

Causal Quantum Mechanics Treating Position and Momentum Symmetrically

S.M. Roy^{*}
CERN-Geneva,

and

Virendra Singh
Tata Institute of Fundamental Research, Bombay

Abstract: De Broglie and Bohm formulated a causal quantum mechanics with a phase space density whose integral over momentum reproduces the position probability density of usual statistical quantum theory. We propose a causal quantum theory with a joint probability distribution such that the separate probability distributions for position and momentum agree with usual quantum theory. Unlike the Wigner distribution the suggested distribution is positive definite and obeys the Liouville condition.

* e-mail: Shasanka@Cernvm.cern.ch On leave (and address after 4 November 1994) from Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India.

1. Introduction. Present quantum theory does not make definite prediction of the value of an observable in an individual observation except in an eigenstate of the observable. Application of quantum rules to two separated systems which interacted in the past together with a local reality principle (Einstein locality) led Einstein, Podolsky and Rosen¹ to conclude that quantum theory is incomplete. Bell² showed that previous proofs of impossibility of a theory more complete than quantum mechanics³ made unreasonable assumptions; he went on however to prove⁴ that a hidden variable theory agreeing with the statistical predictions of quantum theory cannot obey Einstein locality.

Bell's research was influenced by the construction by De Broglie and Bohm⁵ (dBB) of a hidden variable theory which reproduced the position probability density of quantum mechanics but violated Einstein locality for many particle systems. For a single particle moving in one dimension with Hamiltonian

$$H = -\hbar^2/(2m)\partial^2/\partial x^2 + U(x), \quad (1)$$

and wave function $\psi(x, t)$, de Broglie-Bohm proposed the complete description of the state to be $\{\lambda(t), |\psi\rangle\}$, where $\lambda(t)$ is the instantaneous position of the particle, and its momentum is

$$\hat{p}_{dBB}(\lambda, t) = m d\lambda/dt = [\text{Re } \psi^*(-i\hbar \partial\psi/\partial x)/(|\psi|^2)]_{x=\lambda}. \quad (2)$$

In an ensemble the position density $\rho(\lambda, t)$ agrees with $|\psi(\lambda, t)|^2$ for all time. However, Takabayasi⁶ pointed out that the joint probability distribution for position and momentum given by the theory

$$\rho_{dBB}(\lambda, p, t) = |\psi(\lambda, t)|^2 \delta(p - \hat{p}_{dBB}(\lambda, t)) \quad (3)$$

does not yield the correct quantum mechanical expectation value of p^n for integral $n \neq 1$. De Broglie⁵ stated that these values in his theory “correspond to the unobservable probability distribution existing prior to any measurement” and measurement will reveal different values distributed according to standard statistical quantum mechanical formula. On the other hand position measurements have no central role since they simply reveal the existing position. The asymmetrical treatment of position and momentum in the dBB theory constitutes breaking of a fundamental symmetry of the quantum theory and has been considered by some physicists as a defect of the dBB theory (Holland, Ref. 5, p. 21).

Without using hidden variables, Griffiths⁷ and Gell-Mann and Hartle⁸ introduced joint probability distributions for noncommuting observables at different times in the consistent history approach to quantum theory of closed systems. Wigner⁹ had earlier introduced a joint distribution for x and p at the same time,

$$\rho_W(x, p, t) = \int_{-\infty}^{\infty} \frac{dy}{2\pi\hbar} \psi^* \left(x + \frac{y}{2}, t \right) \psi \left(x - \frac{y}{2}, t \right) \exp(ipy/\hbar) \quad (4)$$

which yielded the correct quantum probability distributions separately for x and p on integration over p and x respectively. The Wigner distribution cannot however be considered a probability distribution because it is not positive definite, as seen from the fact that the integral

$$\int dx dp \rho_{W,\psi}(x, p) \rho_{W,\phi}(x, p) = |(\psi, \phi)|^2 / (2\pi\hbar)$$

vanishes for two orthogonal states ψ, ϕ .

We wish now to propose a deterministic quantum theory of a closed system with the following properties. (We consider in this paper only 1 particle in 1 space dimension).

