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Soliton and boundary condition induced fractional fermion number

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Abstract. We show that for a fermion in a bounded background potential in a finite box, eigenvalues of the total charge are independent of whether the potential is solitonic and depend only on the boundary condition: half-odd integral or integral for charge conjugation (C) invariant boundary conditions and an arbitrary fraction for C non-invariant boundary conditions. Fractional fermion numbers for infinite space Jackiw-Rebbi and Goldstone-Wilczek Hamiltonians are reproduced in finite space by using boundary conditions different from the periodic ones of Rajaraman and Bell.

Keywords. Soliton; boundary condition; fermion; charge conjugation.

1. Introduction

Jackiw and Rebbi (1976) discovered that when a Dirac fermion is coupled to a soliton, the soliton has half-integral fermion number. Further, Witten (1979), Grossman (1983) and Yamagishi (1983a, b) have established that for fermion monopole interactions in the presence of CP-violation the monopole becomes a dyon of fractional charge. The occurrence of fractional vacuum charge is now confirmed in several field theoretical models, and condensed matter systems. (Goldstone and Wilczek 1981; Jackiw and Schrieffer 1981; Su *et al* 1979, 1980; Rice 1979; Takayama *et al* 1980; Heeger 1981; Su and Schrieffer 1981; Bardeen *et al* 1983; Shastry 1983; Ikehata *et al* 1980; Goldberg *et al* 1979).

The fractional charge phenomenon nevertheless needs further elucidation. In contrast with the above models in infinite space, Kivelson and Schrieffer (1982), and Rajaraman and Bell (1982) have found that for a Dirac fermion in a box of length L interacting with a soliton, and obeying periodic boundary conditions, the total vacuum charge is *integral*. Only a partial charge operator in a sub-region of length L_1 with a diffuse boundary of width d can have fractional eigenvalues in the carefully defined limit $L \gg L_1 \gg d \to \infty$. This makes the fractional charge phenomenon appear less dramatic and raises the question: is fractional *total charge* a peculiarity of the infinite length case?

Yamagishi's detailed investigations (1983a, b) did not answer the above question. However, in a somewhat more complicated example of three space-dimensional chiral bag models, fractional (but conflicting values) of baryon number within the finite bag have been obtained recently by Rho *et al* (1983), and by Goldstone and Jaffe (1983). These results suggest that fractional charge is not necessarily related to the $L = \infty$ limit.

In this paper therefore we reconsider a fermion in a bounded background potential confined to a one-dimensional box $x \in [L_1, L_2]$. As a first step we do for finite L_1, L_2 what Callias (1977) and Goldhaber (1977) did for $L_1 = 0$, $L_2 = \infty$ (*i.e.* semi-infinite): find the allowed boundary conditions for the Hamiltonian to be self-adjoint. The resulting boundary conditions are characterized by four real parameters of a 2×2 unitary matrix U in contrast with only one parameter for $L_1=0$, $L_2=\infty$ (*i.e.* semi-infinite box case) and contain the previously considered periodic boundary conditions as a special case. A new question immediately arises: If the charge is fractional, on which of the four parameters does it depend?

A regularized charge operator has to be defined. An exponential energy cut-off used by Goldstone and Jaffe (1983), and a regularization identifying the charge operator with minus half the η-invariant of Atiyah *et al* (1973) yield identical results. Our somewhat unexpected conclusions for the eigenvalues of the total charge for finite L are: (i) the total charge eigenvalues are independent of the background potential, solitonic or otherwise, and independent of L (ii) they depend only on the unitary matrix U specifying the boundary condition (iii) the eigenvalues are in general fractional and the fractional part depends only on det U (iv) for C non-invariant boundary conditions charge eigenvalues can be arbitrary fractions but for C-invariant boundary conditions they are half-odd integral or integral.

