Generalized coherent states and the uncertainty principle

S. M. Roy and Virendra Singh
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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We derive from a dynamical symmetry property that the linear and nonlinear Schrödinger equations with harmonic potential possess an infinite string of shape-preserving coherent wave-packet states with classical motion. Unlike the Schrödinger state with $\Delta x \Delta p = \hbar/2$, the uncertainty product can be arbitrarily large for these states showing that classical motion is not necessarily linked with minimum uncertainty. We obtain a generalization of Sudarshan's diagonal coherent-state representation in terms of these states.

A celebrated exception to the nearly all pervasive phenomena of spreading of wave packets in quantum mechanics is the "coherent state" of the harmonic oscillator constructed by Schrödinger[1] and very fruitfully exploited by Glauber, Sudarshan, and others in quantum optics.[2] The Schrödinger coherent state at time t

$$|\alpha\rangle = e^{-|\alpha(t)|^2/2}\left(\sum_{n=0}^{\infty} \frac{[\alpha(t)]^n}{\sqrt{n!}}\,|n\rangle\right) e^{-i\omega t/2}, \tag{1}$$

where ω is the oscillator frequency, $a^\dagger a\,|n\rangle = n\,|n\rangle$, $\langle n|n\rangle = 1$, a and $a^\dagger$ are the annihilation and creation operators, respectively, and

$$\alpha(t) \equiv \alpha(0)e^{-i\omega t} \equiv |\alpha(0)|\,e^{-i(\omega t+\phi)} \tag{2}$$

has the following remarkable properties:

(i) The probability-density wave packet remains unchanged in shape as time progresses and has classical motion, i.e., $\langle x\rangle = x_{\rm cl}(t) \equiv A\cos(\omega t+\phi)$, $\langle p\rangle = p_{\rm cl}(t) \equiv M\dot{x}$, where $A = |\alpha|\,[2\hbar/(M\omega)]^{1/2}$.

(ii) The probability of being in state $|n\rangle$ is independent of time and has a Poisson distribution.

(iii) The uncertainty product is the minimum possible, $\Delta x \Delta p = \hbar/2$.

(iv) The state $|\alpha\rangle$ is an eigenfunction of the annihilation operator with eigenvalue $\alpha(t)$.

(v) A unitary displacement operator on the ground state yields the state $|\alpha\rangle$:

$$|\alpha\rangle = U(\alpha)\exp\left(-\frac{i\omega t}{2}\right)|0\rangle, \tag{3}$$

where

$$U(\alpha) \equiv \exp[\alpha(t)a^\dagger - \alpha^*(t)a] = \exp\left(-\frac{i}{\hbar}(x_{\rm cl}p - p_{\rm cl}x)\right), \tag{4}$$

x and p denoting the position and momentum operators.

States with properties (iii), (iv), and (v) are often called MUCS (minimum-uncertainty coherent states), AOCS (annihilation-operator coherent states), and DOCS (displacement-operator coherent states), respectively. Group-theoretical generalizations of coherent states based on AOCS (Ref. 3) and DOCS (Ref. 4) have been given. Recently, Nieto, Simmons, and Gutschick in a series of beautiful papers[5,6] (and a movie[7]) note that[5] these generalizations "represent a departure from the original motivation for studying coherent states; namely that they obey the classical motion," and propose that the defining criterion for coherent states for general potentials should be Schrödinger's original one, i.e., property (i). Nieto, Simmons, and Gutschick reduce the arbitrary-potential case to a harmonic-oscillator-like problem and produce states which nearly follow classical motion, and are a generalization of the MUCS.

In this paper we adopt the motivation of Schrödinger[1] and Nieto, Simmons, and Gutschick[5,6] and define coherent states as those with *undistorted normalizable wave packets with classical motion.* The states constructed by Nieto, Simmons, and Gutschick are approximately (not exactly) coherent. Further, Mathews and Eswaran[8] constructed "semicoherent" oscillator states with Δx constant though the wave packets suffer distortion with time. We demonstrate for the linear and nonlinear Schrödinger equations with harmonic-oscillator potential (and elsewhere[9] for other potentials) the existence of exact coherent states obeying (i) exactly but not (ii)–(v). For the linear Schrödinger equation the existence of such states was first noticed by Senitzky.[10] We show that for these generalized coherent states the properties (ii)–(v) possess suitable generalizations. In particular, the uncertainty product $\Delta x \Delta p = (n+\frac{1}{2})\hbar$, $n = 0,1,2,\ldots$, with no upper limit for n, showing that classical motion with undistorted wave packets does not imply minimum uncertainty. Other useful properties of the generalized coherent states are noted.