(i) The system point $(x(t), p(t))$ in phase space has a Hamiltonian flow with a c -number causal Hamiltonian $H_C(x, p, \psi(x, t), t)$ so that in an ensemble of mental copies of the system the phase space density $\rho(x, p, t)$ obeys Liouville's theorem

$$d\rho(x, p, t)/dt = 0. \quad (5)$$

Here $\psi(x, t)$ is the solution of the usual Schrödinger equation

$$i\hbar \partial\psi(x, t)/\partial t = H\psi(x, t) \quad (6)$$

with H being the standard quantum mechanical Hamiltonian for the system and H_C being determined from the following criteria.

(ii) Each pure "causal state", i.e., a set of phase space points moving according to a single causal Hamiltonian H_C has phase space density of the deterministic form

$$\rho(x, p, t) = |\psi(x, t)|^2 \delta(p - \hat{p}(x, t)), \quad (7)$$

in which $p - \hat{p}(x, t) = 0$ not only determines p as a function of x , but also determines x as a function of p at each time (step functions being allowed when necessary). Eqn. (7) guarantees

on integration over p the correct quantum probability distribution in x for any real function $\hat{p}(x, t)$. The function is determined from the requirement that on integration over x , $\rho(x, p, t)$ should also yield the correct quantum probability distribution in p . That such a determination is possible and unique apart from a discrete 2-fold ambiguity will be a crucial part of the present theory. It is obvious that our $\hat{p}(x, t)$ will have to be different from the $\hat{p}(x, t)$ of de Broglie-Bohm theory.

(iii) Since the quantum probability distributions for x and p in the statistics of many measurements are exactly reproduced, so are the standard uncertainty relations.

In Secs. II, III we describe the construction of the momentum $\hat{p}(x, t)$ and the causal Hamiltonian H_C , in Sec. IV applications to simple quantum systems, and in Sec. V conceptual features of the new mechanics.

2. Construction of Joint Probability Distribution of position and

momentum. We seek a positive definite distribution of the form (7) where \hat{p} is a monotonic function of x

$$\epsilon \partial \hat{p}(x, t) / \partial x \geq 0, \quad \epsilon = \pm 1 \quad (8)$$

The monotonicity property ensures that for a given t , the δ -function establishes one-to-one invertible correspondence between x and p whenever $\partial \hat{p} / \partial x$ is finite and non-zero. (This is the simplest qualitative assumption about $\hat{p}(x, t)$ which will be shown to result in Hamiltonian evolution; in future development we should try to replace the monotonicity assumption by the assumption of Hamiltonian evolution). The requirement of reproducing the correct quantum probability distribution of p is that

$$\int_{-\infty}^{\infty} \rho(x, p, t) dx = \frac{1}{\hbar} |\tilde{\psi}\left(\frac{p}{\hbar}, t\right)|^2, \quad (9)$$

where $\tilde{\psi}(k, t)$ is the Fourier transform of $\psi(x, t)$. We substitute the ansatz (7) into (9) and integrate over momentum to obtain

$$\int_{-\infty}^p dp' \int_{\hat{p}(x', t) \leq p} dx' |\psi(x', t)|^2 \delta(p' - \hat{p}(x', t)) = \int_{-\infty}^p \frac{dp'}{\hbar} |\tilde{\psi}\left(\frac{p'}{\hbar}, t\right)|^2. \quad (10)$$

The region $\hat{p}(x', t) \leq p$ becomes $x' \leq x$ if $\epsilon = 1$, and $x' \geq x$ if $\epsilon = -1$, where $\hat{p}(x, t) = p$. Thus, we obtain, for $\epsilon = \pm 1$,

$$\int_{-\infty}^{\epsilon x} dx' |\psi(\epsilon x', t)|^2 = \int_{-\infty}^{\hat{p}(x, t)/\hbar} dk' |\tilde{\psi}(k', t)|^2. \quad (11)$$

The left-hand side is a monotonic function of x which tends to 1 for $\epsilon x \rightarrow \infty$ for a normalized wave function; the right-hand side is a monotonic function of \hat{p} tending to 1 for $\hat{p} \rightarrow \infty$ (Parseval's theorem). Hence, for each t , Eq. (11) determines two monotonic functions \hat{p} of x , one for each sign of ϵ . (Note that the curve $\hat{p}(x, t)$ may have segments parallel to x -axis or p -axis corresponding to $\psi(x, t)$ or $\tilde{\psi}(p/\hbar, t)$ vanishing in some segment). The two curves $p = \hat{p}_{\pm}(x, t)$ so determined yield via Eq. (7) phase space densities ρ_{\pm} , with different causal Hamiltonians $(H_C)_{\pm}$ determined below.