It appears that the role of background fields of non-trivial asymptotic behaviour in infinite or semi-infinite space, is played for finite L by appropriate boundary conditions. We agree with Rajaraman and Bell (1983) that the total charge is integral for periodic boundary condition. We show however that the physics of the $L_1=-\infty$, $L_2=+\infty$ solitonic Jackiw-Rebbi case, including half-odd integral total charge eigenvalue, is simply modelled for finite L by a different boundary condition, *viz* vanishing of the lower component of the Dirac wavefunction at the end points. Similarly the Goldstone-Wilczek fractional charge is obtained for finite L *via* appropriate (non-periodic) boundary conditions.

2. Self-adjoint Goldstone-Wilczek hamiltonian in a box

The Jackiw-Rebbi equation for a 2-component fermion field in one-space dimension in a background field $g\phi_1(x)$ is,

$$i\frac{\partial\psi}{\partial t}=H\psi,\ H=-i\sigma_2\frac{\mathrm{d}}{\mathrm{d}x}+g\sigma_1\phi_1(x), \tag{1}$$

where g is a non-negative coupling constant. In infinite space, $\psi^c_\alpha=(\sigma_3)_{\alpha\beta}\psi^+_\beta$ also solves (1) demonstrating charge conjugation (C) invariance: (The σ_i are Pauli matrices). The fermion number operator N odd under C is

$$N=\tfrac{1}{2}\int\mathrm{d}x[\psi^+_\alpha(x),\psi_\alpha(x)]. \tag{2}$$

The vacuum sector $\phi_1(\infty)=\phi_1(-\infty)$, differs from the soliton sector $\phi_1(\infty)=-\phi_1(-\infty)$ in that the latter contains a zero energy c-number solution of (1) associated with a destruction operator a. In the soliton sector

$$N=a^+a+\int\mathrm{d}k\,(b^+_k b_k-d^+_k d_k)-\tfrac{1}{2}, \tag{3}$$

where b_k, d_k are fermion and antifermion destruction operators. This leads to eigenvalues $\pm\frac{1}{2}$ for the degenerate soliton states $|s\rangle$, $a^+|s\rangle$ annihilated by a and a^+ respectively. The fractional eigenvalue is also an immediate consequence of the generalized index theorem of Callias (1978). A generalization of (1) violating C-

invariance is the Goldstone-Wilczek (1981) Hamiltonian

$$H = -i\sigma_2 \frac{\mathrm{d}}{\mathrm{d}x} + g[\sigma_1\phi_1(x) + \sigma_3\phi_3(x)]. \tag{4}$$

The ground state fermion number for $g \neq 0$ is then $[\theta(\infty) - \theta(-\infty)]/(2\pi)$, where

$$\phi(x) \equiv \phi_1(x) + i\phi_3(x) \equiv \rho(x) \exp[i\theta(x)], \tag{5}$$

with ρ, θ real. A generalization to $3+1$ dimensions calculating ground state fermion number of magnetic monopole fermion systems in terms of an asymptotic surface integral of the magnetic flux has been given by Paranjape and Semenoff (1983).

We here consider the Goldstone-Wilczek Hamiltonian (4) with $x \in [L_1, L_2]$, assuming $\phi_1(x)$, $\phi_3(x)$ to be bounded. Denoting a c-number spinor as

$$\psi(x) \equiv \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \equiv \frac{u(x)}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{v(x)}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{6}$$

where

$$u(x) = \frac{f(x) - ig(x)}{\sqrt{2}}, \quad v(x) = \frac{f(x) + ig(x)}{\sqrt{2}}, \tag{7}$$

we deduce,

$$(\psi_2, H\psi_1) - (H\psi_2, \psi_1) = -i(A_2^+ A_1 - B_2^+ B_1), \tag{8}$$

where, for any ψ,

$$A \equiv \begin{pmatrix} u(L_2) \\ v(L_1) \end{pmatrix}, \quad B \equiv \begin{pmatrix} v(L_2) \\ u(L_1) \end{pmatrix}, \tag{9}$$

