Fractionally charged non-leaking dyons and fermions in a bag

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Abstract. We consider a fermion of charge $e$ confined to a spherical bag with a Dirac monopole of strength $g$ at its centre. We find that the boundary conditions making the lowest angular momentum Hamiltonian self-adjoint are characterized by a unitary matrix $U$, and the corresponding vacuum charge has a fractional part $2|\text{det} U|/(2\pi)$ where $\text{det} U = -\exp(2\pi a)$. Boundary conditions for conservation of helicity, $CP$, $CT$ and $PT$ are displayed. We demonstrate the possibility of a fractionally charged dyon whose interaction with a fermion conserves helicity. We also show that the simultaneous validity of helicity, $CP$, $CT$ and $PT$ requires integer vacuum charge.

Keywords. Non-leaking dyons; fermions; spherical bag; unitary matrix.

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The Jackiw-Rebbi (1976) discovery of half-integral fermion number of fermion-monopole systems is further dramatised by Witten's (1979) result that in the presence of a $CP$-violating angle $\theta_0$, the monopole acquires a charge $-e\theta_0/(2\pi)$ where $e$ is the fermion charge. In an apparently different line of research Kazama et al (1977), Callias (1977) and Goldhaber (1977) discovered that the fermion-Dirac monopole Hamiltonian in the lowest angular momentum state is self-adjoint only when a boundary condition at the origin (monopole position) involving the $CP$-violating parameter $\theta_0$ is imposed. Two major consequences are the "inevitable failure of helicity conservation" (Goldhaber 1977) (intimately related to the Rubakov-Callan effect in the non-abelian case (Rubakov 1981, 1982; Callan 1982a, b, 1983)), and the confirmation by Grossman and Yamagishi (1983) of the Witten effect with a precise connection to the $r = 0$ boundary condition. The monopole becomes a helicity leaking dyon of fractional charge, the fraction being irrational in general but a half-integer or an integer when $CP$ is conserved.

The present work demonstrates that these conclusions get radically altered when the fermion monopole system is enclosed in a spherical bag of finite radius $R$. In particular it is possible to have a helicity conserving dyon of fractional charge. $CP$ violation forces fractional charge but does not force helicity violation. Further, the simultaneous conservation of helicity, $CP$, $CT$ and $PT$ forces the monopole charge to be integral. For the lowest angular momentum Hamiltonian we find a simple formula relating the vacuum charge to the boundary conditions. The charge eigenvalues are independent of $R$ but depend non-trivially on boundary conditions at $r = R$ (as well as $r = 0$) and hence have non-unique $R \to \infty$ limit. A similar boundary condition dependence of the vacuum charge was obtained recently (Roy and Singh 1984a, b) for the $1+1$ dimensional Jackiw-Rebbi and Goldstone-Wilczek (1981) Hamiltonians. We generalize here the
Witten-Grossman-Yamagishi results connecting fractional charge to violation of discrete symmetries.

We use the Wu-Yang (1975) vector potentials \( A = A_a(r) \) for \( r \in R_a (\theta \neq \pi) \), and \( A_b(r) \) for \( r \in R_b (\theta \neq 0) \) where

\[
A_a(r) = A_b(-r) = g \frac{\hat{r} \times \hat{p}}{r(1 + \hat{r} \cdot \hat{p})},
\]

with \( g \) = monopole strength. The Dirac wave section \( \psi (= \psi_a \text{ in } R_a, \psi_b \text{ in } R_b) \) obeys,

\[
H \psi(r, t) = i \frac{\partial \psi(r, t)}{\partial t}, \quad H = \alpha \cdot \pi + \beta m,
\]

where

\[
\pi = -i \nabla - eA, \quad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

\( \sigma \) are Pauli matrices and \( e \) and \( m \) denote charge and mass of the fermion, confined to the bag \(|r| \leq R \). Before defining the boundary conditions which make \( H \) self-adjoint, we note, following Goldhaber (1977) the following "formal" symmetry properties

\[
\Sigma \cdot \pi H \Sigma \cdot \pi = H, \quad (C P) H (C P)^{-1} = -H,
\]

\[
(C T) H (C T)^{-1} = -H, \quad (P T) H (P T)^{-1} = H.
\]

