THE BIG BANG AND QUANTUM COSMOLOGY

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ABSTRACT

The notion of the origin of the universe in a big bang is based on the mathematical models constructed from Einstein's field equations of general relativity. However, these equations are classical in nature and they break down when the characteristic radius of curvature of the universe is smaller than $\sim 10^{-33}$ cm; the behaviour of the universe can then be described by quantum physics. The question of the big bang origin of the universe belongs therefore to quantum gravity. This article describes an approach to quantum gravity that throws light on this important question.

INTRODUCTION

In 1922 Friedmann[1] proposed models of the large scale structure of the universe that were obtained as solutions of Einstein's field equations:

$$R_{ik} - \frac{1}{2} g_{ik} R = -\frac{8\pi G}{c^4} T_{ik}. \qquad (1)$$

In this equation it is assumed that the space-time geometry of the universe is of the Riemannian type and the left hand side describes a tensor of geometrical significance. This is equated to the energy momentum tensor T_{ik} of whatever matter/radiation constitutes the universe. The constant multiplying T_{ik} is made up of G, the gravitational constant and c, the speed of light. Here the signature is $(+, -, -, -)$ with $i = 0$ timelike, $i = 1, 2, 3$ spacelike.

Friedmann's solution assumed the universe to be both homogeneous and isotropic. Its line-element is given by

$$ds^2 = c^2 dt^2 - S^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right\}, k = 0, \pm 1. \qquad (2)$$

Here (t, r, θ, ϕ) are the space-time coordinates of typical observers in the universe and ds^2 denotes the square of the interval between two neighbouring points with coordinate differences $(dt, dr, d\theta, d\phi)$. $S(t)$ is the time-dependent scale factor. The line element (2) contains all the symmetries implicit in homogeneity and isotropy. Correspondingly the matter and/or radiation in the universe would also exhibit these properties in T_{ik}.

For each of the three values of k the scale factor $S(t)$ can be determined from Einstein's equations. All solutions exhibit one property in common, however. At some epoch in the past which we may arbitrarily fix as $t = 0$ the scale factor vanished. Since the geometrical parameters like curvature scalar blow up at $t = 0$ and physical expressions like density also diverge, this epoch is called the big bang epoch. The history of the universe cannot be continued beyond $t = 0$, where the spacetime has singularity.

It is debatable whether one should read deep significance into an event that is theoretically ascribed to a break-down of the mathematical and physical frameworks. On one side, one may argue that the $t = 0$ epoch is indeed the epoch of universal creation, an event so fundamental that it is hardly surprising that our frameworks are inadequate to describe it. This argument may be countered by saying that elsewhere in physics the appearance of infinities signals an inadequacy of the underlying theory and that in cosmology also one should try to avoid the conclusion of a singular epoch by searching for a 'better' framework than general relativity.

In the 1950s many relativists thought that the big bang singularity was an artefact of the special assumptions of homogeneity and isotropy. However, many general theorems in the sixties[2] showed that the singularity was an inevitable feature of solutions of (1) unless the T_{ik} admitted unconventional physics such as fields with negative energy or negative stresses.

Thus, if one follows the second point of view in the above debate, the necessity to modify the equation (1) is inevitable. Before looking for totally new approaches, however, it is desirable to explore the consequences of quantum gravity. For, quantum theory has, in other fields, made significant alterations to the classical picture. Can it do so *vis-a-vis* general relativity?

One example will illustrate the point. Classical Maxwell equations lead to the conclusion that an accelerated electric charge radiates energy. Therefore an electron circling round a proton cannot maintain a stationary orbit. The distance R between the electron and the proton will steadily shrink to zero in a finite time of the order

$$\tau = \frac{e^2}{mc^3} \approx 10^{-23}\ \mathrm{s}. \tag{3}$$

This result is evidently in disagreement with the stable existence of the hydrogen atom.

The conundrum is resolved by appeal to quantum mechanics. In the simplest picture we treat R as a quantum variable with a wave function $\psi(R)$ whose square of modules denotes the probability of finding the electron in a unit volume at a distance R from the proton. $\psi(R)$ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}R^2}+\frac{2}{R}\frac{\mathrm{d}\psi}{\mathrm{d}R}\right)-\frac{e^2}{R}\psi = E\psi \tag{4}$$

where E is the energy of the quantum state. It is easy to verify that (4) has a solution

$$\psi(R) = \text{constant} \times \exp(-R/R_0) \tag{5}$$

where

$$R_0 = \frac{\hbar^2}{me^2},\ E = -\frac{me^4}{2\hbar^2}. \tag{6}$$

The probability of finding the electron within a distance R of the proton is given by

$$P(R) = \int_0^R 4\pi x^2 |\psi(x)|^2\,\mathrm{d}x. \tag{7}$$

It is clear that $P \to O$ as $R \to O$. This vanishingly small probability rules out the classical eventuality $R \to O$. In fact the time-independence of $\psi(R)$ ensures that the atom has a stationary, stable structure.