and A_i, B_i denote values of A, B for $\psi = \psi_i$. It follows that the Goldstone-Wilczek Hamiltonian for $x \in [L_1, L_2]$ is self-adjoint provided its domain consists of normalizable ψ obeying the boundary condition,

$$A = UB, \tag{10}$$

where U is any 2×2 unitary matrix. Since U may be parametrized as

$$U = \exp(i\alpha) \begin{pmatrix} \cos\theta \exp(i\beta) & \sin\theta \exp(i\gamma) \\ \sin\theta \exp(-i\gamma) & -\cos\theta \exp(-i\beta) \end{pmatrix}, \tag{11}$$

we obtain a four-parameter family of self-adjoint Hamiltonians. The choice $U = \sigma_1$ corresponds to periodic boundary conditions. For C-invariance we need not only $\phi_3 = 0$, but also that ψ^c obeys the same boundary condition, *i.e.*, $U = U^*$. For future applications with internal symmetry we derive in an appendix self-adjoint extensions of an $N \times N$ Dirac Hamiltonian.

3. Fermion number eigenvalues

For a general U the fermion number operator (2) is ill-defined. Convenient regularized expressions for the eigen-value of N in the ground state (annihilated by fermion and anti-fermion destruction operators) are (Goldstone and Jaffe 1983; Paranjape and

Semenoff 1983; Niemi and Semenoff 1983)

$$N_0 = \lim_{\varepsilon \to 0+} \left(-\tfrac{1}{2} \sum_i \operatorname{sgn} E_i \exp(-\varepsilon |E_i|) \right), \tag{12}$$

and

$$N_0 = -\tfrac{1}{2}\eta(0), \; \eta(s) \equiv \sum_i \operatorname{sgn} E_i |E_i|^{-s}, \tag{13}$$

where the sums go over all eigenvalues of H. Here $\eta(0)$ is the η-invariant of Atiyah *et al* (1973) obtained by continuation from large positive $\operatorname{Re} s$. The two definitions above lead to identical results.

3.1 *Zero mass* ($g = 0$) *case*

We shall see that even in this case we may have fractional N_0 for finite L, and suitable boundary conditions. For any U, the wavefunctions are given by

$$u(x) = c_1 \exp(iEx), \quad v(x) = c_2 \exp(-iEx), \tag{14}$$

with c_1, c_2 determined from the boundary condition,

$$\begin{pmatrix} \exp(iEL) - \exp[i(\alpha+\gamma)] \sin\theta, & -\cos\theta \exp[i(\alpha+\beta-EL)] \\ \exp[i(\alpha-\beta)] \cos\theta & 1 - \sin\theta \exp[i(\alpha-\gamma-EL)] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0. \tag{15}$$

The condition for existence of non-trivial solution for c_1, c_2 is,

$$\cos(EL - \alpha) = \sin\theta \cos\gamma. \tag{16}$$

Hence, the energy eigenvalues are

$$E = \frac{\alpha + \zeta + 2n\pi}{L} \text{ and } \frac{\alpha - \zeta + 2n\pi}{L}, \tag{17}$$

where

$$n = 0, \pm 1, \pm 2, \ldots \tag{18}$$

and ζ is any solution of

$$\cos\zeta = \sin\theta \cos\gamma. \tag{19}$$

The eigenvalues are independent of the parameter β in U; this could have been guessed *a priori*. For zero mass, a rotation $\exp(i\beta_0\sigma_2)$ leaves H invariant except that the rotated ψ obeys a boundary condition with $\beta \to \beta + 2\beta_0$. Hence physics must be independent of β.

