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MAXIMALLY REALISTIC CAUSAL QUANTUM MECHANICS[†,⋆]

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ABSTRACT

We present a causal Hamiltonian quantum mechanics in $2n$-dimensional phase space which is more realistic than de Broglie-Bohm mechanics. The positive definite phase space density reproduces as marginals the correct quantum probability densities of $n+1$ different complete commuting sets of observables e.g. positions, momenta and $n-1$ other sets.

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1. Quantum Contextuality. The most important difference between quantum mechanics and a classical stochastic theory is that quantum probabilities are inherently and irreducibly context (i.e., experimental arrangement) dependent. For any complete commuting set (CCS) of observables A, the quantum state $|\psi\rangle$ specifies the probability of observing the eigenvalues α as $|\langle\alpha|\psi\rangle|^2$, if A were to be observed. If B is another CCS with eigenvalues β, but $[A,B]\neq 0$, the analogous probabilities $|\langle\beta|\psi\rangle|^2$, if B were to be measured refer to a different context or experimental situation. Each context corresponds to the experimental arrangement to measure one CCS of observables. Due to this inherent context dependence quantum mechanics does not specify joint probabilities of noncommuting observables. Moreover, the context dependence is irreducible, i.e., quantum mechanics cannot be embedded in a classical context independent stochastic theory. This is the lesson from decades of work, e.g. Gleason's theorem[1], Kochen-Specker theorem[2], Bell's theorem[3] (where the contextuality corresponds to violation of "local realism") and Martin-Roy theorem[4] (which is a direct phase space proof relevant to the present work). The contextuality theorems circumscribe the extent to which dynamical variables in quantum mechanics can be ascribed simultaneous 'Reality' independent of observations.

2. De Broglie-Bohm. The De Broglie-Bohm (dBB) causal quantum mechanics[5] has shaped a paradigm of causal quantum mechanics in which the Position Observable occupies a favoured status of a "beable"[6] with values independent of context or observation, whereas other observables may have context dependent values. The state of the individual system is characterized by $\{|\psi(t)\rangle, \vec{x}(t)\}$ where $|\psi(t)\rangle$ and $\vec{x}(t)$ are the state vector and the configuration space coordinates, whereas an ensemble of these states corresponds to $|\psi(t)\rangle$. For a many particle system with the quantum Hamiltonian

$$H = -\sum_i \frac{\hbar^2}{2m_i}\nabla_i^2 + U(\vec{x}),$$

the individual $\vec{x}_i(t)$ move according to

$$(\vec{p}_i)_{\mathrm{dBB}} = m_i\frac{d\vec{x}_i}{dt} = \vec{\nabla}_i S(\vec{x}(t),t),$$

where $\psi = R\exp(iS/\hbar)$, with R and S being real and $\vec{x}$ denoting $(\vec{x}_1,\vec{x}_2,\cdots)$. This means that the phase space dynamics is determined by the causal Hamiltonian

$$H_{\mathrm{dBB}}(\vec{x},\vec{p},t) = \sum_i \frac{\vec{p}_i^2}{2m_i} + U(\vec{x}) - \sum_i \frac{\hbar^2}{2m_i R}\vec{\nabla}_i^2 R$$

and corresponds to the phase space density

$$\rho_{\mathrm{dBB}}(\vec{x},\vec{p},t) = |\psi(\vec{x},t)|^2\delta(\vec{p}-\vec{p}_{\mathrm{dBB}}(\vec{x},t)).$$

(We shall set $\hbar = 1$ henceforth). Integration over momentum shows that the ensemble position density agrees with $|\psi(\vec{x},t)|^2$. However, as pointed out by Takabayasi[7], integration over position does not yield the quantum momentum density $|\tilde{\psi}(\vec{p},t)|^2$, where $\tilde{\psi}$ is the Fourier transform of ψ. This disagreement exhibits the context dependence of momentum in dBB thoery: the preexisting momentum probability density given by the dBB theory is assumed to be converted into the correct quantum density by means of a measurement interaction appropriate to the context of a momentum measurement. On the other hand, position measurements simply reveal the existing position. Thus the position measurement interaction does not play the same role of altering the existing probability distribution.