QUANTUM COSMOLOGY

Normally cosmology would hardly expect to gain anything from quantum theory, since the former deals with large microscopic structures while the latter is significant only for microscopic systems. However, the very early universe would have been microscopic enough to be affected by quantum theory. To see this we consider the general criterion that decides when a system is subject to quantum laws. The criterion is simply expressed as an inequality:

$$|J| \lesssim \hbar. \tag{8}$$

Here J is the classical action describing the system, which in the case of gravity is the Hilbert action:

$$J_H = \frac{c^3}{16\pi G}\int_V R\sqrt{-g}\,\mathrm{d}^4x, \tag{9}$$

where R is the Ricci scalar curvature and g the determinant of the metric tensor. V is the characteristic spacetime region.

In the very early universe, R itself is zero; but we can use a typical component of the curvature tensor which at time t after the big bang behaves as $1/c^2t^2$. Taking the typical length scale as ct, we estimate the 4-volume of V as c^4t^4. Hence,

$$|J_H| \approx \frac{c^3}{16\pi G} \times \frac{1}{c^2t^2} \times c^4t^4 = \frac{c^5t^2}{16\pi G}. \tag{10}$$

The inequality (8) therefore tells us that quantum gravity will be important *before* the socalled Planck epoch

$$t_P = (G\hbar/c^5)^{1/2}. \tag{11}$$

For $t < t_P$, we cannot take the validity of classical cosmology for granted. Note that the important big bang epoch belongs to this interval. "Did the universe have a big bang origin?" This question can and should properly be answered only within the framework of quantum cosmology.

To construct a framework for quantum gravity and cosmology is, however, not so easy. There are conceptual as well as operational problems. The usual concepts of field quantization cannot be straight away carried over to gravity because field theorists normally consider quantization in a background flat spacetime. Even if one assumes, as in general relativity, that the background spacetime is curved, the fields themselves in this case determine spacetime geometry! Thus the problem is conceptually more involved than say quantizing the electromagnetic field in a curved spacetime background. At technical level, the classical equations (1) are nonlinear and contain equations of motion as part of them. Thus there is no clear separation between the free field term and the interaction term as in Maxwell-Lorentz electrodynamics. Further, the weak-field linearized theory of quantum gravity is not renormalizable: that is, its perturbation expansion contains integrals that keep on diverging more strongly at each step of the expansion and hence is physically meaningless.

These difficulties are the reason why no satisfactory theory of quantum gravity has yet emerged. Some typical approaches are found in references[3-7]. It is against this formally unsatisfactory background that the following approach is to be assessed.

CONFORMAL QUANTIZATION

Let us adopt a pragmatic approach wherein our method of quantization is such as to answer specific physical questions. In particular, we wish to know if the universe had a singular beginning. The classical solution leads us to believe that singularity arose in the spacetime given by (2) because $S \to O$. Can a quantum version avoid this situation?

The analogy with the hydrogen atom is clear. The $R \to O$ collapse of the H-atom was avoided by quantum theory. We may therefore expect analogous result by quantizing S. There is, however, a better way than quantizing S, known as *conformal* quantization. Conformal quantization brings in the notion of scale in a coordinate invariant way whereas the function $S(t)$ of (2) depends on the coordinates chosen.

Conformal transformations scale all spacetime intervals uniformly at any point:

$$ds \to \tilde{d}s = \Omega\, ds. \qquad (12)$$

Here Ω is a function of space and time. Thus the scaling of intervals varies from point to point. By such a scaling we can construct new spacetimes whose geometries do not necessarily satisfy Einstein's equations if the geometry of the original spacetime did. This result is often stated by the remarks: 'general relativity is not invariant under local scale transformations', or 'general relativity is not conformally invariant'.

We can thus have the following general situation. Take a spacetime geometry described by the metric tensor $\bar{g}_{ik}$ and from it construct another with the metric

$$g_{ik} = (1+\phi)^2\, \bar{g}_{ik}. \qquad (13)$$

Suppose the $\bar{g}_{ik}$ satisfies Einstein's equations. For an arbitrary spacetime function ϕ, the g_{ik} obviously will not be a solution of those equations. Refer to $\bar{g}_{ik}$ as the classical geometry and g_{ik} as a nonclassical geometry whose fluctuation from the former is given by ϕ. In conformal quantization, ϕ is made into a quantum variable. How should we proceed with its quantization?

