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UNITARITY TESTS AT HIGH ENERGIES

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ABSTRACT

We derive unitarity inequalities involving ReF(s,t) and ImF(s,t) in a region $0 \leq |t| \leq |T|$ which exhaust the content of unitarity given only the elastic amplitude F(s,t) in this range of t. If the Froissart bound is saturated, these inequalities lead to inequalities of Martin's scaling functions. The inequalities may be used to check compatibility with unitarity of theoretical models or experimental data.

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Recent $\bar{p}p$ elastic scattering experiments at the collider[1,2] show a change in slope at $|t| \sim 0.15 - 0.2$ $(GeV/c)^2$ and a shoulder around $|t| \sim 0.8$ $(GeV/c)^2$. The "preliminary results" in the shoulder region[2] "seem to be in disagreement" with some theoretical models[3] as well as with an analysis based on geometrical scaling[4]. This situation has stimulated us to develop a method to test unitarity at high energies. The method we report here can be used to check unitarity of theoretical models as well as of experimental data at any energy, directly. We also derive asymptotic unitarity restrictions in the case when the Froissart bound on σ_{tot} is saturated. These restrictions can be tested against experimental data using a modification of the method of Dias de Deus and Kroll[4], in which the asymptotically small contribution of ReF(s,t) is estimated using the scaling theorem of Auberson et al. and of Martin[5]. For $s \to \infty$ we obtain non-trivial unitarity restrictions on the scaling function.

At low energies the partial wave expansion of the elastic amplitude F(s,t)

$$F(s,t) = \frac{\sqrt{s}}{k} \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(s) P_\ell(\cos\theta) \qquad (1)$$

is a very good tool for checking the unitarity condition

$$\mathrm{Im} f_\ell(s) \geq |f_\ell(s)|^2 \qquad (2)$$

(We use s,t to denote squares of the c.m. energy and momentum transfer respectively, and k = c.m. momentum). The expansion (1) converges within the Lehmann-Martin (L-M) ellipse[6] in the $\cos\theta$-plane with foci -1 and +1 and semimajor axis $1 + t_o/(2k^2)$, where t_o is a positive constant independent of s ($t_o = 4m_\pi^2$ for many interesting cases). At high energies, as the ellipse shrinks to the real line $\cos\theta = -1$ to +1, a very large number of partial waves is needed to approximate F(s,t). For example, using

$$\sigma_{tot} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \mathrm{Im} f_\ell(s) \qquad (3)$$

and $0 \leqslant \mathrm{Im}f_\ell \leqslant 1$, we see that for $\sqrt{s} \sim 540$ GeV and $\sigma_{tot} \sim 60$ mb, we need more than 800 partial waves. A second difficulty is that to compute $f_\ell(s)$ we need $F(s,t)$ for all $\cos\theta$ in $(-1,1)$, and frequently the data exist only in a near forward cone of small opening angle. Therefore, so far, indirect tests, such as the McDowell-Martin bound[7] on the derivative of the absorptive part at $t=0$, and the Singh-Roy bound[8] on the absorptive part at negative t have been used[9] to check consistency of high energy data with unitarity.

Our object is to derive unitarity inequalities involving both the real and imaginary parts of the amplitude which can be tested given only the elastic amplitude in a limited t-range $0 \leqslant |t| \leqslant |T|$; further, given only this information the inequalities exhaust the content of unitarity.

The unitarity relation for the elastic amplitude F_{ba} where $\vec{n}_a$, $\vec{n}_b$ are initial and final directions in c.m. is

$$\mathrm{Im}F_{ba} = \frac{k}{4\pi\sqrt{s}} \left(\int d\Omega_c \; F_{bc}F^*_{ac} + \int d\Omega_x \sum_x F_{bx}F^*_{ax} \right) , \tag{4}$$

where, in the contribution of the inelastic states, the symbol $\sum_x$ stands for summations over all kinds of particles as well as phase space factors, and identical particle factors if necessary.

Suppose we know only the amplitude F_{ba} inside a narrow cone of half-angle α within which $\vec{n}_a.\vec{n}_b$ varies from $\cos(2\alpha)$ to 1, i.e.,

$$0 \leq |t| \leq 2k^2(1 - \cos(2\alpha)) \tag{5}$$

Multiplying Eq.(4) by $U(\vec{n}_a)U^*(\vec{n}_b)$, where $U(\vec{n})$ is an arbitrary function, and integrating over $\vec{n}_a$ and $\vec{n}_b$ over the region R inside the cone, we have,

$$\int_R d\Omega_a \int_R d\Omega_b \; \mathrm{ImF}_{ba} U(\vec{n}_a) U^*(\vec{n}_b)$$