The above eigenvalues yield

$$N_0 = \frac{\alpha}{\pi} - 1 - \operatorname{Int}\left(\frac{\alpha+\zeta}{2\pi}\right) - \operatorname{Int}\left(\frac{\alpha-\zeta}{2\pi}\right), \tag{20}$$

where $\operatorname{Int}(x)$ denotes the largest integer $\leqslant x$. The ground-state fermion number is fractional for all boundary conditions with α/π fractional. Note that $\det U = -\exp(2i\alpha)$, and hence the fractional part of N_0 depends only on $\det U$. For periodic boundary conditions α/π is integral and so is the fermion number.

3.2 *Jackiw-Rebbi and Goldstone-Wilczek cases* ($g \neq 0$, $L_2 > 0 > L_1$)

For simplicity of explicit calculations we shall use

$$\phi(x) = \phi_+ \operatorname{step}(x) + \phi_- \operatorname{step}(-x), \tag{21}$$

where step (x) is the step function, and $\phi_\pm$ are complex constants. This is similar to the Rajaraman-Bell (1982) simplification of the Jackiw-Rebbi soliton function. The Dirac equation $H\psi = E\psi$ then requires that $u(x)$, $v(x)$ be continuous at $x = 0$ and obey

$$\begin{aligned} &\left(-i\frac{\mathrm{d}}{\mathrm{d}x} - E\right)u(x) = ig\, v(x)\,\phi(x), \\ &\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k_+^2 \text{ step}(x) + k_-^2 \text{ step}(-x)\right)u(x) = 0, \end{aligned} \tag{22}$$

where $k_\pm$ are real or pure imaginary solutions of,

$$k_\pm^2 \equiv E^2 - g^2|\phi_\pm|^2. \tag{23}$$

The solution is

$$u(x) = \begin{cases} c_2 \cos k_+x + c_2 \sin k_+x, \text{ for } x > 0 \\ c_1 \cos k_-x + \dfrac{1}{k_-\phi_+}[k_+\phi_-c_2 + iE(\phi_+ - \phi_-)c_1] \sin k_-x, \\ \qquad\qquad\qquad\qquad \text{for } x < 0, \ldots \end{cases} \tag{24}$$

and

$$v(x) = \begin{cases} \dfrac{1}{ig\phi_+}(-(ik_+c_2 + Ec_1)\cos k_+x + (ik_+c_1 - Ec_2)\sin k_+x), \text{ for } x > 0, \\ \dfrac{1}{\mathrm{i}g\phi_+}\left(-(ik_+c_2 + Ec_1)\cos k_-x + \dfrac{1}{k_-}\{ic_1(E^2 - g^2\phi_+\phi_-^*) \right. \\ \qquad\qquad \left. - Ek_+c_2\} \sin k_-x\right), \text{ for } x < 0. \end{cases} \tag{25}$$

The constants c_1, c_2 are found, apart from a normalization, from the boundary condition:

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0, \tag{26}$$

where,

$$V_{11} = \cos(k_+L_2) + U_{11}\alpha_1 + U_{12}\alpha_2, \tag{27}$$

$$V_{12} = \sin(k_+L_2) + U_{11}\alpha_3 + U_{12}\alpha_4, \tag{28}$$

$$\begin{aligned} V_{21} = &\left(iE\cos(k_-L_1) + (E^2 - g^2\phi_+\phi_-^*)\sin(k_-L_1)k_-\right)/(g\phi_+) \\ &+ U_{21}\alpha_1 + U_{22}\alpha_2, \end{aligned} \tag{29}$$

$$\begin{aligned} V_{22} = &\, k_+[-\cos(k_-L_1) + iE\sin(k_-L_1)/k_-]/(g\phi_+) \\ &+ U_{21}\alpha_3 + U_{22}\alpha_4, \end{aligned} \tag{30}$$

with U_{ij} given in (11) and the constants α_i defined by,

$$\alpha_1 = i[-E\cos(k_+L_2) + ik_+\sin(k_+L_2)]/(g\phi_+), \tag{31}$$

$$\alpha_2 = -[\cos(k_-L_1) + i\sin(k_-L_1)(\phi_+ - \phi_-)E/(k_-\phi_+)], \tag{32}$$

$$\alpha_3 = [k_+\cos(k_+L_2) - iE\sin(k_+L_2)]/(g\phi_+). \tag{33}$$

$$\alpha_4 = -\sin(k_-L_1)k_+\phi_-/(k_-\phi_+). \tag{34}$$

The eigenvalue condition is the condition for existence of nontrivial solution c_1, c_2 to (26) (see Appendix B).