3. Motivations For A Causal Quantum Mechanics More Realistic than De Broglie-Bohm Theory. The asymmetrical treatment of position and momentum constitutes the breaking of a fundamental symmetry of quantum theory which has sometimes been considered as a defect of the dBB theory (Holland, Ref. 5, p. 21). We recently constructed a causal quantum mechanics[8,9] in one dimension, in which Takabayasi's objection as well as the asymmetric treatment of position and momentum are removed. Without invoking the measurement interaction, the new causal theory reproduces both position and momentum probability distributions of usual quantum theory, and is therefore more realistic than the dBB theory.

Can we formulate a notion of a maximally realistic causal mechanics (for spinless particles with a configuration space of n dimensions) which respects quantum contextuality theorems? A mechanics which yields Hamiltonian evolution of phase space variables with a positive definite phase space density will be called a 'Causal Hamiltonian Mechanics' or a 'Causal Mechanics' in brief. A 'Causal Mechanics' which simultaneously reproduces the quantum probability densities of the maximum number of different (mutually noncommuting) CCS of obserables as marginals of the same phase space density will be called a 'Maximally Realistic Causal Quantum Mechanics'. The definition is nontrivial because the contextuality theorems do not allow probability distributions of all possible CCS to be simultaneously reproduced.

What constraints can we impose selfconsistently on the phase space probability density $\rho(\vec{x},\vec{p},t)$ of an ensemble of phase space points of a causal theory? Motivated by the success in one dimension[8,9] we may require that the quantum position and momentum probability densities are reproduced as 'marginals', i.e.,

$$\int \rho(\vec{x},\vec{p},t)d\vec{p} = |\psi(\vec{x},t)|^2, \tag{1}$$

$$\int \rho(\vec{x},\vec{p},t)d\vec{x} = |\tilde{\psi}(\vec{p},t)|^2, \tag{2}$$

where $\tilde{\psi}$ denotes the Fourier transform of the wave function ψ. The probability interpretation necessitates the positivity condition,

$$\rho(\vec{x},\vec{p},t) \geq 0, \tag{3}$$

which rules out many phase space distribution functions such as the Wigner function.[10] Moreover, positive distribution functions obtained by smoothing the Wigner function[11] do not in general reproduce the correct marginals. Neverthless, as emphasized by Cohen and Zaparovanny[12], the uncertainty principle does not preclude the existence of a phase space density obeying conditions (1) – (3). A simple example is

$$\rho_0(\vec{x},\vec{p},t) = |\psi(\vec{x},t)|^2|\tilde{\psi}(\vec{p},t)|^2.$$

For a causal theory a further condition is necessary if the phase space density is to arise from an underlying Hamiltonian dynamics, viz. the "Liouville condition",

$$\frac{\partial\rho}{\partial t}(\vec{x},\vec{p},t)+\sum_{i=1}^{n}\left(\dot{x}_i\frac{\partial\rho}{\partial x_i}+\dot{p}_i\frac{\partial\rho}{\partial p_i}\right)=0, \tag{4}$$

where a dot denotes time-derivative. As is well known, this condition is an immediate consequence of the phase space continuity eqn.

$$\frac{\partial\rho(\vec{x},\vec{p},t)}{\partial t}+\sum_{i=1}^{n}\left\{\frac{\partial}{\partial x_i}(\dot{x}_i\rho)+\frac{\partial}{\partial p_i}(\dot{p}_i\rho)\right\}=0 \tag{4a}$$

and the existence of a causal Hamiltonian $H_c(\vec{x},\vec{p},t)$ such that

$$\partial H_c/\partial p_i=\dot{x}_i,\ \partial H_c/\partial x_i=-\dot{p}_i. \tag{4b}$$