A DYNAMICAL SYMMETRY

Let $\phi(x,t)$ be a solution of the nonlinear or linear ($\lambda=0$) Schrödinger equation with a harmonic-oscillator potential

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\tfrac{1}{2}M\omega^2x^2\psi+\lambda\psi g(|\psi|)$$
$$\equiv H\psi+\lambda\psi g(|\psi|)\,, \qquad (5)$$

where $g(|\psi|)$ is an arbitrary real function [the choices $g=|\psi|^2$ and $1-\exp(-\epsilon|\psi|^2)$ being frequently encountered[11] when $\omega=0$]. Then we prove by direct substitution that

$$\psi(x,t)=\phi(x-x_{\rm cl}(t),t)\times\exp\left(i\frac{p_{\rm cl}(t)}{\hbar}[x-\tfrac{1}{2}x_{\rm cl}(t)]\right) \qquad (6)$$

is also a solution, with $x_{\rm cl}$ and $p_{\rm cl}$ as defined in (i). We see readily that

$$\psi(x,t)=U(\alpha)\phi(x,t)\,, \qquad (7)$$

where U is the unitary operator (4). This general symmetry property of Eq. (5) is of special interest when ϕ is an eigensolution:

$$\phi(x,t)=\phi_n(x)\exp(-iE_nt/\hbar)\,. \qquad (8)$$

The new solution

$$\psi_{n,\alpha}(x,t)=\phi_n(x-x_{\rm cl}(t))\exp\left[\frac{ip_{\rm cl}}{\hbar}\left(x-\frac{x_{\rm cl}}{2}\right)-\frac{iE_nt}{\hbar}\right], \qquad (9)$$

then, has a probability or number-density packet $|\phi_n(x-x_{\rm cl}(t))|^2$ *which retains its shape and moves classically*. This particlelike or "lump" property is common to the nonlinear ($\lambda\neq0$) and linear ($\lambda=0$) Schrödinger equations. We have thus a "perturbative lump" whose motion $x_{\rm cl}(t)$ is independent of λ and whose shape ϕ_n obeys $(E_n-H)\phi_n(x)=\lambda\phi_n g(|\phi_n|)$. Clearly for $\lambda=0$, $\phi_n=cu_n(x)$, where $u_n(x)$ denote normalized oscillator eigenfunctions. Perturbatively ϕ_n and E_n can be given explicitly to any order in λ, e.g.,

$$E_n=(n+\tfrac{1}{2})\hbar\omega+\lambda\int_{-\infty}^{\infty}dx\,u_n^2(x)g(|c|u_n(x))+O(\lambda^2)\,. \qquad (10)$$

We now discuss the linear case ($\lambda=0$) in detail.

GENERALIZED COHERENT STATES OF THE HARMONIC OSCILLATOR

The states

$$|n,\alpha\rangle\equiv U(\alpha(t))\,|n\rangle\exp[-i(n+\tfrac{1}{2})\omega t],\quad n=0,1,2,\ldots\,, \qquad (11)$$

which are displaced excited eigenstates are exact coherent states since they obey the defining property (i) as demonstrated by Eq. (9). The state $|0,\alpha\rangle$ is just the Schrödinger state. The Fock-space representation of these states is found to be

$$|n,\alpha\rangle=e^{-|\alpha(0)|^2/2}\sum_{m=0}^{\infty}\left(\frac{n!}{m!}\right)^{1/2}L_n^{m-n}(|\alpha(0)|^2)\times[\alpha(0)]^{m-n}|m\rangle e^{-i\omega t(m+1/2)}\,, \qquad (12)$$

where $L_n^{m-n}(x)$ are Laguerre polynomials with the normalization $L_0(x)=1$, $L_n^{-n}(x)=(-x)^n/n!$. The Fock-space expansion of $|n,\alpha\rangle$ contains states $|m\rangle$ with $m<n$ also, a fact which might not be immediately obvious. The generalizations of (ii)–(v) and some other interesting properties are given below.

(ii′) The probability for being in state $|m\rangle$, given by (12), has for $n=0$ the Poisson distribution, and for $n=1,2,3,\ldots$ the appropriate generalization of it.

(iii′) The uncertainties in state $|n,\alpha\rangle$ are

$$m\omega(\Delta x)^2=\frac{1}{m\omega}(\Delta p)^2=|\Delta x\Delta p|=(n+\tfrac{1}{2})\hbar\,. \qquad (13)$$

Since $n=0,1,2,3,\ldots$, minimum uncertainty is not necessary for coherence. The higher uncertainty product is a manifestation of the fact that the wave packet $|n,\alpha\rangle$ has $n+1$ humps and is not as localized as the Schrödinger coherent state.

(iv′) The generally valid characterization of coherent states $|n\alpha\rangle$ is that they are eigenfunctions of $(a^\dagger-\alpha^*(t))(a-\alpha(t))$, with eigenvalue n, $n=0,1,2,3,\ldots$, where $\alpha(t)$ is any complex number with time dependence given by (2). Only for $n=0$ are they eigenfunctions of the destruction operator.

(v′) The generalization of Eq. (3) is Eq. (11).

(vi) It is remarkable that the expectation value of energy is a sum of a purely classical and a purely quantum term:

$$\langle n,\alpha|H|n,\alpha\rangle=E_{\rm cl}+(n+\tfrac{1}{2})\hbar\omega\,, \qquad (14)$$

where $E_{\rm cl}=\frac{1}{2}M\omega^2A^2$ is the classical energy for oscillation amplitude A.