$$= \frac{k}{4\pi\sqrt{s}} \left[\int d\Omega_c \left(\int_R F_{bc} d\Omega_b U^*(\vec{n}_b)\right)\left(\int_R F^*_{ac} d\Omega_a U(\vec{n}_a)\right) \right.$$

$$\left. + \int d\Omega_x \sum_x \left(\int_R F_{bx} d\Omega_b U^*(\vec{n}_b)\right)\left(\int_R F^*_{ax} d\Omega_a U(\vec{n}_a)\right) \right]$$

$$\geq \frac{k}{4\pi\sqrt{s}} \int_R d\Omega_c \left| \int_R F^*_{ac} d\Omega_a U(\vec{n}_a) \right|^2 , \tag{6}$$

which follows because the two factors in each integrand on the right-hand side are complex conjugates of each other. In writing (6) we have thrown away the unknown but positive contributions of elastic scattering outside the cone, as well as of inelastic scattering, such that the inequality (6) involves only known elastic amplitudes. For every unit vector $\vec{n}$ inside the cone R

$$\vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta), \tag{7}$$

define another unit vector $\vec{n}'$ which has $\phi' = \phi$, and a larger θ' given by

$$1 - \cos\theta' = \frac{2}{1-\cos\alpha}\,(1-\cos\theta). \tag{8}$$

Then as $\vec{n}$ varies over the cone, $\vec{n}'$ varies over the unit sphere, so that

$$\int_R d\Omega_a = \int d\Omega'_a \,\frac{1-\cos\alpha}{2} . \tag{9}$$

Expanding $U(\vec{n}_a)$ in spherical harmonics $Y_{\ell m}(\vec{n}'_a)$,

$$U(\vec{n}_a) = \sum_{\ell,m} U_{\ell m} Y_{\ell m}(\vec{n}'_a) \tag{10}$$

and denoting

$$\mathrm{Imf}_{\ell' m';\ell m} \equiv \int d\Omega'_a \int d\Omega'_b \; Y^*_{\ell' m'}(\vec{n}'_b)\,\mathrm{ImF}_{ba} Y_{\ell m}(\vec{n}'_a) , \tag{11}$$

$$f_{\ell m}(\vec{n}_c) \equiv \int d\Omega_a' \, F_{ac} Y^*_{\ell m}(\vec{n}_a'), \tag{12}$$

the inequality (6) becomes,

$$\sum_{\ell, m, \ell'} U^*_{\ell' m} \, \mathrm{Im} f_{\ell' m; \ell m} U_{\ell m} \geq \frac{k}{4\pi\sqrt{s}} \, \frac{1-\cos\alpha}{2} \int d\Omega_c' \left| \sum_{\ell, m} U^*_{\ell m} f_{\ell m}(\vec{n}_c) \right|^2 \tag{13}$$

We may call the matrix with elements $\mathrm{Im} f_{\ell' m'; \ell m}$ the "incomplete absorptive partial wave matrix". This matrix is symmetric under $\ell \leftrightarrow \ell'$, diagonal in m, m', and real because in Eq.(11) $\mathrm{Im} F_{ba}$ depends only on $\vec{n}_a \cdot \vec{n}_b$. Note that if $\cos\alpha = -1$, the $\mathrm{Im} f_{\ell' m; \ell m}$ become diagonal also in ℓ, ℓ' and independent of m, and the relation (13) then coincides with the usual partial wave unitarity relation (2). Of course, the point is that at high energies, $1-\cos\alpha = O(1/s)$, and hence the usual partial waves cannot be computed. The unitarity inequalities (13), however, involve only the amplitude in the limited $|t|$ range (5). Note that in this range the $Y_{\ell m}(\vec{n}_a)$ with small ℓ-values are practically independent of θ_a because $\theta_a \approx 0$; hence in (10) we have used the $Y_{\ell m}(\vec{n}_a')$ to expand $U(\vec{n}_a)$. For practical computations the following more explicit forms of Eqs.(11) and (12) will be useful.

$$\mathrm{Im} f_{\ell' m'; \ell m} = \delta_{mm'} \int_{\theta_{ab}=0}^{\min(2\alpha, \pi)} \sin\theta_{ab} d\theta_{ab} \, \mathrm{Im} F(t_{ab}) K_{\ell\ell' m}(\theta_{ab}), \tag{11a}$$

where $\cos\theta_{ab} \equiv \vec{n}_a \cdot \vec{n}_b$, $t_{ab} = -2k^2(1-\cos\theta_{ab})$, and

$$K_{\ell\ell' m}(\theta_{ab}) \equiv N_{\ell' m} N_{\ell m} \frac{8\pi}{(1-\cos\alpha)^2} \int_0^{\alpha} d\theta_a \sin\theta_a \int_{\max(\theta_{ab}-\theta_a, \theta_a)}^{\min(\theta_{ab}+\theta_a, 2\pi-\theta_{ab}-\theta_a, \alpha)} d\theta_b \sin\theta_b$$

$$\times \frac{\left\{P^m_{\ell'}(\cos\theta_b')P^m_{\ell}(\cos\theta_a') + P^m_{\ell'}(\cos\theta_a')P^m_{\ell}(\cos\theta_b')\right\}\cos\left\{m(\phi_a-\phi_b)\right\}}{\left[\left\{\cos(\theta_a-\theta_b) - \cos\theta_{ab}\right\}\left\{\cos\theta_{ab} - \cos(\theta_a+\theta_b)\right\}\right]^{1/2}} \tag{11b}$$