The achievement of de Broglie and Bohm[5] was to construct a causal mechanics obeying (1), (3) and (4). The mechanics we constructed[8,9] for $n=1$ is more realistic because it obeys Eq. (2) in addition. The new mechanics has the phase space density

$$\rho(x,p,t)=|\psi(x,t)|^2|\psi(p,t)|^2\delta\left(\int_{-\infty}^{p}dp'|\psi(p',t)|^2-\int_{-\infty}^{\epsilon x}dx'|\psi(\epsilon x',t)|^2\right) \tag{5}$$

where $\epsilon=\pm1$, and we have omitted the tilda denoting Fourier transform (i.e., $\psi(p,t)$ actually stands for $\tilde{\psi}(p,t)=\langle p|\psi(t)\rangle$). Eqs. (1), (2) and (3) are obviously satisfied; further, it also has a c-number causal Hamiltonian of the form[8,9]

$$H_c=\frac{1}{2m}(p-A(x,t))^2+V(x,t)$$

with two quantum potentials A and V (instead of just one in the de Broglie-Bohm theory) which depend on the wave function ψ. Hence the Liouville condition (4) is also obeyed.

We shall see that in higher dimensions, we can and should be even more ambitious.

4. <u>Maximally Realistic Causal Quantum Mechanics</u>. The purpose of the present work is to construct a new mechanics which we tentatively call "maximally realistic causal quantum mechanics" in 2n-dimensional phase space. In this theory a single phase space probability density reproduces the quantum probability densities of $n+1$ different CCS of observables in spite of the fact that no two sets are mutually commuting. The pleasant surprise is that not only the quantum probability densities of position and momentum, but also those of $n-1$ other CCS of observables can be simultaneously realized. The choice of the $n+1$ CCS whose probabilities are simultaneously realized is not unique; the appropriate choice can depend on the context. Different contexts have the same wave function but different phase space probability densities; thus the wave function is not a complete description of these probabilities.

The phase space quantum contextuality theorem of Martin and Roy[4] proves that it is impossible to realize quantum probability densities for all possible choices of the CCS of observables as marginals of one positive definite phase space density. We conjecture that the simultaneous realization of quantum probability densities of more than $n+1$ different CCS is impossible and hence that the causal theory presented here is maximally realistic.

Let the state of the individual system be characterized by $\{|\psi(t)>, \vec{x}(t)\}$ or $\{|\psi(t)>, \vec{p}(t)\}$ since any n independent phase space coordinates are now on equal footing. Due to the freedom of canonical transformations, we may assume without loss of generality that the CCS $(X_1, \cdots X_n)$ is among the $n+1$ CCS whose quantum probability densities are reproduced in the new causal theory. We assume (without any fundamental justification) a one-to-one relation between coordinates and momenta, as this played a crucial role in our construction of a causal hamiltonian in one dimension[8,9]. The phase space density must then be of the general form,

$$\rho(\vec{x}, \vec{p}, t) = |\psi(\vec{x}, t)|^2 \prod_{j=1}^{n} \delta(p_j - \hat{p}_j(\vec{x}, t)) \tag{6}$$

which returns the correct marginal $|\psi(\vec{x}, t)|^2$ on integration over the momenta. We shall now show that the functions $\hat{\vec{p}}(\vec{x}, t)$ can be chosen so as to reproduce the quantum probability densities of a 'chain' of $n+1$ different CCS, e.g.