In the large $|E|$ limit we have, for positive or negative integers n of large magnitude,

$$E = E_{n+} + \frac{e_+}{n} + 0(1/n^2) \text{ and } E_{n-} + \frac{e_-}{n} + 0(1/n^2), \tag{35}$$

where,

$$E_{n\pm} \equiv (2n\pi + \alpha \pm \zeta)/L, \tag{36}$$

and

$$\begin{aligned} e_\pm \equiv & \frac{g^2}{4\pi}(L_2|\phi_+|^2 - L_1|\phi_-|^2) \\ & \pm \frac{g}{2\pi}\frac{\cos\theta}{\sin\zeta}\operatorname{Re}\Big(\exp(i\beta)\{\sin(L_2E_{n\pm})\exp(-iL_1E_{n\pm})\phi^*_\mp \\ & - \sin(L_1E_{n\pm})\exp(-iL_2E_{n\pm})\phi^*_\pm\}\Big). \end{aligned} \tag{37}$$

Thus the large eigenvalues of the Goldstone-Wilczek Hamiltonian ($g \neq 0$) differ from those of the zero mass case by terms of order $1/E$, *i.e.*

$$E(g) - E(0) = 0[1/E(0)]. \tag{38}$$

Let M be a large enough integer such that in some ordering of the eigenvalues $E_j(g)$, $|E_j(g)| \neq 0$ for $j > M$, as g varies from 0 to g. Consider, $N_0(g) = N_1(g) + N_2(g)$, where,

$$N_1(g) \equiv -\frac{1}{2}\sum_{j=1}^{M} \operatorname{sgn} E_j,\ N_2(g) \equiv -\frac{1}{2}\lim_{\varepsilon\to 0+}\sum_{j=M+1}^{\infty} \operatorname{sgn} E_j \exp(-\varepsilon|E_j|). \tag{39}$$

Then,

$$\frac{dN_2}{dg} = \frac{1}{2}\lim_{\varepsilon\to 0+}\sum_{j=M+1}^{\infty} \varepsilon\frac{dE_j}{dg}\exp(-\varepsilon|E_j|) = 0, \tag{40}$$

where the last step follows from $|E_j| \sim \text{const}\, j$, and $dE_j/dg = 0(1/j)$. However, at values g_i of g where one or more levels $E_j(g_i) = 0$, $N_1(g)$ jumps by an integer. Thus,

$$N_0(g) - N_0(0) = -\sum_{g_i < g}\ \sum_{j:E_j(g_i)=0} \operatorname{sgn}\left(\frac{dE_j}{dg}\right)\Bigg|_{g=g_i}, \tag{41}$$

where $N_0(0) = N_0$ of (17). Since the right side of (41) is integral, our calculation of the fractional part of the ground state fermion number of the Goldstone-Wilczek Hamiltonian is complete.

4. Discussion

For the Jackiw-Rebbi Hamiltonian, $\phi_3 = 0, \phi_+ = -\phi_-$, we find that the condition for zero energy levels to exist is,

$$\det(1 - U\sigma_1) = 0, \tag{42}$$

which is independent of the value of g. Hence, no levels will pass through $E = 0$ as g is varied. Thus, for the Jackiw-Rebbi Hamiltonian