Here $P^m_{\ell}(x)$ denote associated Legendre functions, with $\cos\theta_i'$ and

$\cos\theta_i$ (i = a or b) related as in Eq.(8), and

$$N_{\ell m} \equiv \left[\frac{2\ell+1}{4\pi}\cdot\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2}, \quad \cos(\phi_a-\phi_b) = \frac{\cos\theta_{ab}-\cos\theta_a\cos\theta_b}{\sin\theta_a\sin\theta_b} \tag{11c}$$

Similarly Eq.(12) yields:

$$f_{\ell m}(\vec{n}_c) = e^{-im\phi_c}\frac{4}{1-\cos\alpha}N_{\ell m}\int_0^{\min(\alpha+\theta_c,\pi)} d\theta_{ac}\sin\theta_{ac}F(t_{ac})\cdot$$

$$\times\int_{|\theta_{ac}-\theta_c|}^{\min(\theta_c+\theta_{ac},2\pi-\theta_c-\theta_{ac},\alpha)} d\theta_a\sin\theta_a \frac{P_\ell^m(\cos\theta_a')\cos\{m(\phi_a-\phi_c)\}}{\sqrt{\{\cos(\theta_a-\theta_c)-\cos\theta_{ac}\}\{\cos\theta_{ac}-\cos(\theta_a+\theta_c)\}}} \tag{12a}$$

where θ_{ac} and $\cos(\phi_a-\phi_c)$ are defined similarly to θ_{ab} and $\cos(\phi_a-\phi_b)$.

Since the relations (13) hold for arbitrary $U_{\ell m}$, they imply that the incomplete absorptive partial wave matrix is a positive matrix. Further, they give positive lower bounds on the eigenvalues. For example, if we choose all $U_{\ell m}$ beyond a certain ℓ=L to vanish, and the $U_{\ell m}$ for $\ell \leq$ L to coincide with a normalized ($\sum|U_{\ell m}|^2 = 1$) eigenvector $U_{\ell m}^{(i)}$ of the incomplete absorptive partial wave matrix (with ℓ, $\ell' \leq$ L) corresponding to eigenvalue λ_i, then

$$\lambda_i \geq \frac{k}{4\pi\sqrt{s}}\,\frac{1-\cos\alpha}{2}\int d\Omega_c'\left|\sum_{\ell,m} U_{\ell m}^{(i)*} f_{\ell m}(\vec{n}_c)\right|^2 \tag{14}$$

Inequalities (14) constitute an optimum form of (13). The unitarity test involves computing the eigenvalues λ_i and the normalized eigenvectors $U_{\ell m}^{(i)}$ of the partial wave matrix truncated at ℓ=L, ℓ'=L. If some λ_i is negative the amplitude violates unitarity; if all λ_i are non-negative they still have to satisfy (14) for unitarity to be valid.

The unitarity inequality (14) can be used directly to test theoretical models, as well as experimental data if both ReF(s,t) and ImF(s,t) are measured. Often only $d\sigma/dt$ is measured. If we assume that in the range (5) $\mathrm{Im}F(s,t) \neq 0$ and $\mathrm{Re}F(s,t)$ is negligible, we can test using (14) whether these assumptions are compatible with unitarity. Instead of neglecting $\mathrm{Re}F(s,t)$ we shall use the scaling theorem of Ref. 5) to estimate it. Ref. 5) implies that if $\sigma_{tot}/(\ln s)^2 \to$ const and the odd signature amplitude is negligible for $s \to \infty$, then for $s \to \infty$ with $\tau \equiv t\sigma_{tot}$ fixed,

$$\mathrm{Im}F(s,t)/\mathrm{Im}F(s,o) \to \phi(\tau), \quad \mathrm{Re}F(s,t)/\mathrm{Re}F(s,o) \to \frac{d}{d\tau}(\tau\phi) \qquad (15)$$