3. Determination of the Causal Hamiltonian. We view $\rho(x, p, t)$ as describing an ensemble of system trajectories in the phase space. We saw in the last section that such a description is possible at each time. We would now like to find causal Hamiltonian such that the time evolution in phase space implied thereby is consistent with the time dependent Schrödinger equation.

In order that the total number of trajectories is conserved in time we must have the continuity equation

$$\partial\rho/\partial t + \partial(\rho\dot{x})/\partial x + \partial(\rho\dot{p})/\partial p = 0 \quad (12)$$

If the dynamics of the trajectories is of Hamiltonian nature i.e.

$$\dot{x} = \partial H_C / \partial p, \quad \dot{p} = -\partial H_C / \partial x \quad (13)$$

then we have Liouville's theorem that the phase space density is conserved,

$$\partial\rho/\partial t + \dot{x}\partial\rho/\partial x + \dot{p}\partial\rho/\partial p = 0 \quad (14)$$

i.e.

$$\partial\rho/\partial t + (\partial H_C / \partial p) \partial\rho/\partial x - (\partial H_C / \partial x) \partial\rho/\partial p = 0. \quad (15)$$

The c -number Hamiltonian H_C describing the causal time evolution of the trajectories in the phase space will be allowed to be different from the usual q -number Hamiltonian H describing the time evolution of the Schrödinger wave function ψ according to Eq. (6).

On substituting into Eq. (15) the ansatz (7) discussed in the last section, we obtain

$$\xi\delta(p - \hat{p}) + \frac{\partial}{\partial p}(\eta\delta(p - \hat{p})) = 0 \quad (16)$$

where

$$\begin{aligned} \xi &= \partial|\psi|^2/\partial t + (\partial H_C/\partial p) \partial|\psi|^2/\partial x - \partial\eta/\partial p \\ \eta &= -|\psi|^2 \{ \partial\hat{p}/\partial t + (\partial\hat{p}/\partial x) \partial H_C/\partial p + \partial H_C/\partial x \}. \end{aligned}$$

We thus need for consistency

$$\xi = 0 \quad \text{and} \quad \eta = 0 \quad \text{if} \quad p = \hat{p}. \quad (17)$$

We now specialise to the usual case when H is given by Eq. (1). We find that this situation is taken care of with the choice of $H_C(x, p, t)$

$$H_C = \frac{1}{2m}(p - A(x, t))^2 + V(x, t). \quad (18)$$

The causal Hamiltonian is of the Newtonian form apart from the introduction of a vector potential $A(x, t)$ and allowing the potential $V(x, t)$ to differ from $U(x)$. Eqs. (17) lead to the following equations to determine V and A (after using Schrödinger eqn. to substitute for $\partial|\psi|^2/\partial t$),

$$-\partial V(x, t)/\partial x = \partial\hat{p}(x, t)/\partial t + (2m)^{-1}\partial(\hat{p}(x, t) - A(x, t))^2/\partial x, \quad (19)$$

$$\partial \left[|\psi|^2(\hat{p} - A - mv) \right] / \partial x = 0, \quad (20)$$

where v is given by

$$v(x, t) = \hbar/(2im) \partial \ln(\psi/\psi^*)/\partial x \quad (21)$$

which is just the de Broglie-Bohm velocity. Eq. (20) implies that the quantity in square brackets must be a function of t alone. We choose this function of t to be zero in order to avoid a singularity of the vector potential at the nodes of the wave function. We thus obtain

$$A(x, t) = \hat{p}(x, t) - mv(x, t) \quad (22)$$

With the calculation of the causal Hamiltonian thus completed via Eqs. (18), (19) and (22) a consistent Liouville description emerges.

It should be stressed that the qualitatively new feature of the theory, $\hat{p}(x, t) \neq mv(x, t)$ is quite independent of the specific ansatz (18) for H_c . Comparison of the continuity equation for the spatial probability density $\rho(x, t)$ in a deterministic theory with that following from Schrödinger eqn. plus the requirement $\rho(x, t) = |\psi(x, t)|^2$ imply that $dx/dt = v(x, t)$, the dBB velocity. The assumption $\hat{p} = mv$ will then lead to the dBB answer for momentum probability density which conflicts with the quantum answer. Hence, to reproduce both position and momentum probability densities correctly we need $\hat{p} \neq mv$. On taking ensemble average, the equality is restored in our theory in agreement with Ehrenfest's theorem.