Here \( \Sigma \cdot \pi \) is related to the helicity, with \( \Sigma = \text{diag}(\sigma, \sigma) \). The discrete transformations \( C, P, T \) are defined by

\[
\psi_{a, b}(x, t) \xrightarrow{P} \psi_{a, b}(x, t) = \eta_p \beta \psi_{b, a}(-x, t),
\]

\[
\psi(x, t) \xrightarrow{C} \psi^c(x, t) = \eta_c \beta x_2 \psi^* (x, t),
\]

\[
\psi(x, t) \xrightarrow{T} \psi^T(x, t) = \eta_T \Sigma_2 \psi^* (x, t),
\]

\[
PA(x)P^{-1} = CA(x)C^{-1} = TA(x)T^{-1} = -A(x),
\]

where \(|\eta_p| = |\eta_c| = |\eta_T| = 1\). The subscripts \( a, b \) in (5) refer to regions \( R_a, R_b \). The corresponding subscripts in (6)-(8) are omitted, since the same subscript occurs throughout each equation. It follows that the wave sections,

\[
\psi^p_{a, b}(x, t) = \eta_p \eta_T \beta \Sigma_2 \psi^p_{b, a}(-x, -t),
\]

\[
\psi^c T(x, t) = -\eta_c \eta_T \beta \Sigma_2 \psi^* (x, -t),
\]

and

\[
\psi^p T(x, t) = -\eta_c \eta_T \beta \Sigma_2 \psi^*_{b, a}(-x, t)
\]

obey the same Dirac equation as \( \psi(x, t) \). In the lowest angular momentum state \( j = |q| - \frac{1}{2}, \ q \equiv e\gamma \),

\[
\psi(x, t) = \frac{1}{r} \begin{pmatrix} F(r) \\ G(r) \end{pmatrix} \eta_{j_m}(\theta, \phi) \exp(-iEt) \equiv \frac{1}{r} \chi(r) \otimes \eta_{j_m}(\theta, \phi) \exp(-iEt),
\]

\[\frac{1}{r} \]
(24) must imply that \( \chi_2 \in D(H_0) \). We thus find that \( D(H_0) \) consists of \( \chi \) which apart from obeying,
\[
\int_0^R dr \chi^+(r) \chi(r) < \infty, \quad \int_0^R dr (H_0 \chi)^+(H_0 \chi) < \infty, \tag{25}
\]
also obey the boundary conditions
\[
\begin{pmatrix}
F(R) + G(R) \text{sgn}q \\
F(0) - G(0) \text{sgn}q
\end{pmatrix} = U \begin{pmatrix}
F(R) - G(R) \text{sgn}q \\
F(0) + G(0) \text{sgn}q
\end{pmatrix}, \tag{26}
\]
where \( U \) is the \( 2 \times 2 \) unitary matrix
\[
U \equiv \exp \left( i \alpha \begin{pmatrix}
\cos \lambda \exp \left[ i(\beta + \pi/2) \right] & \sin \lambda \exp (i\gamma) \\
\sin \lambda \exp (-i\gamma) & -\cos \lambda \exp \left[ -i(\beta + \pi/2) \right]
\end{pmatrix} \right), \tag{27}
\]
with \( \alpha, \beta, \gamma, \lambda \) being arbitrary real parameters. These boundary conditions define a four parameter family of self-adjoint Hamiltonians similar to the Jackiw-Rebbi and Goldstone-Wilczek one space dimension cases considered previously (Roy and Singh 1984a, b).

**Self-adjointness of helicity**

Similarly, the boundary conditions which make \( h_0 \) self-adjoint are
\[
\begin{pmatrix}
F(R) + G(R) \text{sgn}q \\
F(R) - G(R) \text{sgn}q
\end{pmatrix} = V \begin{pmatrix}
F(0) + G(0) \text{sgn}q \\
F(0) - G(0) \text{sgn}q
\end{pmatrix}, \tag{28}
\]
where \( V \) is an arbitrary \( 2 \times 2 \) unitary matrix.

**Simultaneous self-adjointness of Hamiltonian and helicity**

For this purpose the parameters \( \alpha, \beta, \gamma, \lambda \) must be such that (26) and (28) are equivalent (i.e., they imply each other). This happens if and only if \( \sin \lambda = \pm 1 \). Hence the common domain of self-adjointness of the Hamiltonian and helicity is specified by the simple boundary condition
\[
\begin{pmatrix}
F(R) \\
G(R) \text{sgn}q
\end{pmatrix} = \exp (i\gamma_1 + i\alpha \sigma_1) \begin{pmatrix}
F(0) \\
G(0) \text{sgn}q
\end{pmatrix}, \tag{29}
\]
where
\[
\exp (i\gamma_1) \equiv \exp (i\gamma) \sin \lambda, \quad \sin \lambda = \pm 1. \tag{30}
\]

Both (26) and (28) are equivalent to (29) with,
\[
U = \exp \left( i \alpha \begin{pmatrix}
0 & \exp (i\gamma_1) \\
\exp (-i\gamma_1) & 0
\end{pmatrix} \right), \quad V = \exp (i\gamma_1) \begin{pmatrix}
\exp (i\alpha) & 0 \\
0 & \exp (-i\alpha)
\end{pmatrix}. \tag{31}
\]

The departure of the boundary condition (29) from the quasi-periodic ones mentioned by Grossman (1983) corresponding to \( \alpha = n\pi \) \( (n \text{ integer}) \) will be crucial for the existence of fractional Witten charge.

