.,

$$UL=-L,\; L\equiv(\underbrace{1,\ldots 1}_{d-2},\underbrace{-1,\ldots\;-1}_{d-2},\underbrace{0,\ldots 0}_{2(D-d)}). \tag{22}$$

Step 2. Space rotational symmetry in the $d-2$ transverse dimensions alone requires that the boundary condition matrix U obeys

$$[U,W]=0,\; W\equiv\text{“diagonal”}\;(\omega,\omega,0,0), \tag{23}$$

where ω is an arbitrary $(d-2)$ dimensional rotation matrix, and "diagonal" denotes block-diagonal. It now follows readily that (i) U has zero matrix elements connecting the A and B group of indices, (ii) that in the A sector the boundary conditions do not couple different transverse dimensions and (iii) that the decoupled boundary conditions for the $(d-2)$ transverse dimensions are identical. Finally the decoupled boundary conditions are further restricted by the translational invariance requirement (22). The allowed boundary conditions become for the $(D-d)$ dimensions,

$$\begin{pmatrix}(x'(0,\tau)+x(0,\tau))^{B}\\(x'(2\pi,\tau)-x(2\pi,\tau))^{B}\end{pmatrix}=V\begin{pmatrix}(x'(0,\tau)-x(0,\tau))^{B}\\(x'(2\pi,\tau)+x(2\pi,\tau))^{B}\end{pmatrix},$$
$$V=V^{T},\quad V^{T}V=1, \tag{24}$$

where V is a real $2(D-d)\times 2(D-d)$ dimensional matrix. For the first $(d-2)$ transverse dimensions we obtain, either "closed", i.e.,

$$x^{i}(2\pi,\tau)=x^{i}(0,\tau),\; x'^{i}(2\pi,\tau)=x'^{i}(0,\tau), \tag{25a}$$

or "open", i.e.,

$$x'^i(2\pi, \tau) = x'^i(0, \tau) = 0, \tag{25b}$$

as the only allowed boundary conditions. Here $i = 2, \ldots, d-1$. Translational and rotational invariance in the $(d-2)$ transverse dimensions have restricted the boundary condition in that sector to be the usual ones.

Step 3. Lorentz transformations can now be studied using (21), and the boundary conditions (23)–(25). Assume first the closed string boundary conditions (25a) on the $(d-2)$ transverse dimensions. Since

$$\omega^i_\nu x^\nu = x^+\omega^{i-} + x^-\omega^{i+} + \sum_{j=2}^{d-1} \omega^i_j x^j, \tag{26}$$

$$\omega^+_\nu (x^\nu - p^\nu\tau) = \sum_{j=2}^{d-1} \omega^+_j (x^j - \tau p^j), \tag{27}$$

$$\omega^+_\nu x^\nu = \tau p^+\omega^{+-} + \sum_{j=2}^{d-1} \omega^+_j x^j, \tag{28}$$

and the $x^j (j = 2, \ldots, d-1)$ and their τ-derivatives obey closed string boundary conditions, we see by inspection of equations (19)–(21) that the $y^i(\sigma, \tau)$ for $i \in [2, \ldots, d-1]$ obey closed string boundary conditions provided only that $x^-(\sigma, \tau)$ does so. We also see that for $i > d-1$ the $y^i(\sigma, \tau)$ obey the same boundary conditions as the $x^i(\sigma, \tau)$ provided that the $x'^i(\sigma, \tau)$ obey the same boundary conditions as $x^i(\sigma, \tau)$. First for x^-, using (7) we find that

$$x'^-(\sigma, \tau)\Big|_0^{2\pi} = 0 \tag{20}$$

for arbitrary real U obeying (15), and

$$x^-(\sigma, \tau)\Big|_0^{2\pi} = \frac{1}{p^+}\int_0^{2\pi} d\sigma'\, \dot{x}^{\mathrm{Tr}}(\sigma', \tau)\cdot x'^{\mathrm{Tr}}(\sigma', \tau), \tag{30}$$

$$\frac{d}{d\tau}\left(x^-(\sigma, \tau)\Big|_0^{2\pi}\right) = \frac{1}{2p^+}((x'^{\mathrm{Tr}}(\sigma, \tau))^2 + (\dot{x}^{\mathrm{Tr}}(\sigma, \tau))^2)\Big|_0^{2\pi}. \tag{31}$$

The vanishing of the right side of (30) at $\tau = 0$ is a well-known condition for closed string theory even for $D = d$. Its vanishing at all τ follows provided that the right side of (31) vanishes; that happens if and only if

$$x^B(2\pi, \tau) = R\, x^B(0, \tau),\ x'^B(2\pi, \tau) = R\, x'^B(0, \tau), \tag{32}$$

where R is a $(D-d)\times(D-d)$ dimensional real orthogonal matrix,

$$R^T = R^{-1}. \tag{33}$$

Apart from the discrete ambiguity det $R = \pm 1$, R has $(D-d)(D-d-1)/2$ free parameters if $D > d+1$. Equation (32) means that V must have the special form,

$$V = \begin{pmatrix} 0 & R^T \\ R & 0 \end{pmatrix}, \tag{34}$$

where each entry on the right side is a $(D-d)\times(D-d)$ dimensional matrix. Orthogonality of R guarantees orthogonality of V. The last condition that $x'^B(\sigma,\tau)$ should obey the same boundary condition as $x^B(\sigma,\tau)$ follows automatically from the equation of motion (9) and the boundary conditions (32).