4. Illustrative Examples. (i) Quantum Free Particle. Let the quantum free particle be described by the Gaussian momentum space wave function

$$\tilde{\psi}(p/\hbar, t) = (\alpha\pi)^{-1/4} \exp \left[-(p - \beta)^2 / (2\alpha\hbar^2) - ip^2t / (2m\hbar) \right] \quad (23)$$

so that the coordinate space wave function is

$$\psi(x, t) = (\pi\alpha)^{-1/4} (m\alpha / (m + i\alpha\hbar t))^{1/2} \exp f, \quad (24)$$

$$f = -(\alpha/2) \left[(x - \beta t/m)^2 - i \left(\frac{\alpha\hbar t}{m} x^2 + \frac{2\beta x}{\alpha\hbar} - \frac{\beta^2 t}{m\alpha\hbar} \right) \right] / \left(1 + \frac{\alpha^2\hbar^2 t^2}{m^2} \right).$$

Our procedure yields

$$\hat{p} - \beta = \pm\hbar \sqrt{\frac{m^2\alpha^2}{m^2 + \alpha^2\hbar^2 t^2}} \left(x - \frac{\beta t}{m} \right), \quad (25)$$

$$A = (\hat{p} - \beta) \left[1 \mp \frac{\hbar\alpha t}{\sqrt{m^2 + (\hbar\alpha t)^2}} \right],$$

and

$$\partial V / \partial x = \pm (m^2 + \alpha^2\hbar^2 t^2)^{-3/2} [xt(\alpha\hbar)^2 + \beta m] (\hbar\alpha m) \quad (26)$$

The determination of the causal Hamiltonian is now complete apart from an irrelevant additive function of t . The quantum potentials A and V are seen to be proportional to \hbar in this example. An interesting feature of Eq. (25) is that for $\epsilon = +1$, for $t \gg m / (\alpha\hbar) \approx 2(\hbar E / (\Delta E)^2)$, it agrees with the naive classical expectation corresponding to zero vector potential.

(ii) Quantum Oscillator. For the minimum uncertainty coherent state of the harmonic oscillator of mass m , angular frequency ω and amplitude of oscillation a we find

$$\rho(x, p, t) = \sqrt{\frac{m\omega}{\pi\hbar}} \exp \left[-\frac{1}{2} \frac{m\omega}{\hbar} (x - a \cos(\omega t))^2 \right] \delta(p - \hat{p}(x, t)), \quad (27)$$

where

$$\hat{p}(x, t) = -m\omega a \sin(\omega t) \pm m\omega(x - a \cos(\omega t)), \quad (28)$$

and

$$A(x, t) = \pm m\omega(x - a \cos(\omega t)), \quad (29)$$

$$-\partial V(x, t)/\partial x = -m\omega^2 a \cos(\omega t) \pm m\omega^2 a \sin(\omega t) \quad (30)$$

The causal Hamiltonian yields the equation of motion

$$m d^2 x / dt^2 = -m\omega^2 a \cos \omega t \quad (31)$$

which results in exact harmonic motion even for x away from the centre of the packet. We do not of course expect this for solutions of the Schrödinger eqn. different from the coherent state here considered.

5. New conceptual features. (a) We have derived corresponding to every quantum wave function ψ , two joint probability distributions for position and momentum of the form (7) which are (i) positive definite, (ii) have Hamiltonian evolution with causal Hamiltonians $(H_C)_\pm$ and obey (iii)

$$\int (f(x) + g(p)) \rho_\pm(x, p, t) dx dp = \left(\psi, \left(f(x) + g \left(-i\hbar \frac{\partial}{\partial x} \right) \right) \psi \right), \quad (32)$$

for arbitrary functions $f(x)$ and $g(p)$. Eq. (32) is the major advantage of the present theory over the dBB theory. We postpone the discussion of measurements until we present a generalization of the theory to many particles.