$$(X_1, X_2, \cdots, X_n), (P_1, X_2, \cdots, X_n), (P_1, P_2, X_3, \cdots, X_n), \cdots, (P_1, P_2, \cdots, P_n), \tag{7}$$

where each CCS in the chain is obtained from the preceding one by replacing one phase space variable by its canonical conjugate. In the one dimensional case the rquirement of one-to-one relation between x and $\hat{p}(x, t)$ yields two discrete solutions $\hat{p}(x, t)$ (corresponding to $\epsilon = \pm 1$ in Eq. (5)) which are non-decreasing and non-increasing functions of x respectively. Analogously, in the n-dim case, there is a 2-fold ambiguity in determining the phase space variables of each CCS in the chain (7) in terms of the preceding CCS, and hence 2^n discrete solutions $\hat{\vec{p}}(\vec{x}, t)$. These solutions can be read off from the δ-functions in the 2^n phase space densities, each of which reproduces the desired $n+1$ quantum probability densities as marginals:

$$\rho(\vec{x}, \vec{p}, t) = \prod_{i=0}^{n} |\psi(\Omega_i, t)|^2 \ \prod_{j=1}^{n} \delta(A_j). \tag{8}$$

Here, each Ω_i denotes phase space variables corresponding to one CCS :

$$\begin{aligned}
\Omega_0 &= (x_1, x_2, \cdots, x_n), \\
\Omega_i &= (p_1, p_2, \cdots, p_i, x_{i+1}, \cdots, x_n), \quad \text{for} \ \ 1 \le i \le n-1, \\
\Omega_n &= (p_1, p_2, \cdots, p_n),
\end{aligned}$$

and the $\psi(\Omega_i, t)$ denote appropriate Fourier transforms of $\psi(\Omega_o, t)$. Each $\delta(A_j)$ serves to determine Ω_j in terms of Ω_{j-1} ,

$$\begin{aligned}
A_1 &= \int_{-\infty}^{p_1} |\psi(p_1', x_2, \cdots, x_n, t)|^2 dp_1' - \int_{-\infty}^{\epsilon_1 x_1} |\psi(\epsilon_1 x_1', x_2, \cdots, x_n, t)|^2 dx_1', \\
A_j &= \int_{-\infty}^{p_j} |\psi(p_1, \cdots, p_{j-1}, p_j', x_{j+1}, \cdots, x_n, t)|^2 dp_j' \\
&\quad - \int_{-\infty}^{\epsilon_j x_j} |\psi(p_1, \cdots, p_{j-1}, \epsilon_j x_j', x_{j+1}, \cdots, x_n, t)|^2 dx_j', \quad \text{for} \ \ 1 < j < n,
\end{aligned}$$

$$
\begin{aligned}
A_n &= \int_{-\infty}^{p_n} |\psi(p_1,\cdots,p_{n-1},p_n',t)|^2 dp_n' \\
&\quad - \int_{-\infty}^{\epsilon_n x_n} |\psi(p_1,\cdots,p_{n-1},\epsilon_n x_n',t)|^2 dx_n',
\end{aligned}
$$

with

$$\epsilon_i = \pm 1, \quad \text{for} \;\; 1 \le i \le n.$$

Since there are 2^n possible choices of the $\epsilon_1,\cdots,\epsilon_n$ we have here 2^n phase space densities. Direct integration over n variables, (using the n δ-functions), yields

$$\int \rho(\vec{x},\vec{p},t)d\bar{\Omega}_i = |\psi(\Omega_i,t)|^2$$

which are the correct marginals. Here $\bar{\Omega}_i$ denotes the n-tuple of phase space variables complementary to Ω_i, i.e., $\bar{\Omega}_i = (x_1,x_2,\cdots,x_i,p_{i+1},\cdots,p_n)$, with $\bar{\Omega}_0 = (p_1,\cdots,p_n)$ and $\bar{\Omega}_n = (x_1,x_2,\cdots,x_n)$. The condition $A_1 = 0$ determines p_1 in terms of $x_1,x_2,\cdots,x_n$ and t, i.e., $\hat{p}_1(x_1,\cdots,x_n,t)$; $A_2 = 0$ determines p_2 in terms of p_1 and $x_2,\cdots,x_n,t$ and hence $\hat{p}_2(x_1,\cdots x_n,t)$ after substituting $p_1 = \hat{p}_1$, and so on. Hence all the momenta are determined via the $\delta(A_j)$ in terms of $x_1,\cdots,x_n,t$, and the phase space density (8) can be rewritten in the form (6). (The δ-functions $\delta(A_j)$ can of course be used equally well to determine the coordinates in terms of momenta. E.g. $A_n = 0$ yields x_n in terms of $p_1,\cdots,p_n,t$; $A_{n-1} = 0$ yields x_{n-1} in terms of $p_1,\cdots,p_{n-1},x_n,t$ and hence in terms of $p_1,\cdots,p_n,t$ after substituting for x_n, and so on).

