J. Astrophys. Astr. (1998) **19,** 141–146

Nonconformal Fluctuations in Radiation Dominated Anisotropic Cosmology*

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Received 1999 February 3; accepted 1999 March 2

Abstract. Using simple path integral, Feynman propagator method and the relation between conformal time η and scale factor τ, we investigate the non-conformal quantum fluctuations (of expansion and shear) and axisymmetric singularity case in radiation dominated anisotropic cosmology. We show that near the classical singularity the quantum fluctuations tend to diverge.

Key words. Quantum Cosmology—Anisotropic universes.

1. Introduction

It has been recently claimed that (Keifer *et al.* 1998) the epoch of the origin of the universe can be traced back to unavoidable fluctuations of some scalar field ϕ. As a consequence, these fluctuations, together with scalar fluctuations in the metric, are expected to lead to anisotropies in the cosmic background radiation. One would also expect that there are gravitational waves originating from tensor fluctuations in the metric. It is also generally assumed (Ellis and Wainwright 1997) that the quantum gravity epoch, occurred before the Planck epoch. Prior to this epoch the effects of classical general relativity may not be taken to be valid (Narlikar 1978, 1979, 1981, 1984). In an earlier paper (Naing & Narlikar 1998, hereafter Paper I) we had studied the quantum conformal fluctuations of radiation dominated Bianchi Type I models. This paper extends that work further to non-conformal fluctuations.

The Bianchi Type I models are the simplest anisotropically expanding cosmological models. As in the isotropic Friedmann models here too the universe is expected to be radiation dominated in the early epochs (close to $t = 0$). In this paper we shall therefore be concerned with the Bianchi Type I radiation dominated cosmology for our study of non-conformal quantum fluctuations. In section 2 we will give a brief account of the asymptotic behavior of our result obtained in Paper I. In section 3, applying the Feynman path integral method to Gaussian wavepackets describing non-conformal fluctuations (i.e., including expansion and shear) of these models we will show that classical singularity is made very unlikely. In section 4 an axisymmetric singularity case will be considered. The concluding remarks will be given in section 5.

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* Since the December 1998 issue of this journal was delayed, we were able to include this paper in this issue.

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2. Classical asymptotic solution for radiation dominated Bianchi Type I cosmology

One of the advantages of using the path integral method is that the Lagrangian density is *a scalar* quantity (Kashiwa *et al* 1997) that can be handled more simply than the Hamiltonian behaving as *a vector*, and that, symmetries in a system are simply incorporated in terms of *the classical Lagrangian.* Dealing with dynamical and curved spacetime is also conceptually easier with this approach. The use of the path integral method in quantum gravity is now well established and very common in literature. In our previous paper the notation we used for the probability amplitude or propagator K was

$$K[\mathcal{G}_2, \Sigma_2 : \mathcal{G}_1, \Sigma_1] = \int \exp\left[\frac{iS(\mathcal{G})}{\hbar}\right] \mathcal{D}\mathcal{G}, \tag{1}$$

where Σ_1 and Σ_2 are two spacelike hypersurfaces, $\mathcal{G}_1$, $\mathcal{G}_2$ three geometries, $S(\mathcal{G})$ the classical action computed for geometry $\mathcal{G}$, Planck's constant, and $\mathcal{D}\mathcal{G}$ the measure of the integration on the right hand side. This formal expression for K becomes considerably simplified in the case of homogeneous cosmologies. In our case K can even be explicitly evaluated. The classical solution for the anisotropic Bianchi Type I cosmology is given by the line element,

$$ds = dt^2 - X^2(t)\,(dx)^2 - Y^2(t)\,(dy)^2 - Z^2(t)\,(dz)^2 \tag{2}$$

where X, Y and Z are functions of time only.
The 'non-classical' metric conformal tensor (DeWitt 1976) can be written as,

$$g_{ik} = (1+\phi)^2 \bar{g}_{ik}, \tag{3}$$

where g_{ik} is non-classical metric tensor, and g_{ik} classical metric tensor. The scalar ϕ, a function of the spacetime coordinates denotes the conformal fluctuation of the metric. The Hilbert Einstein action principle is given as,

$$S = \frac{c^3}{16\pi G} \int R\sqrt{-\bar{g}}\,d^4x + S_m, \tag{4}$$

where R is the scalar curvature of the spacetime, c the speed of light, g the determinant of the metric g_{ik} while S_m denotes the action of the matter term. Making use of equations (4), (3), (2) in (1) gives,

$$K[\phi_2, \Sigma_2; \phi_1, \Sigma_1] = \sum \exp\left(\frac{3ic^3}{8\pi G\hbar} \int -\dot{\phi}^2 \sqrt{-\bar{g}}\,d^4x\right). \tag{5}$$

