

Maximally causal quantum mechanics

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Abstract. We present a new causal quantum mechanics in one and two dimensions developed recently at TIFR by this author and V Singh. In this theory both position and momentum for a system point have Hamiltonian evolution in such a way that the ensemble of system points leads to position and momentum probability densities agreeing exactly with ordinary quantum mechanics.

Keywords. Causal quantum mechanics; de Broglie–Bohm quantum mechanics; hidden variable theory; quantum mechanics without observers.

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1. Nonexistence of history in ordinary quantum mechanics

At the 1927 Solvay conference Einstein discussed the example of a particle passing through a narrow hole on to a hemispherical fluorescent screen which records the arrival of the particle. Suppose that a scintillation is seen at a point P at time $t = T$, and suppose that the hole is so narrow that the wave packet corresponding to the particle is uniformly spread all over the screen at t slightly less than T . Was the particle somewhere near P at $t = T - \epsilon$ (ϵ small)? Ordinary quantum mechanics says that the probabilities at $t = T - \epsilon$ for the particle being anywhere on the screen are uniform (and not particularly large in the vicinity of P). Thus the naive history corresponding to the idea of a particle with a trajectory (*any* trajectory) is denied.

There have been recent attempts to define ‘consistent histories’ in quantum mechanics of open systems [1]. Apart from detailed features found unattractive by some [2], there is the basic proposition by the authors themselves that only very special sets of histories are ‘consistent’, and only these can be assigned probabilities. For example, in a double slit interference experiment we cannot assign a probability that the particle reached a region of the screen having earlier passed through slit 1 (except in the case of vanishing interference).

2. Lack of causality in ordinary quantum mechanics

One of the definitions of causality (e.g. that advocated by Jauch) is that “Different results should have different causes”. Consider a quantum superposition $\alpha|+\rangle + \beta|-\rangle$ for a spin-

1/2 particle, where $|+\rangle$ and $|-\rangle$ are eigenstates of σ_z with eigenvalues $+1$ and -1 respectively. When this state is prepared repeatedly and passed through a Stern–Gerlach apparatus to measure σ_z , a fraction $|\alpha|^2$ of the particles ends up at the detector corresponding to $\sigma_z = +1$, and a fraction $|\beta|^2$ goes to the detector corresponding to $\sigma_z = -1$. Thus different results (going to one detector or the other) arise from exactly the same cause (the same initial state). Of course this lack of causality might be restored in a theory in which the wave function is not a complete description of the state of the system.

3. Context dependence of quantum reality

In quantum mechanics,

$$|\psi(\vec{x}, t)|^2 d\vec{x}$$

is the probability of ‘observing’ position to be in $d\vec{x}$ if position were measured. It is not the probability of position ‘being’ in $d\vec{x}$ independent of observation. In fact, the same state vector also yields

$$|\tilde{\psi}(\vec{p}, t)|^2 d\vec{p}$$

which is the probability of observing momentum to be in the interval $d\vec{p}$ if momentum were to be measured. The standard dogma is that a simultaneous measurement of position and momentum is not possible. For, if it were possible, it would collapse the state vector into a simultaneous eigenstate of position and momentum which does not exist. Thus, quantum mechanics does not give probabilities for position and momentum in the same experimental situation or ‘context’. Moreover, quantum mechanics cannot be embedded in a stochastic hidden variable theory in which ‘reality’ is context independent [3]. Consider for example Bell’s theorem [3] in the context of Einstein–Podolsky–Rosen type measurements of $\vec{\sigma}_1 \cdot \vec{a} \vec{\sigma}_2 \cdot \vec{b}$ for a system of two spin 1/2 particles in the singlet state. It shows that even if the measurements of $\vec{\sigma}_1 \cdot \vec{a}$ and $\vec{\sigma}_2 \cdot \vec{b}$ are made at spacelike separation, statistical predictions of quantum mechanics are inconsistent with the assumption that the measured value of $\vec{\sigma}_1 \cdot \vec{a}$ has a reality that is independent of whether $\vec{\sigma}_2 \cdot \vec{b}$ or $\vec{\sigma}_2 \cdot \vec{b}'$ is measured together with it. In this example context dependence takes the form of violation of Einstein locality.