The phase space density (8) corresponds to the choice (7) of the $n+1$ CCS. The form (6) is however more general since it results for any choice of the chain of $n+1$ CCS which includes $(X_1,X_2,\cdots,X_n)$. E.g. for $n=2$, all the three chains $\{(X_1,X_2),(P_1,X_2),(P_1,P_2)\}$, $\{(X_1,P_2),(X_1,X_2),(P_1,X_2)\}$, and $\{(X_1,X_2),(X_1,P_2),(P_1,P_2)\}$ will lead to Eq. (6), of course with different functions $\hat{\vec{p}}(\vec{x},t)$.

<u>Consistency Condition on Velocities due to Schrödinger Eqn.</u> The density $\rho(\vec{x},\vec{p},t)$ of the ensemble of system points depends on the $n+1$ marginals $|\psi(\Omega_i,t)|^2$. Hence the velocities of the individual system points will also be constrained by the time dependent Schrödinger Eqn. We work out these constraints explicitly when the Schrödinger Eqn. is of the form :

$$i\hbar\partial\psi/\partial t = \left(\sum_i \frac{P_i^2}{2m_i} + U(\vec{x})\right)\psi$$

in terms of the chosen coordinates and momenta. Starting from the general form (6) of the phase space density and taking a partial derivative w.r.t. t with $\vec{x}$ and $\vec{p}$ fixed we obtain,

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho(\vec{x},\vec{p},t) &= \left(\frac{\partial}{\partial t}|\psi(\vec{x},t)|^2\right)\prod_{j=1}^{n}\delta(p_j - \hat{p}_j(\vec{x},t)) \\
&\quad - \sum_{i=1}^{n}\frac{\partial}{\partial p_i}\left[\frac{\partial \hat{p}_i(\vec{x},t)}{\partial t}\rho(\vec{x},\vec{p},t)\right]. \qquad (9)
\end{aligned}
$$

The time dependent Schrödinger Eqn. yields the probability current conservation eqn.

$$\frac{\partial}{\partial t}|\psi(\vec{x},t)|^2 + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}j_i(\vec{x},t) = 0, \qquad (10)$$

where

$$j_i(\vec{x},t) = \mathrm{Re}\left[\psi^\star(\vec{x},t)\frac{-i}{m_i}\,\frac{\partial}{\partial x_i}\;\psi(\vec{x},t)\right]. \tag{11}$$

Further, the conservation of the number of system points in phase space yields the phase space continuity eqn. (4a) for $\partial\rho/\partial t$, with $\dot{x}_i = \dot{x}_i(\vec{x},\vec{p},t),\ \dot{p}_i = \dot{p}_i(\vec{x},\vec{p},t)$. Substituting Eqs. (4a) and (10) into Eq. (9) we obtain,

$$\begin{aligned}\sum_{i=1}^{n}\left[\frac{\partial}{\partial x_i}\left(v_i|\psi(\vec{x},t)|^2 - j_i(\vec{x},t)\right)\right]\prod_{j=1}^{n}\delta(p_j-\hat{p}_j(\vec{x},t))\\ +\frac{\partial}{\partial p_i}\left\{\left(\frac{dp_i}{dt}-\frac{d\hat{p}_i(\vec{x},t)}{dt}\right)|\psi(\vec{x},t)|^2\prod_{j=1}^{n}\delta(p_j-\hat{p}_j(\vec{x},t))\right\} = 0,\end{aligned} \tag{12}$$