This finishes the consideration of "closed" boundary conditions (25a) on the $d-2$ transverse x^i. The "open" case (25b) may be considered similarly. It leads to the condition $x'^- = 0$ at $\sigma = 0$ and 2π, and hence to a boundary condition (24) with

$$V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}, \quad V_i = V_i^T = V_i^{-1}, \quad \text{for } i = 1, 2, \tag{35}$$

where V_1 and V_2 are $(D-d)\times(D-d)$ dimensional real symmetric orthogonal matrices. Thus, in the "open" case V has N free parameters, where $N = (D-d)^2/2$ if $(D-d)$ is even and $N = [(D-d)^2 - 1]/2$ if $(D-d)$ is odd.

Our final results are summarized by the following two theorems in the "closed" and "open" cases respectively.

Theorem 2. In a D-dimensional string theory with Nambu-Goto action Poincaré invariance in the first d dimensions hold if (i) the first $(d-2)$ transverse co-ordinates have the closed boundary conditions (25a), (ii) the remaining $(D-d)$ transverse co-ordinates obey the quasi-periodic boundary conditions (32) involving the real orthogonal matrix R with $(D-d)(D-d-1)/2$ free parameters, and (iii) the right side of (30) vanishes at $\tau = 0$.

Theorem 3. In a D-dimensional string theory with Nambu-Goto action, Poincaré invariance in the first d-dimensions holds if (i) the first $(d-2)$ transverse co-ordinates obey the Neumann ("open") boundary conditions (25b), and (ii) the remaining $(D-d)$ transverse co-ordinates obey the boundary condition (24) with the matrix V given by (35) involving N free parameters, where $N = (D-d)^2/2$ if $D-d$ is even, and $N = [(D-d)^2 - 1]/2$ if $D-d$ is odd.

Remarks: (i) If the matrix R in the expression (34) has k eigenvalues equal to unity then the theory discussed in theorem 2 is actually Poincaré invariant in $d+k$ dimensions. Similarly for special choices of V_1 and V_2 the theory given by theorem 3, could be Poincaré invariant in a dimension larger than d.

(ii) The theorems 2 and 3, enumerate all possible *linear* boundary conditions on x^{Tr} which permit Poincaré invariant theories.

Quantization of the string theory based on the new family of boundary conditions here obtained has been carried out consistent with Poincaré invariance in d dimensions. The results are presented separately (Roy and Singh 1985). Before writing this work we became aware of a completely different approach to string boundary conditions developed by Vafa and Witten (1985); also see Govindarajan *et al* 1985, based on multiple valued currents on the string world sheet. It is intriguing to compare the boundary conditions we derived (theorem 2) with those postulated by Vafa and Witten.

The ideas presented here were conceived in January 1985 during a visit here of B Sakita. It is a pleasure to thank him and J L Gervais for discussions. One of us (SMR) wishes to thank warmly P Majumdar for his interest in this work and for a collaboration on extending it to the case with fermions.

References

Brink L 1984 CERN-TH-4006/84
Casher A, Englert F, Nicolai H and Taormina A 1985 CERN-TH-4220/85
Freund P G O 1985 *Phys. Lett.* **B151** 387
Goddard P, Goldstone J, Rebbi C and Thorn C B 1973 *Nucl. Phys.* **B56** 109
Goto T 1971 *Progr. Theor. Phys.* **46** 1560
Govindarajan T R, Jayaraman T, Mukherjee A and Wadia S 1985 in preparation
Green M B 1983 *Surv. High Energy Phys.* **3** 127
Green M B and Schwarz J H 1984 *Phys. Lett.* **B149** 117
Green M B and Schwarz J H 1985 *Phys. Lett.* **B151** 21
Gross D J, Harvey J A, Martinec E and Rohm R 1985 *Phys. Rev. Lett.* **54** 502
Hara O 1971 *Progr. Theor. Phys.* **46** 1549
Nambu Y 1970 Lectures at Copenhagen Symposium (unpublished)
Neveu A and Schwarz J H 1971 *Nucl. Phys.* **B31** 86
Neveu A and Schwarz J H 1971b *Phys. Rev.* **D4** 1109
Roy S M and Singh V 1985 TIFR/TH/85-21
Ramond P 1971 *Phys. Rev.* **D3** 2415
Schwarz J M 1982 *Phys. Rep.* **89** 223
Vafa C and Witten E 1985 Princeton preprint
Witten E 1984 *Phys. Lett.* **B149** 35