$$N_0(g) - N_0(0) = 0. \tag{43}$$

For periodic boundary conditions $N_0(0)$ and hence $N_0(g)$ is integral, the Rajaraman-Bell (1982) result; the crucial difference from the Jackiw-Rebbi infinite space result is due to the existence of two-zero energy normalized C-number solutions, (choosing $\phi_+ = -\phi_-$, $m \equiv g|\phi_\pm|$, $L_2 = -L_1 = L/2$),

$$\psi_{01} = \begin{pmatrix} \exp(-m|x|) \\ 0 \end{pmatrix} \left(\frac{m}{1 - \exp(-mL)} \right)^{1/2},$$

$$\psi_{02} = \begin{pmatrix} 0 \\ \exp(m|x|) \end{pmatrix} \left(\frac{m}{\exp(mL) - 1} \right)^{1/2} \tag{44}$$

one of which (ψ_{02}) disappears in the infinite space limit. For finite L the only boundary conditions which preserve C-invariance and allow $E = 0$ solutions (features of infinite space) are,

$$U = \sigma_1, \; U = \pm \cos\theta_0 + i\sigma_2 \sin\theta_0. \tag{45}$$

Except for $U = \sigma_1$, the others have $\det U = +1$ and hence half-odd integral fermion number. For $U = \pm\cos\theta_0 + i\sigma_2 \sin\theta_0$ there is only one zero energy solution:

$$\psi_0 = \text{const}((1 + \sin\theta_0 \pm \cos\theta_0)\psi_{01} + i(1 + \sin\theta_0 \mp \cos\theta_0)\exp(-mL/2)\psi_{02}). \tag{46}$$

Only for $U = +1$ (*i.e.*, $\cos\theta_0 = \pm 1$) is the admixture of ψ_{02} (which is non-normalizable in infinite space) absent. (For $U = -1$, $\psi_0 = \psi_{02}$). Thus, the physics of the infinite space solitonic Jackiw-Rebbi case corresponds for finite L to the boundary condition $U = 1$. Other (non-periodic) boundary conditions correspond to the infinite space Goldstone-Wilczek results. It seems likely that in 3 dimensions also, different allowed boundary conditions deduced from self-adjointness might capture the physics of different nontrivial asymptotic behaviours of background fields.

A brief account of these results is being submitted elsewhere (Roy and Singh 1984) for publication.

Appendix A

Self-adjoint extensions of an $N \times N$ Dirac Hamiltonian

Consider

$$H = \alpha p + \beta(x), \; p \equiv -i\frac{\mathrm{d}}{\mathrm{d}x}, \tag{A1}$$

where α is a hermitean $N \times N$ Dirac matrix, and $\beta(x)$ is a bounded hermitean $N \times N$ matrix function of x representing a background field. On $x \in [L_1, L_2]$ we seek boundary conditions which will make the above formal Hamiltonian self-adjoint. From (A1) we deduce,

$$(\psi_2, H\psi_1) - (H\psi_2, \psi_1) = -i\psi_2^+(x)\alpha\psi_1(x)\Big|_{x=L_1}^{L_2}, \tag{A2}$$

if ψ_1, ψ_2, $H\psi_1$ and $H\psi_2$ are fermion wavefunctions (not fields) obeying

$$\int_{L_1}^{L_2} \psi^+ \psi \, dx < \infty, \int_{L_1}^{L_2} (H\psi)^+ (H\psi) < \infty. \tag{A3}$$

Let $\chi^{(i)}$ be the orthonormal eigenfunctions of α with eigenvalue λ_i such that $\lambda_i \geqslant 0$ for $i = 1, \ldots M$, and $\lambda_i < 0$ for $i = M+1, \ldots N$,

$$\sum_{s=1}^{N} \alpha_{rs} \chi_s^{(i)} = \lambda_i \chi_r^{(i)}, \text{ (no sum over } i). \tag{A4}$$