where $\vec{v}$ deotes the system point velocities. Thus

$$\vec{v}(\vec{x},t) = \dot{\vec{x}}(\vec{x},\vec{p},t)\Big|_{\vec{p}=\hat{\vec{p}}(\vec{x},t)}, \tag{13}$$

$$\frac{\partial}{\partial x_i}\left(v_i|\psi(\vec{x},t)|^2\right) = \left(\frac{\partial}{\partial x_i}+\sum_{k=1}^{n}\frac{\partial\hat{p}k}{\partial x_i}\,\frac{\partial}{\partial p_k}\right)\left(\dot{x}_i|\psi(\vec{x},t)|^2\right)\Big|_{\vec{p}=\hat{\vec{p}}},$$

and

$$\frac{d\hat{p}_i(\vec{x},t)}{dt} = \frac{\partial\hat{p}_i(\vec{x},t)}{\partial t}+\sum_{k=1}^{n}\frac{\partial\hat{p}_i(\vec{x},t)}{\partial x_k}\dot{x}_k(\vec{x},\vec{p},t)\Big|_{\vec{p}=\hat{\vec{p}}}$$

Integrating Eq. (12) after multiplying by $dp_1\cdots dp_n$ we obtain the constraint on velocities,

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(v_i|\psi(\vec{x},t)|^2 - j_i(\vec{x},t)\right) = 0. \tag{14}$$

Similarly, multiplying Eq. (12) by $p_k dp_1\cdots dp_n$ and integrating, we obtain

$$\left(\frac{dp_k}{dt}-\frac{d\hat{p}_k(\vec{x},t)}{dt}\right)\Big|_{\vec{p}=\hat{\vec{p}}(\vec{x},t)} = 0,$$

which is identically satisfied.

For $n=1$, Eq. (14) implies that the dBB velocity is the unique solution if we wish to avoid singularities of the velocity at nodes of the wave function. For $n>1$, Eq. (14) can be solved for the velocities to yield,

$$(v_i - v_{i,B})|\psi(\vec{x},t)|^2 = \sum_{\ell}\frac{\partial W_{i\ell}(\vec{x},t)}{\partial x_\ell}, \tag{15}$$

where

$$W_{i\ell} = -W_{\ell i}, \tag{16a}$$

and $v_{i,B}$ denotes the dBB velocity

$$v_{i,B} = j_i(\vec{x},t)/|\psi(\vec{x},t)|^2 \tag{16b}$$

The velocities given by (15) differ from the dBB velocities due to the term involving the antisymmetric tensor W. Eq. (15) was derived directly from Eq. (10) by Deotto and Ghirardi and by Holland in their search for alternatives to dBB trajectories[13]. Our derivation shows that the Schrödinger Eqn. places no other constraints, for example on $\hat{p}_j(\vec{x},t)$. We shall now see that in general W has to be non-zero in order that a causal Hamiltonian exist.

Partial Differential Equations For Velocities From Existence of the Causal Hamiltonian. In addition to the constraint (14) due to the Schrödinger Eqn., the velocities must also obey partial differential eqns. which follow from the requirement of existence of a c-number causal Hamiltonian $H_c(\vec{x},\vec{p},t)$. If $\dot{x}_i$ and $\dot{p}_i$ are derived via Hamilton's equations,

$$v_k(\vec{x},t) = \dot{x}_k|_{\vec{p}=\hat{\vec{p}}} = \left(\frac{\partial H_c(\vec{x},\vec{p},t)}{\partial p_k}\right)\Big|_{\vec{p}=\hat{\vec{p}}}, \tag{17}$$

$$\frac{d\hat{p}_i(\vec{x},t)}{dt} = \frac{dp_i}{dt}\Big|_{\vec{p}=\hat{\vec{p}}} = -\frac{\partial H_c(\vec{x},\vec{p},t)}{\partial x_i}\Big|_{\vec{p}=\hat{\vec{p}}}. \tag{18}$$