4. Causal quantum mechanics of de Broglie and Bohm

De Broglie and Bohm [4] (dBB) proposed a theory with position as a ‘hidden variable’ so that $\{\vec{x}, |\psi\rangle\}$, i.e., the state vector supplemented by the instantaneous position is the complete description of the state of the system. Here $\vec{x} = (\vec{x}_1, \dots, \vec{x}_N)$ denotes the configuration space co-ordinate which evolves according to

$$\frac{d\vec{x}_i}{dt} = \frac{1}{m_i} \vec{\nabla}_i S(\vec{x}(t), t), \quad (1)$$

where m_i denotes the mass of particle i , and the Schrödinger wave function is given by,

$$\langle \vec{x} | \psi(t) \rangle \equiv R \exp iS, \quad (2)$$

with R and S real functions of (\vec{x}, t) . DBB show that if we start at $t = 0$ with an ensemble of particles whose position density coincides with $|\psi(\vec{x}, 0)|^2$ at $t = 0$, and evolves with

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time according to (1), then the position density coincides with $|\psi(\vec{x}, t)|^2$ at any arbitrary time t . Thus, the phase space density is

$$\rho_{dBB}(\vec{x}, \vec{p}, t) = |\psi(\vec{x}, t)|^2 \delta(\vec{p} - \vec{\nabla}S(\vec{x}, t)) \quad (3)$$

whose marginal at arbitrary time reproduces the position probability density,

$$\int \rho_{dBB}(\vec{x}, \vec{p}, t) d\vec{p} = |\psi(\vec{x}, t)|^2. \quad (4)$$

Further, the time evolution (1) corresponds to evolution according to a c -number causal Hamiltonian $H_c(\vec{x}, \vec{p}, t)$ in which the potential has an added term "the quantum potential" which depends on the wave function.

As far as the position variable is concerned the dBB theory restores history and causality without altering the statistical predictions of quantum mechanics. The lack of Einstein locality is however an essential feature of quantum mechanics. It is in sharp focus in eq. (1): the velocity of the i th particle depends on the instantaneous position of all the particles however far they may be.

The momentum and other variables besides position do not have the same favoured status as position however. As Takabayasi [5] pointed out the dBB phase space density does not yield the correct quantum momentum density, i.e.,

$$\int \rho_{dBB}(\vec{x}, \vec{p}, t) d\vec{x} \neq |\vec{\psi}(\vec{p}, t)|^2.$$

To overcome this problem dBB introduce a measurement interaction whose purpose is to convert the preexisting momentum prior to measurement into one whose distribution agrees with the quantum distribution. In contrast, for position, the value observed is the same as the preexisting value. 'Momentum' therefore has not the same reality as 'position'.

5. Maximally realistic causal theory

We asked the question [6,7], is it possible to remove this asymmetrical treatment of position and momentum and build a new causal quantum mechanics in which momentum and position can have simultaneous reality? In ref. [6,7] we spelt out an affirmative answer in one dimensional configuration space. here we recall the one dimensional construction and also give the two dimensional generalization.

The point of departure is to seek a phase space density of the form

$$\rho(x, p, t) = |\psi(x, t)|^2 \delta(p - \hat{p}(x, t)), \quad (5)$$

where $\hat{p}(x, t)$ is not given by the dBB formula. Rather, $\hat{p}(x, t)$ is to be determined by the requirement

$$\int \rho(x, p, t) dx = |\tilde{\psi}(p, t)|^2. \quad (6)$$

If we assume that $\hat{p}(x, t)$ is a monotonic function of x (non-decreasing or non-increasing),

$$\delta(p - \hat{p}(x, t)) = \frac{\delta(x - \hat{x}(p, t))}{\left| \frac{\partial \hat{p}(x, t)}{\partial x} \right|} \quad (7)$$

and

$$\rho(x, p, t) = \frac{|\psi(x, t)|^2}{\left| \frac{\partial \hat{p}(x, t)}{\partial x} \right|} \delta(x - \hat{x}(p, t)) \quad (8)^*$$