**Discrete symmetries**

For \( CP \) invariance (4) to hold if \( \chi \in D(H_0) \), then \( \chi^{CP} \in D(H_0) \), and vice versa. Similar conditions hold for \( CT \) and \( PT \) invariance. Thus conditions for the discrete symmetries
to hold are given by

\[ PT: U = U^\dagger; \quad CT: U = -\sigma_3 U^\dagger \sigma_3; \quad CP: U = -\sigma_3 U^* \sigma_3. \] (32)

The infinite space results follow from (32) by setting \(|U_{32}| = 1\), and other \(U_{ij} = 0\), showing that PT invariance allows arbitrary \(\theta_0\) in (23) whereas CT and CP invariance give identical conditions \(\theta_0 = n\pi\). For finite \(R\), (32) yields

\[
PT: \sin \lambda \sin \gamma = 0 \\
CT: \text{either } \sin \alpha = \cos \lambda \sin \beta = 0 \text{ or } \cos \alpha = \cos \beta = \sin \lambda = 0 \\
CP: \text{either } \sin \alpha = \cos \lambda \sin \beta = \sin \lambda \sin \gamma = 0, \text{ or } \cos \alpha = \cos \lambda \cos \beta = \sin \lambda \cos \gamma = 0.
\] (33)

For CT or CP invariance we need \(\sin \alpha \cos \alpha = 0\). If \(\sin \alpha = 0\), CP invariance implies CT invariance. If \(\cos \alpha = 0\), CT invariance implies CP invariance. The most symmetric case, with helicity self-adjoint and conserved, and all the discrete symmetries PT, CT, CP valid corresponds to:

\[ h_0, PT, CT, CP: \sin \alpha = \cos \lambda = \sin \gamma = 0, \text{ i.e. } U = \pm \sigma_1, \] (34)

i.e., periodic or antiperiodic boundary conditions.

**Vacuum charge eigenvalues**

The Dirac field operator \(\Psi(x, t)\) in the Heisenberg representation obeys the same differential equation as the c-no. wave function \(\Psi(x, t)\) and has the charge conjugation property

\[ \Psi^c(x, t) = \eta_c(\beta \alpha)_{ab} \Psi^a(x, t). \] (34)

The total charge operator \(N\) (odd under \(C\)) and its eigenvalue \(N_0\) in a vacuum state (with a convenient regularization (Goldstone and Jaffe 1983; Paranjape and Semenoff 1983; Niemi and Semenoff 1983)) are given by

\[ N = \frac{1}{2} \int d^3x \left[ \Psi^a(x, t) \Psi^a(x, 0) \right], \] (35)

\[ N_0 = \lim_{\varepsilon \to 0^+} \left[ -\frac{1}{2} \sum_{E_n \neq 0} (\text{sgn}E_n) \exp(-\varepsilon |E_n|) \right] + N_{0,0}, \] (36)

where \(N_{0,0}\) is the contribution of zero energy levels to vacuum charge. When there are \(v\) zero energy c-no. wave functions the vacuum state is \(2^v\) fold degenerate and the corresponding \(N_{0,0}\) vary from \(-v/2\) to \(v/2\) in steps of unity. In such a case we shall calculate the \(N_0\) corresponding to the choice of the vacuum with \(N_{0,0} = -v/2\); the other vacua will have charges differing from this by integers.

We shall only calculate the contribution to vacuum charge \(N_0\) of the lowest angular momentum levels. The results will however give the full vacuum charge whenever the boundary conditions for the higher \(j\) levels respect CP or CT leading to \(E \to -E\) symmetry of these levels and hence their zero contribution to \(N_0\).

For the most general boundary condition given by (26) and (27), we find that the energy levels are given by

\[ \cos(kR)\cos \alpha + \frac{\sin(kR)}{k} \left[ E \sin \alpha + M \cos \lambda \cos \beta \right] = \cos \zeta, \] (37)
where \[ k^2 = E^2 - M^2, \cos \zeta \equiv \sin \lambda \cos \gamma. \] (38)

\textit{Vacuum charge for } M = 0

In this case,

\[ E(M = 0) = \frac{\alpha + \zeta + 2n\pi}{L} \text{ and } \frac{\alpha - \zeta + 2n\pi}{L}, \quad n = 0, \pm 1, \pm 2, \ldots \quad (39) \]

Each of these levels is \( 2|q| = 2|q| \) fold degenerate. A simple calculation yields

\[ N_0(M = 0) = 2|q| \left( \frac{\alpha}{\pi} - 1 - \text{Int} \left( \frac{\alpha + \zeta}{2\pi} \right) - \text{Int} \left( \frac{\alpha - \zeta}{2\pi} \right) \right), \quad (40) \]

with \( \text{Int } x \equiv \text{largest integer } \leq x. \)