Defining

$$\hat{H}_c(\vec{x},t) = H_c(\vec{x},\vec{p},t)|_{\vec{p}=\hat{\vec{p}}(\vec{x},t)}, \tag{19}$$

and substituting Eqs. (17) and (18), we have,

$$\begin{aligned}\frac{\partial \hat{H}_c(\vec{x},t)}{\partial x_i} &= \left(\frac{\partial H_c}{\partial x_i} + \frac{\partial \hat{p}_k}{\partial x_i}\frac{\partial H_c}{\partial p_k}\right)\Big|_{\vec{p}=\hat{\vec{p}}} \\ &= -\frac{d\hat{p}_i(\vec{x},t)}{dt} + \frac{\partial \hat{p}_k}{\partial x_i} v_k(\vec{x},t).\end{aligned} \tag{20}$$

In order that a function $\hat{H}_c(\vec{x},t)$ obeying the partial differential eqns. (20) exist, the integrability conditons

$$\frac{\partial^2 \hat{H}_c}{\partial x_i \partial x_j} = \frac{\partial^2 \hat{H}_c}{\partial x_j \partial x_i} \tag{21}$$

must hold. Substituting (20) into (21) we obtain the $n(n-1)/2$ conditions on the velocities, $(1 \leq i < j \leq n)$,

$$\frac{\partial}{\partial x_i}(f_{kj}v_k) - \frac{\partial}{\partial x_j}(f_{ki}v_k) + \frac{\partial}{\partial t} f_{ij}(\vec{x},t) = 0, \tag{22}$$

where

$$f_{ij}(\vec{x},t) = \frac{\partial}{\partial x_i}\hat{p}_j(\vec{x},t) - \frac{\partial}{\partial x_j}\hat{p}_i(\vec{x},t). \tag{23}$$

Note that these partial differential eqns. for the velocities for existence of a causal Hamiltonian are derived without any assumption about the functional form of the Hamiltonian. When we substitute Eqs. (15) for the velocities (given by Schrödinger Eqn.) into Eqs. (22), we obtain $n(n-1)/2$ partial differential eqns. for the $n(n-1)/2$ functions $W_{i\ell}(\vec{x},t)$:

$$\frac{\partial}{\partial x_i}\left(\frac{f_{kj}}{|\psi(\vec{x},t)|^2}\frac{\partial W_{k\ell}}{\partial x_\ell}\right) - \frac{\partial}{\partial x_j}\left(\frac{f_{ki}}{|\psi(\vec{x},t)|^2}\frac{\partial W_{k\ell}}{\partial x_\ell}\right) + F_{ij} = 0, \tag{24}$$

where

$$F_{ij} \equiv \frac{\partial f_{ij}}{\partial t} + \frac{\partial}{\partial x_i}(f_{kj} v_{k,B}) - \frac{\partial}{\partial x_j}(f_{ki} v_{k,B}). \tag{25}$$

Except for very special wave functions (e.g. factorizable wave functions), the $\hat{p}_i$ determined to fit quantum probability distributions of $n+1$ CCS yield $f_{ij} \neq 0$ and $F_{ij} \neq 0$, and Eqs. (24) do not have the trivial solution $W_{k\ell} = 0$. In contrast, in the dBB theory which reproduces only the position probability density $(\hat{p}_i)_{\rm dBB} = \partial_i S$ which yields $f_{ij} = F_{ij} = 0$ and hence Eq. (24) is obeyed with $W_{k\ell} = 0$. Thus, for general wave functions, departure from dBB velocities (i.e., $W_{k\ell} \neq 0$) is needed for existence of a causal Hamiltonian when we insist on reproducing quantum probability distributions of $n+1$ CCS ($n > 1$) of observables.