If we determine $\hat{p}(x, t)$ such that

$$|\psi(x, t)|^2 = \left| \frac{\partial \hat{p}(x, t)}{\partial x} \right| |\tilde{\psi}(p, t)|^2, \quad (9)$$

we obtain

$$\rho(x, p, t) = |\tilde{\psi}(p, t)|^2 \delta(p - \hat{p}(x, t)) \quad (10)$$

which obeys the desired eq. (6). Two explicit solutions to eq. (9), corresponding to non-decreasing ($\epsilon = 1$) and non-increasing ($\epsilon = -1$) functions $\hat{p}(x, t)$ are given by,

$$\int_{-\infty}^{\hat{p}(x, t)} dp' |\tilde{\psi}(p', t)|^2 = \int_{-\infty}^{\epsilon x} dx' |\psi(\epsilon x', t)|^2. \quad (11)$$

Instead of eq. (5) or eq. (10), the phase space density may now be written in the symmetric form,

$$\rho(x, p, t) = |\psi(x, t)|^2 |\tilde{\psi}(p, t)|^2 \delta \left(\int_{-\infty}^p dp' |\tilde{\psi}(p', t)|^2 - \int_{-\infty}^{\epsilon x} dx' |\psi(\epsilon x', t)|^2 \right). \quad (12)$$

We have shown [6,7] that this phase space density corresponds to evolution of position and momentum according to a *c*-number causal Hamiltonian of the form

$$H_c(x, p, t) = \frac{1}{2m} (p - A(x, t))^2 + V(x, t), \quad (13)$$

with both $A(x, t)$ and $V(x, t)$ having parts which depend on the wave function. Thus we have two quantum potentials instead of just one in the *dBB* theory. The existence of H_c ensures that the phase space density obeys the Liouville condition. In one dimension it turns out that

$$\frac{dx}{dt} = \left(\frac{dx}{dt} \right)_{dBB},$$

but

$$\hat{p}(x, t) = m \frac{dx}{dt} - A(x, t) \neq \left(m \frac{dx}{dt} \right)_{dBB}$$

It may be recalled that like the phase space distribution (12), the Wigner distribution [8] also reproduces $|\psi(x, t)|^2$ and $|\tilde{\psi}(p, t)|^2$ as marginals. However the Wigner distribution is not positive definite and cannot therefore have a probability interpretation; further the condition of Hamiltonian evolution of x, p and the consequent Liouville property are valid for the phase space density (12) but not for the Wigner distribution.

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In higher dimensions there is a surprise. Example for $n = 2$, we can construct a causal quantum mechanics in which the quantum probability densities corresponding to three different complete commuting sets (CCS) of observables, e.g. $(X_1, X_2), (P_1, X_2), (P_1, P_2)$ is simultaneously realized. Explicitly, the positive definite phase space density

$$\rho(\vec{x}, \vec{p}, t) = |\psi(x_1, x_2, t)|^2 |\psi(p_1, x_2, t)|^2 |\psi(p_1, p_2, t)|^2 \delta(A_1) \delta(A_2) \quad (14)$$

where

$$A_1 \equiv \int_{-\infty}^{p_1} |\psi(p'_1, x_2, t)|^2 dp'_1 - \int_{-\infty}^{x_1} |\psi(x'_1, x_2, t)|^2 dx'_1, \quad (15)$$

$$A_2 \equiv \int_{-\infty}^{p_2} |\psi(p_1, p'_2, t)|^2 dp'_2 - \int_{-\infty}^{x_2} |\psi(p_1, x'_2, t)|^2 dx'_2, \quad (16)$$

reproduces as marginals the correct quantum probability densities $|\psi(x_1, x_2, t)|^2$, $|\psi(p_1, x_2, t)|^2$ and $|\psi(p_1, p_2, t)|^2$. The corresponding velocities are however in general different from the *dB* velocities. The calculation of the velocities, and the explicit causal Hamiltonian for $n \geq 2$ will be given in detail in a later publication [9] where we show that the quantum probability densities corresponding to $n + 1$ CCS can be simultaneously realized (i.e., with one phase space density). A recent quantum phase space contextuality theorem proves that it is impossible to simultaneously realize quantum probability densities corresponding to all possible CCS [10]. We conjecture that the causal quantum mechanics we have constructed is maximally realistic.

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