Denoting

$$\psi(x) = \sum_{i=1}^{N} u_i(x) \chi^{(i)}, \tag{A5}$$

and introducing the N component spinors constructed out of the boundary values of the $u_i(x)$,

$$A = \begin{bmatrix} (\lambda_1)^{1/2} & u_1(L_2) \\ (\lambda_2)^{1/2} & u_2(L_2) \\ \vdots & \\ (\lambda_M)^{1/2} & u_M(L_2) \\ (|\lambda_{M+1}|)^{1/2} & u_{M+1}(L_1) \\ (|\lambda_{M+2}|)^{1/2} & u_{M+2}(L_1) \\ \vdots & \\ (|\lambda_N|)^{1/2} & u_N(L_1) \end{bmatrix}, \quad B = \begin{bmatrix} (|\lambda_{M+1}|)^{1/2} & u_{M+1}(L_2) \\ (|\lambda_{M+2}|)^{1/2} & u_{M+2}(L_2) \\ \vdots & \\ (|\lambda_N|)^{1/2} & u_N(L_2) \\ (\lambda_1)^{1/2} & u_1(L_1) \\ (\lambda_2)^{1/2} & u_2(L_1) \\ \vdots & \\ (\lambda_M)^{1/2} & u_M(L_1) \end{bmatrix}, \tag{A6}$$

we see that (A2) becomes,

$$(\psi_2, H\psi_1) - (H\psi_2, \psi_1) = -i(A_2^+ A_1 - B_2^+ B_1), \tag{A7}$$

where A_i and B_i ($i = 1, 2$) denote values of A and B for $\psi = \psi_i$. From (A7) we deduce easily that the boundary condition on $\psi(x)$ in order that the formal Hamiltonian (A1) becomes self-adjoint is that (in addition to (A3)),

$$A = UB, \tag{A8}$$

where U is an arbitrary $N \times N$ unitary matrix. Thus we have an N^2 parameter family of self-adjoint Hamiltonians. For a semi-infinite box one of the end points, say $L_2 \to \infty$, and the corresponding $u_i(L_2)$ must vanish due to (A3). Hence A will have $(N-M)$ components, B will have M components and U will be a rectangular $(N-M) \times M$ matrix obeying $U^+U = 1_{M \times M}$, $UU^+ = I_{(N-M)\times(N-M)}$. For $L_1 \to -\infty$, $L_2 \to +\infty$, A and B must vanish and the Hamiltonian is essentially self-adjoint.

Appendix B

Eigenvalue equation for Jackiw-Rebbi and Goldstone-Wilczek cases

It is worthwhile to note down the exact eigenvalue equation corresponding to

$$\phi(x) = \phi_+ \,\text{step}\, x + \phi_- \,\text{step}\,(-x), \tag{B1}$$

which is given by the condition that equation (26) of the text leads to nontrivial values of c_1 and c_2. This is given by

$$G(E) = 0, \tag{B2}$$

where

$$\begin{aligned} G(E) \equiv & -\sin\theta\cos\gamma + \cos(k_+ L_2)\cos(k_- L_1)\cos\alpha \\ & + \frac{\sin(k_+ L_2)}{k_+}\cdot\frac{\sin(k_-|L_1|)}{k_-}\{\mathrm{Re}\,[(g^2\phi_-\phi_+^* - E^2)\exp(i\alpha)] \\ & + E\cos\theta\,\mathrm{Im}\,[\exp(i\beta)(g\phi_-^* - g\phi_+^*)]\} \\ & + \frac{\sin k_+ L_2}{k_+}\cdot\cos(k_- L_1)\{E\sin\alpha + \mathrm{Re}(g\phi_+^*\exp(i\beta))\cos\theta\} \\ & + \cos k_+ L_2 \frac{\sin(k_-|L_1|)}{k_-}\{E\sin\alpha + \mathrm{Re}(g\phi_-^*\exp(i\beta))\cos\theta\}. \end{aligned} \tag{B3}$$

Note that the phase β occurs only in the combination $\phi_\pm^* \exp(i\beta)$. The large eigenvalues are then given by (35)–(37) of the text.

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