Determination of Causal Hamiltonian. With velocities so determined, $\hat{H}_c$ is found by integrating Eqs. (20) without any assumption about the form of the Hamiltonian H_c. We shall show that a causal Hamiltonian exists without making any claim of its uniqueness. We now make the ansatz,

$$H_c(\vec{x}, \vec{p}, t) = \sum_{i=1}^{n} \frac{(p_i - A_i(\vec{x}, t))^2}{2m_i} + V(\vec{x}, t). \tag{26}$$

We find from Hamilton's eqns. (17),

$$A_i(\vec{x}, t) = \hat{p}_i(\vec{x}, t) - m_i v_i \tag{26a}$$

which yield the A_i; we then calculate $V(\vec{x}, t)$ from,

$$V(\vec{x}, t) = \hat{H}_c(\vec{x}, t) - \sum_i \frac{(\hat{p}_i - A_i)^2}{2m_i}. \tag{26b}$$

This completes the determination of a causal Hamiltonian (26) which contains $n+1$ quantum potentials. The explicit forms of the potentials for $n = 1$ have been given in Ref. 9. For $n = 2$ they follow from the velocity formula (15), and Eqs. (25) and (28).

Explicit Formulae For Velocities For $n = 2$. In 2 dimensions Eq. (24) simplifies to a first order partial differential eqn.

$$\frac{\partial}{\partial x_2}(g_{12}) \frac{\partial W_{12}}{\partial x_1} - \frac{\partial}{\partial x_1}(g_{12}) \frac{\partial W_{12}}{\partial x_2} - F_{12} = 0, \tag{27}$$

where

$$g_{12}(\vec{x}, t) = f_{12}(\vec{x}, t) / |\psi(\vec{x}, t)|^2. \tag{27a}$$

Since no time derivatives occur, t may be considered as a fixed parameter and Eq. (27) solved by Lagrange's method by considering a curve $x_1 = x_1(s)$, $x_2 = x_2(s)$ with

$$ds = \frac{dx_1}{\partial g_{12}/\partial x_2} = -\frac{dx_2}{\partial g_{12}/\partial x_1} = \frac{dW_{12}}{F_{12}}.$$

On this curve

$$\frac{dg_{12}}{ds} = 0, \ \frac{dW_{12}}{ds} = F_{12}.$$

Inserting back the fixed parameter t, we obtain the most general solution,

$$W_{12}(x_1, x_2, t) = h(g_{12}, t) + \int_0^{x_1} \left(\frac{F_{12}}{\partial g_{12}/\partial x_2} \right) (x_1', x_2', t) dx_1', \tag{28}$$

where (i) the argument x_2' along the path of integration equals x_2 at $x_1' = x_1$ and is determined for other values of x_1' by the condition $dg_{12} = 0$ along the curve, and (ii) h is an arbitrary function of g_{12} and t. The velocities given by (15) will contain a degree of arbitrariness corresponding to the choice of the function $h(g_{12}, t)$.

5. Conclusions. The new causal quantum mechanics in n dim. configuration space has the following important properties. (i) It reproduces quantum probability distributions of $n+1$ CCS of observables with one positive definite phase space density. (ii) It has a c-no. causal Hamiltonian which contains $n+1$ quantum potentials. (iii) It has velocities (and hence trajectories) which (for general wave functions) differ from dBB velocities for $n > 1$ due to the insistence on reproducing quantum probabilities of $n+1$ CCS. The velocities contain some arbitrariness (e.g. the function h in 2 dim.) in spite of these constraints. (iv) It has position-momentum correlations in individual events (given by $\hat{\vec{p}}(x, t)$) different from dBB theory. Applications to quantum chaos and possible experimental tests need further work. We also hope to compare numerically the trajectories implied by the present work with those given by the de Broglie-Bohm theory in a future communication.

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