(b) Since both ρ_+ and ρ_- obey Eq. (32) so will $\rho = C\rho_+ + (1-C)\rho_-$ with $0 \leq C \leq 1$. But since ρ_+ and ρ_- correspond to different causal Hamiltonians $(H_C)_\pm$, ρ will not correspond to a ‘pure causal state’. We are led to the concept of a pure causal state as being more fine grained than a pure wave function ψ . All $\rho = C\rho_+ + (1-C)\rho_-$ correspond to ψ ($\rho \leftrightarrow \psi$) for a continuum

of values of C , but only $C = 0, 1$ correspond to pure causal states. To quantum density matrix states $\Sigma C_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$ correspond phase space densities $\Sigma C_\alpha \rho_\alpha$ if $\rho_\alpha \leftrightarrow \psi_\alpha$.

(c) One can ask if Eq. (32) can be generalized to more general quantum observables. Here we face the old problem that there exist nonclassical observables e.g. $x \left(-i\hbar \frac{\partial}{\partial x}\right) x, \left(\left(-i\hbar \frac{\partial}{\partial x}\right) x x h.c.\right) / 2$ which have different expectation values but the same ‘naive’ classical analogue $x^2 p$. A trivial but nonunique way followed already for the *dBB* distribution is: given a nonclassical observable A the phase space analogue can be $f(x, p, \psi)$ such that $f(x, \hat{p}, \psi) = \psi^* A \psi / |\psi|^2$.

(d) It can be proved that for the individual trajectories, Newton’s first law [$d\hat{p}/dt = 0$ for $U(x) = 0$] holds, unlike *dBB* theory.

(e) A qualitative advantage of our theory over the *dBB* theory is the symmetric treatment of x and p obvious from our phase space density

$$\rho(x, p, t) = |\psi(x, t)|^2 |\tilde{\psi}(p/\hbar, t)|^2 \hbar^{-1} \delta \left(\int_{-\infty}^{\epsilon x} dx' |\psi(\epsilon x', t)|^2 - \int_{-\infty}^{\hat{p}(x, t)/\hbar} dk' |\tilde{\psi}(k', t)|^2 \right)$$

We are deeply indebted to André Martin for many discussions and for a decisive contribution¹⁰ to the elucidation of conditions that can be imposed on the phase space density $\rho(\vec{x}, \vec{p}, t)$ in higher dimensions. We thank C.V.K. Baba, R. Cowsik, D. Dhar, P. Eberhard, B. d’Espagnat, J. Finkelstein, P. Holland, D. Home, S.S. Jha, L. Kadanoff, A. Kumar, P. Mitra, G. Rajasekaran, N.F. Ramsey, D. Sahoo and E.J. Squires for discussions and correspondence on an earlier version of the work¹¹.

References

1. A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47, 777 (1935).
2. J.S. Bell, Rev. Mod. Phys. 38, 447 (1966).
3. E.g. J. Von Neumann, “Mathematische Grundlagen der Quanten Mechanik”, Julius Springer Verlag, Berlin (1932) (English Transl.: Princeton Univ. Press, Princeton, N.J. 1955).
4. J.S. Bell, Physics (Long Island City, N.Y.) 1, 195 (1964).
5. L. de Broglie, “Nonlinear Wave Mechanics, a causal interpretation”, (Elsevier, Amsterdam 1960); D. Bohm, Phys. Rev. 85, 166, 180 (1952); D. Bohm, B.J. Hiley and P.N. Kaloyerou, Phys. Rep. 144, 349 (1987); D. Dürr, S. Goldstein and N. Zanghi, J. Stat. Phys. 67, 843 (1992); P.R. Holland, “The Quantum Theory of Motion” (Cambridge Univ. Press, Cambridge 1993).
6. T. Takabayasi, Progr. Theoret. Phys. 8, 143 (1952).
7. R.B. Griffiths, J. Stat. Phys. 36, 219 (1984); Phys. Rev. Letts. 70, 2201 (1993).
8. M. Gell-Mann and J.B. Hartle, in Proc. 25th Int. Conf. on High Energy Physics, Singapore, 1990, Eds. K.K. Phua and Y. Yamaguchi (World Scientific, Singapore 1991).
R. Omnès, Rev. Mod. Phys. 64, 339 (1992).
9. E. Wigner, Phys. Rev. 40, 749 (1932).
10. A. Martin and S.M. Roy (in preparation).
11. S.M. Roy and V. Singh, ‘Deterministic Quantum Mechanics in One Dimension’, to appear in the Proceedings of the Conference “Bose and Twentieth Century Physics”, Calcutta, January 1994 (Kluwer Academic, Dordrecht, Netherlands).