In getting the above equation, we assume that, in our case of radiation dominated cosmology, both the curvature term and the matter action term vanish or are negligible. It has been shown elsewhere (Saha *et al* 1997, also Paper I) that the second order differential equation for the expansion scale factor $\tau = XYZ = \sqrt{-\bar{g}}$ can be given as,

$$\frac{\tau''}{\tau^3} - \frac{5}{3}\frac{\tau'^2}{\tau^4} + \frac{B}{\tau^2} + \frac{A}{\tau^{4/3}} = 0,$$

or,

$$\tau''\tau - \frac{5}{3}\tau'^2 + B\tau^2 + A\tau^{8/3} = 0. \tag{6}$$

where $'$ denotes the differentiation with respect to η. A, B are constants. We use the conformal time coordinate,

$$\eta = \int \frac{\mathrm{d}t}{\tau(t)}, \quad \text{i.e,} \quad \dot{\eta} = \frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{1}{\tau}. \tag{7}$$

Using simple calculus we can show that the general solution of the equation (6) is given as,

$$\tau^{-2/3} = p\sinh(a\eta) + q\cosh(a\eta) - \frac{A}{B}, \tag{8}$$

where p, q are arbitrary constants. Now look for the limit as $\eta \to -\infty$ and the asymptotic solution is given by,

$$\tau^{-2/3} \sim \frac{p}{2}e^{-a\eta}. \tag{9}$$

This is the relation between the scale factor and the conformal time coordinate close to the spacetime singularity($\tau \longrightarrow 0$) of the classical solution. From equation (7), $\dot{\eta} = 1/\tau$, and we have,

$$\eta \sim \text{const.} + \frac{2}{3a}\ln t. \tag{10}$$

The above relation will be used in our next sections for computing the diffusion of Gaussian wavepackets.

3. Nonconformal fluctuations in radiation dominated anisotropic cosmology

The Bianchi Type I cosmology is characterized by two properties, namely, shear and expansion. In Paper I, quantum fluctuations in the expansion (scale factor) τ (t) or (η) were dealt with using the conformal techniques. To deal with the shear fluctuations we have to use a different technique. We will follow the transformations used by Misner (1972). One defines

$$X = e^{\xi+\zeta+\sqrt{3}\lambda}; \quad Y = e^{\xi+\zeta-\sqrt{3}\lambda};$$
$$Z = e^{\xi-2\zeta}; \quad \tau = XYZ = \sqrt{-g} = e^{3\xi}, \tag{11}$$

where ξ, ζ and λ are functions of time. In our Paper I, we took into consideration the fluctuation of ξ, that is, only the expansion part. We will now consider the fluctuations in ζ and λ while ξ is kept at its classical value. Ignoring the surface term, the action functional integral over a 3-volume $\mathcal{V}$ now takes the form,

$$S \sim -\frac{3\mathcal{V}}{8\pi}\int_{t_1}^{t_2} e^{3\bar{\xi}}(\dot{\bar{\xi}}^2 - \dot{\zeta}^2 - \dot{\lambda}^2)\mathrm{d}t$$

$$\sim -\frac{3\mathcal{V}}{8\pi}\int_{t_1}^{t_2} \tau(\eta)(-\dot{\zeta}^2 - \dot{\lambda}^2)\mathrm{d}t, \tag{12}$$

where the ξ term does not produce any variable quantity and has been ignored. But, ζ and λ are functions of time (conformal time), hence

$$-(\dot{\zeta}^2 + \dot{\lambda}^2)\mathrm{d}t = -(\zeta'^2 + \lambda'^2)\dot{\eta}\,\mathrm{d}\eta$$

and $\dot{\eta} = 1/\tau$. Here $'$ denotes the differentiation with respect to η and, we have from (12)

$$S = \frac{3\mathcal{V}}{8\pi}\int_{\eta_1}^{\eta_2} (\zeta'^2 + \lambda'^2)\,\mathrm{d}\eta. \tag{13}$$