The fits of Block and Cahn[10], for instance, suggest that saturation of the Froissart bound is a reasonable hypothesis. Dias de Deus and Kroll[4] then use

$$\frac{d\sigma}{dt}(s,t) = \frac{1}{16\pi}\sigma^2_{tot}(s)\left\{\phi^2(\tau) + \rho^2(s)\left(\frac{d}{d\tau}(\tau\phi)\right)^2\right\} \qquad (16)$$

where

$$\rho(s) \equiv \mathrm{Re}F(s,o)/\mathrm{Im}F(s,o) , \qquad (17)$$

to calculate $\phi(\tau)$ from $d\sigma/dt$ data. This method is interesting, but it ignores the possibility, allowed by Eq.(15), that the non-scaling part of $\mathrm{Im}F(s,t)/\mathrm{Im}F(s,o)$ could be comparable to $\rho(s)\mathrm{Re}F(s,t)/\mathrm{Re}F(s,o)$. Hence we suggest a modification of the method of Ref.4) in which the scaling Eqs. (15) are used <u>only to estimate the small term $\rho(s)\mathrm{Re}F(s,t)/\mathrm{Re}F(s,o)$</u>. We write

$$\frac{d\sigma}{dt}(s,t) = \frac{1}{16\pi}\sigma^2_{tot}(s)\left\{\phi^2(s,t) + \rho^2(s)\left(\frac{d}{dt}(t\phi(s,t))\right)^2\right\} \qquad (18)$$

where

$$\phi(s,t) \equiv \mathrm{Im}F(s,t)/\mathrm{Im}F(s,o) \qquad (19)$$

At each s, $\phi(s,t)$ can be calculated from $d\sigma/dt$ data if $\rho(s)$ is known. We give below unitarity restrictions on $\phi(s,t)$.

The unitarity inequality (13) yields,

$$\sum_{\ell,m,\ell'} U^*_{\ell'm}\phi_{\ell'm;\ell m}U_{\ell m} \geq \frac{k^2(1-\cos\alpha)\sigma_{tot}}{32\pi^2}$$

$$\times \int d\Omega'_c \left| \sum_{\ell m} U^*_{\ell m}\left\{\rho(s)\chi_{\ell m}(\vec{n}_c) + i\phi_{\ell m}(\vec{n}_c)\right\}\right|^2, \qquad (20)$$

where

$$\phi_{\ell'm;\ell m} \equiv \int d\Omega'_a \int d\Omega'_b \; Y^*_{\ell'm}(\vec{n}'_b)\phi(s,t_{ba})Y_{\ell m}(\vec{n}'_a), \qquad (21)$$

$$\phi_{\ell m}(\vec{n}_c) \equiv \int d\Omega'_a \; Y^*_{\ell m}(\vec{n}'_a)\phi(s,t_{ac}), \qquad (22)$$

and

$$\chi_{\ell m}(\vec{n}_c) \equiv \int d\Omega'_a \; Y^*_{\ell m}(\vec{n}'_a) \frac{d}{d(t_{ac})} (t_{ac}\phi(s,t_{ac})). \qquad (23)$$

We have used the scaling approximation only for the asymptotically small term $\rho(s)\mathrm{Re}F(s,t)/\mathrm{Re}F(s,o)$. For $s \to \infty$ this term is negligible, $\phi(s,t) \to \phi(\tau)$ and $k^2(1-\cos\alpha)\sigma_{tot}$ is a finite non-zero constant for τ fixed; hence inequality (20) becomes a non-trivial unitarity constraint on the scaling function $\phi(\tau)$. At any s, Eq.(20) states that the matrix $\phi_{\ell'm;\ell m}$ is a positive matrix. Further, if we choose $U_{\ell m} = 0$ for $\ell > L$, and $U_{\ell m}$ for $\ell \leq L$ equal to $U^{(i)}_{\ell m}$, the normalized eigenvectors with eigenvalue λ_i of the truncated matrix $\phi_{\ell'm;\ell m}$, then,

$$\lambda_i \geq \frac{k^2(1-\cos\alpha)\sigma_{tot}}{32\pi^2} \int d\Omega'_c \left| \sum_{\ell m} U^{(i)*}_{\ell m} \left\{\rho(s)\chi_{\ell m}(\vec{n}_c)+i\phi_{\ell m}(\vec{n}_c)\right\}\right|^2 \qquad (24)$$

which can be experimentally tested. The simplest inequality for $L = 0$ is,

$$\int d\Omega'_a \int d\Omega'_b \phi(s,t_{ba}) \geq \frac{k^2(1-\cos\alpha)\sigma_{tot}}{32\pi^2}$$

$$\times \int d\Omega'_c \left| \int d\Omega'_a \left\{\rho(s) \frac{d}{dt_{ac}} (t_{ac}\phi(s,t_{ac})) + i\phi(s,t_{ac})\right\}\right|^2 \qquad (25)$$

In summary, we have presented optimum unitarity inequalities given only the amplitude in the near forward region, both in the exact version (13) and in the asymptotic version (20) where scaling formulae are used to estimate the contribution of ReF(s,t). Their practical utility remains to be explored.

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