(vii) *Time-energy uncertainty relation.* If $\tau_x=M\Delta x/|\langle p\rangle|$ is the time required for the wave packet $|n\alpha(t)\rangle$ to move by Δx, and ΔH the energy dispersion, then

$$\frac{(\Delta H)^2}{2\omega E_{\rm cl}}=2\omega E_{\rm cl}\sin^2(\omega t+\phi)\tau_x^{\,2}=(n+\tfrac{1}{2})\hbar\,, \qquad (15)$$

and hence at any time

$$\Delta H\tau_x\geq(n+\tfrac{1}{2})\hbar\,, \qquad (16)$$

the equality corresponding to $\sin^2(\omega t+\phi)=1$. Again, minimum uncertainty can be reached only for $n=0$, though coherence holds for all n.

(viii) *Overcompleteness and reproducing kernels.* Since $U(\alpha)$ is unitary, for any given α, the set of states $|n, \alpha\rangle$, $n = 0, 1, 2, \ldots$ forms a complete set just like the set $|n\rangle$. It is more interesting that for any given n, the set $|n, \alpha\rangle$ with all complex α's forms an overcomplete set. Using Eq. (12) and orthonormality of the Laguerre polynomials we obtain the "resolution of the identity"

$$I = \int \frac{d^2\alpha}{\pi} |n, \alpha\rangle\langle n, \alpha| , \tag{17}$$

where $d^2\alpha = d(\mathrm{Re}\alpha)d(\mathrm{Im}\alpha)$ and the integration is over the whole complex plane. Using $U^\dagger(\beta)U(\alpha) = U(\alpha - \beta)\exp(i\,\mathrm{Im}\alpha\beta^*)$ and Eq. (12), we get

$$\langle n, \beta | n, \alpha\rangle = L_n(|\alpha - \beta|^2)\exp(\beta^*\alpha - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2) . \tag{18}$$

Equations (17) and (18) establish overcompleteness. They imply for an arbitrary state $|\psi\rangle$ the set of integral equations

$$\langle n, \beta|\psi\rangle = \int \frac{d^2\alpha}{\pi} L_n(|\alpha - \beta|^2) \times \exp(\beta^*\alpha - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2)\langle n, \alpha|\psi\rangle , \tag{19}$$

$n = 0, 1, 2, \ldots$, with the reproducing kernels $K_n(\beta, \alpha)$ being just the right-hand side of (18). When $|\psi\rangle$ is itself a coherent state, (19) yields the idempotent condition for the kernel K_n. These relations are already well exploited[2] for $n = 0$. Their utility for higher n is obvious from the next result.

(ix) *Diagonal coherent-state representation.* The Sudarshan representation[2] for the density operator in terms of the Schrödinger coherent states, useful in quantum optics, has the following generalization in terms of the generalized coherent states. Let $G = G(a, a^\dagger)$ be an operator with the Fourier representation[12]

$$G = \int \frac{d^2\alpha}{\pi} U(\alpha)g(\alpha) , \tag{20}$$

where $g(\alpha) = \mathrm{Tr}(GU^\dagger(\alpha))$. [A sufficient condition for its validity is $\mathrm{Tr}(GG^\dagger) < \infty$.] Then we prove

$$G = \int \frac{d^2\alpha}{\pi} |n, \alpha\rangle\langle n, \alpha| p_n(\alpha) , \tag{21}$$

where

$$p_n(\alpha) = \frac{P}{\pi} \int d^2\beta \frac{g(\beta)}{L_n(|\beta|^2)} \exp(\tfrac{1}{2}|\beta|^2 + \beta\alpha^* - \beta^*\alpha) \tag{22}$$

provided that the integral converges. Here the principal value is taken at zeros of $L_n(|\beta|^2)$. Further $g(\beta)$ may be expressed in terms of the diagonal elements $\langle n, \alpha|G|n, \alpha\rangle$:

$$g(\beta) = \frac{\exp(\frac{1}{2}|\beta|^2)}{L_n(|\beta|^2)} \int \frac{d^2\alpha}{\pi} \langle n\alpha|G|n\alpha\rangle \exp(\alpha\beta^* - \alpha^*\beta) . \tag{23}$$

Clearly, convergence of the integral (22) at $\beta = \infty$ is assured for a larger class of $g(\beta)$'s for states with $n \neq 0$ than for $n = 0$. Equations (21)–(23) constitute the new diagonal coherent-state representation.

The above properties of generalized coherent states raise hopes of their utility in quantum optics similar to that of the Schrödinger state. Since the electromagnetic field Hamiltonian $\sum \hbar\omega_k a_k^\dagger a_k$ is essentially a sum of oscillator Hamiltonians, its coherent states $\prod_k |n_k, \alpha_k\rangle$ may be defined such that for each mode k, the criterion (i) of classical motion is obeyed. The consequences are under study.

Since the work of Nieto and Simmons[5] on coherent states for general potentials takes off from an oscillatorlike problem, and since exact oscillator coherent states with non-minimum uncertainty exist, it is possible that Ref. 5 might be fruitfully generalized to yield non-MUCS.

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