The propagator (for details see Feynman and Hibbs 1965) for action (13) is,

$$K[\zeta_2, \lambda_2, \eta_2; \zeta_1, \lambda_1, \eta_1] = \int\!\!\int \exp\left[\frac{3i\mathcal{V}}{8\pi}\int_{\eta_1}^{\eta_2} (\zeta'^2 + \lambda'^2)\mathrm{d}\eta\right]\mathcal{D}\zeta\mathcal{D}\lambda. \tag{14}$$

The solution of the above integral is,

$$K[\zeta_2, \lambda_2, \eta_2; \zeta_1, \lambda_1, \eta_1] = -\frac{8\pi^2}{3\mathcal{V}}(\eta_2 - \eta_1)\exp 3i\mathcal{V}\left(\frac{(\zeta_2 - \zeta_1)^2 + (\lambda_2 - \lambda_1)^2}{8\pi(\eta_2 - \eta_1)}\right). \tag{15}$$

We now apply the above propagator to study the evolution of the wavefunction of the universe. We have,

$$\Psi(\phi_2, \eta_2) = \int K[\phi_2, \eta_2; \phi_1, \eta_1]\Psi(\phi_1, \eta_1)\mathrm{d}\phi_1, \tag{16}$$

which connects the wavefunction at epoch η_1 to the wavefunction at epoch η_2.

Normally this technique is used to compute the probability amplitude for evolution of a wavefunction *forward* in time. Thus we normally have $t_2 > t_1$. Here, however, we use the Feynman propagator *backward* in time to compute the probability amplitude for the wavefunction to have evolved from a state ϕ_2 to a state ϕ_1 i.e. $t_2 < t_1$. This technique was used by Narlikar (1981) for estimating the probable range of states at an earlier epoch t_2 from which the universe could have evolved to its present state ϕ_1 at time t_1 . Since the present state is very nearly classical, an appropriate expression for $\Psi(\phi_1, \eta_1)$ is a Gaussian wavepacket (expressed in the η-time coordinate),

$$\Psi(\phi_1, \eta_1) = (2\pi\Delta_1^2)^{-1/4}\exp\left(-\frac{\phi_1^2}{4\Delta_1^2}\right). \tag{17}$$

The wavepacket moves in the (ζ, λ) plane with the constant velocity $(\bar{\lambda}', \bar{\zeta}')$ and at the same time diffuses so that its dispersion at $\eta = \eta_2$ is Δ_2, where,

$$\Delta_2{}^2 = \Delta_1{}^2 + \frac{4\pi^2(\eta_2 - \eta_1)^2}{9\mathcal{V}^2\Delta_1{}^2}. \tag{18}$$

In our case of the radiation dominated cosmology with the conformal time η given by equation (10) and, since $\Delta_1 \ll 1$, we get,

$$\Delta_2 \approx \frac{2\pi}{3\mathcal{V}\Delta_1}(\eta_2 - \eta_1) = \frac{4\pi}{9a\mathcal{V}\Delta_1}\ln\left(\frac{t_2}{t_1}\right). \tag{19}$$

Thus, as in our previous case, Δ_2 diverges logarithmically as $t_2 \to 0$. In the case of the dust driven universe (Narlikar 1979) the dispersion Δ_2 is found to be,

$$\Delta_2 \approx \frac{4\pi}{27M\Sigma V\Delta_1}|\ln t_2|, \tag{20}$$

where M, Σ are constants > 0 and Δ_2 will diverge as $t_2 \longrightarrow 0$.

4. Axisymmetric singularity case

It is also interesting to examine whether a similar divergence of quantum uncertainty occurs in the axisymmetric case of radiation dominated cosmology. One of the possible methods to deal with the axisymmetric case with $X(t) = Y(t)$ is as follows. Define,

$$\alpha(t) = \tfrac{1}{2}\sqrt{X}(X+Z), \quad \beta(t) = \tfrac{1}{2}\sqrt{X}(X-Z). \tag{21}$$

where, α and β are functions of time.