\textit{Vacuum charge for } M > 0

In this case, for a given \( U, E = E(M = 0) + O(1/n), \) for \( n \to \infty. \) We then show that \( dN_0(M)/dM = 0 \) except at those values \( M = M_i \) where one or more eigenvalues \( E \) pass through zero leading to an integer jump in \( N_0(M) \equiv N_0 \) (Roy and Singh 1984a, b). We find,

\[ N_0(M) = N_0(M = 0) + I, \quad (41) \]

where \( I \) is an integer given by

\[ I = -2|q| \sum_{i=1}^{2} \text{sgn} \left( \frac{dE}{dM} \right)_{M = M_i} \theta(M - M_i)\theta(x_i)(-y_i), \quad (42) \]

with \( \theta(x) \) being the step function, and

\[ x_i \equiv \frac{1}{D} (\cos \lambda \cos \beta \cos \zeta + (-1)^i S \cos \alpha), \quad (43) \]

\[ y_i \equiv \frac{1}{D} (\cos \alpha \cos \zeta + (-1)^i S \cos \lambda \cos \beta), \quad (44) \]

\[ S \equiv + (\cos^2 \zeta + \cos^2 \lambda \cos^2 \beta - \cos^2 \alpha)^{1/2}, \quad (45) \]

\[ D \equiv \cos^2 \lambda \cos^2 \beta - \cos^2 \alpha, \quad (46) \]

\[ M_i \equiv \frac{1}{R} \sin h^{-1} x_i, \quad (47) \]

and

\[ \left( \frac{dE}{dM} \right)_{M = M_i} \equiv -\frac{M_i R}{\sin \alpha} (\cos \alpha + \cos \lambda \cos \beta \coth (M_i R)). \quad (48) \]

Equation (41) gives the exact formula for the vacuum charge \( N_0(M). \) Its fractional part is given by the simple formula

\[ \text{Fractional part of } (N_0(M) - 2|q|\alpha/\pi) = 0. \quad (49) \]

Helicity conservation allows arbitrary \( \alpha \) and hence arbitrary fractional value for vacuum (monopole) charge. So does \( PT \) conservation. For either \( CP \) or \( CT \) to be conserved \( \sin \alpha \cos \alpha = 0, \) and hence the vacuum charge is \( \frac{1}{2} \times \text{integer}, \) where the integer
Table 1. Dependence of fractional part of vacuum charge (equation (49)) on symmetries of the lowest angular momentum fermion-monopole bag Hamiltonian assuming $CP$ (or $CT$) invariance of the higher angular momentum Hamiltonians.

<table>
<thead>
<tr>
<th>Conserved quantities</th>
<th>Helicity</th>
<th>$CP$</th>
<th>$CT$</th>
<th>$PT$</th>
<th>Helicity, $CP$, $CT$, $PT$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fractional part of vacuum charge</td>
<td>arbitrary</td>
<td>0, $\frac{1}{2}$</td>
<td>0, $\frac{1}{2}$</td>
<td>arbitrary</td>
<td>0</td>
</tr>
</tbody>
</table>

can be even or odd. This is due to the fact that the non-zero energy levels then occur in pairs due to $E \rightarrow -E$ symmetry and do not contribute to the vacuum charge. If helicity, $CP$, $CT$ and $PT$ are conserved then $\sin \alpha = 0$, and the vacuum charge must be integral (table 1).

Our extension of the Callias-Goldhaber and Witten-Grossman-Yamagishi results demonstrate that the change from infinite to finite space volume has interesting consequences for the vacuum charge eigenvalues and symmetry properties e.g., if we consider $U = \text{diag} \exp \left[ i\eta (\theta_0 - \pi/2) \right], \exp \left[ -i\eta (\theta_0 - \pi/2) \right]$ which Yamagishi used in discussing the $R \rightarrow \infty$ limit, we find that $N_0$ has a fractional part $2q(\theta_0-\theta_0 + \pi)/(2\pi)$, with a non-trivial dependence on the boundary condition parameter $\theta_0$ at $r = R$, no matter how large $R$ may be. This fact escaped notice previously because $N_0$ was not calculated for finite $R$. Further, more symmetry properties are possible to satisfy for finite $R$ than for $R = \infty$, each symmetry restricting boundary conditions and vacuum charge eigenvalues.

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References

Callan C G Jr 1982b Phys. Rev. D26 2058
Callias C J 1977 Phys. Rev. D16 3068
Goldhaber A S 1977 Phys. Rev. D16 1815
Roy S M and Singh V 1984a Pramana 23 333
Yamagishi H 1983a Phys. Rev. D27 2383
Yamagishi H 1983b Phys. Rev. Lett. 50 458