The action functional for the radiation dominated universe becomes,

$$S \sim -\frac{\mathcal{V}}{6\pi}\int(\dot{\alpha}^2 - \dot{\beta}^2)\tau(t)\,\mathrm{d}t. \tag{22}$$

Since both α and β are functions of time η, we have

$$(\dot{\alpha}^2 - \dot{\beta}^2)\mathrm{d}t = (\alpha'^2 - \beta'^2)\dot{\eta}\,\mathrm{d}\eta,$$

and, the propagator K for the equation (22) is,

$$K[\alpha_2, \beta_2, \eta_2; \alpha_1, \beta_1, \eta_1] = \int\!\!\int \exp\left[-\frac{i\mathcal{V}}{6\pi}\int_{\eta_1}^{\eta_2}(\alpha'^2 - \beta'^2)\mathrm{d}\eta\right]\mathcal{D}\alpha\mathcal{D}\beta. \tag{23}$$

The solution of the equation (23) is,

$$K[\alpha_2, \beta_2, \eta_2; \alpha_1, \beta_1, \eta_1] = \frac{6\pi^2}{\mathcal{V}}(\eta_2 - \eta_1)\exp -i\mathcal{V}\left(\frac{(\alpha_2-\alpha_1)^2-(\beta_2-\beta_1)^2}{6\pi(\eta_2-\eta_1)}\right). \tag{24}$$

In our present case, the dispersion in probability grows as,

$${\Delta_2}^2 = {\Delta_1}^2 + \frac{9\pi^2}{4\mathcal{V}^2{\Delta_1}^2}(\eta_2 - \eta_1)^2. \tag{25}$$

The classical singularity here is identified with $t_2 \longrightarrow 0$. Using the equation (9), and since $\Delta_1 \ll 1$, we get

$$\Delta_2 = \frac{3\pi}{2\mathcal{V}\Delta_1}(\eta_2 - \eta_1) = \frac{\pi}{a\mathcal{V}\Delta_1}\ln\left(\frac{t_2}{t_1}\right). \quad (26)$$

Here again, Δ_2 diverges logarithmically as time $t \longrightarrow 0$, i.e. as t approaches its classical singularity. This scenario is exactly the same when one of us (J.V.N.) investigated the case of axisymmetric singularity for a dust driven universe.

5. Concluding remarks

In this paper we have investigated the non-conformal fluctuations in radiation dominated cosmology using the simple path integral method and Gaussian wavepackets. Here we considered the quantum fluctuations in the shear part (ζ and λ), while the volume expansion ξ is kept at its classical value. We also carried out our investigation of the axisymmetric singularity case. In both cases, it has been found that the quantum fluctuations around the classical singularity, are divergent and as a consequence, non-singular and finite solutions can exist near the classical spacetime singularity (see also Narlikar & Padmanabhan 1986). These investigations show that in the early universe (radiation and dust driven epochs etc.) the quantum effects are much more pronounced and classical relativity cannot be taken to be valid.

Acknowledgements

One of us (T.Z.N.) would like to express his gratitude to IUCAA for the hospitality and financial support which he received during his stay at IUCAA. J.V.N, acknowledges the Homi Bhabha Research Professorship of the Department of Atomic Energy.

References

DeWitt,B.S.1976,*Ann.Phys.*, **97**, 307.

Ellis, G.F.R., Wainwright,J.1997, *Dynamical Systems in Cosmology* (Cambridge:CUP).

Feynman, R. P., Hibbs, A. R. 1965, *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill).

Kashiwa, T.,Ohnuki, Y.,Suziki, M. 1997, *Path Integral Methods* (New York:Clarendon).

Keifer,C,Polarski, D.,Starobinski,A. 1998, *Int. J. Mod.Phys.*,**D7**, 455.

Misner, C. W. 1972, In *Magic Without Magic*, ed. J. R. Klauder (San Francisco: W. H. Freeman), p. 441.

Naing, T. Z., Narlikar, J. V. 1998, *J. Astrophys.Astr*, **19**, 133.

Narlikar, J. V. 1978, *Mon. Not. R. Astr. Soc*, **183**, 159.

Narlikar, *J* V. 1979, *Gen. Rel. Gra.*, **10**, 883.

Narlikar, J.V.1981, *Found. Phys.*,**11**,473.

Narlikar, J.V.1984, *Found.Phys.*, **14**, 443.

Narlikar, J. V., Padmanabhan, T. 1986, *Gravity, Gauge Theory and Quantum Cosmology* (Dordrecht: D. Reidel).

Sana, B., Shikin, G. N. 1997, *Class. Quan. Grav.*, **29**, 1099.