An example from classical mechanics will help understand this situation. Imagine a particle under no forces moving in one spatial dimension. Denote by $x(t)$ its displacement from a fixed origin, at any time t. Suppose it is given that the particle was at $x = x_1$ at $t = t_1$ and at $x = x_2$ at $t = t_2 > t_1$. If m is the mass of the particle then in Newtonian mechanics its action is given by

$$J = \int_{t_1}^{t_2} \frac{1}{2} m \dot{x}^2\, dt, \qquad (14)$$

where $\dot{x} \equiv dx/dt$. The stationary action principle

$$\delta J = 0 \qquad (15)$$

gives us the equation of motion

$$m\ddot{x} = 0 \qquad (16)$$

whose solution for the prescribed boundary conditions is

$$x(t) \equiv \bar{x}(t) = x_1 + \frac{x_2 - x_1}{t_2 - t_1}(t - t_1). \quad (17)$$

Thus $\bar{x}(t)$ is the classical solution of the problem. What is its quantum counterpart? We define a general *nonclassical* trajectory by

$$x(t) = \bar{x}(t) + \phi(t), \quad (18)$$

where $\phi = 0$ at $t = t_1$, $t = t_2$; but otherwise $\phi(t)$ is arbitrary. By virtue of (15) we find that

$$J \equiv \int_{t_1}^{t_2} \frac{1}{2} m \dot{x}^2 \, \mathrm{d}t = \bar{J} + \int_{t_1}^{t_2} \frac{1}{2} m \dot{\phi}^2 \, \mathrm{d}t. \quad (19)$$

In quantum mechanics the deterministic trajectory (17) is replaced by a probabilistic statement of how the particle moves from x_1 to x_2. In the quantum version all particle trajectories are permissible. For each trajectory $x(t)$, we associate a probability amplitute

$$P(x) = \exp(i J/\hbar). \quad (20)$$

Notice that $|P(x)|^2 = 1$ for all paths but the net probability amplitude for the particle to go from x_1 at t_1 to x_2 at t_2 is obtained by summing $P(x)$ over all trajectories:

$$K[x_2, t_2; x_1, t_1] = \int \exp(i J/\hbar) D x(t). \quad (21)$$

This path integral expression was first proposed by Feynman[8] as the starting point for non-relativistic quantum mechanics. It provides a natural way of transition from classical to quantum theory. For example for the free-particle (21) gives the solution

$$K[x_2, t_2; x_1, t_1] = \left\{ \frac{m}{2\pi i \hbar (t_2 - t_1)} \right\}^{3/2} \times \exp\left\{ \frac{im(x_2 - x_1)^2}{2\hbar(t_2 - t_1)} \right\}. \quad (22)$$

This is the Green's function for the Schrödinger equation of a free massive particle. It tells us how the wave function $\psi(x, t)$ evolves with time:

$$\psi(x_2, t_2) = \int_{-\infty}^{\infty} K[x_2, t_2; x_1, t_1] \psi(x_1, t_1) \mathrm{d}x_1. \quad (23)$$

It is instructive (in view of our later result) to apply (23) to ψ in the form of a wavepacket with dispersion Δ_1 at $t = t_1$:

$$\psi(x_1, t_1) = (2\pi\Delta_1)^{-1/4} \exp(-x_1^2/4\Delta_1^2). \quad (24)$$

(22) and (23) together give for (24) a $\psi(x_2, t_2)$ with dispersion Δ_2 increased to

$$\Delta_2^2 = \Delta_1^2 + \frac{\hbar^2 (t_2 - t_1)}{4m^2 \Delta_1^2}. \quad (25)$$

The growing dispersion reflects the growing uncertainty in the location of the particle.

These ideas can be readily extended to conformal quantization as we will see next.

QUANTUM FLUCTUATIONS OF HOMOGENEOUS ISOTROPIC MODELS

Let us first consider the quantum cosmological version of the Friedmann models. Taking (2) as the classical solution of Einstein's equations, we have a definite function $\bar{S}(t)$ of time that satisfies the relations

$$3 \frac{\dot{\bar{S}}^2 + kc^2}{\bar{S}^2} = \frac{8\pi G \varepsilon(t)}{c^2}, \quad (26)$$

$$2 \frac{\ddot{\bar{S}}}{\bar{S}} + \frac{\dot{\bar{S}}^2 + kc^2}{\bar{S}^2} = -\frac{8\pi G p(t)}{c^2}, \quad (27)$$

where $\varepsilon(t)$ and $p(t)$ are the energy density and pressure of the cosmic matter. For matter dominated models $p = 0$ while for the radiation dominated models $p = \varepsilon/3$.

What is the equivalent action for the problem? We take the Hilbert action with noninteracting massive particles $a, b, c \ldots$ with masses $m_a, m_b, m_c, \ldots$

$$J = \frac{c^3}{16\pi G} \int_V R\sqrt{-g} \, \mathrm{d}^4 x - \sum_a \int m_a \mathrm{d}s_a. \quad (28)$$

The substitution (13) with $\bar{g}_{ik}$ given by (2) can be made to compute J. Since we are dealing with homogeneous isotropic models ϕ can depend on

t only. After some manipulation we find[9].

$$J = \bar{J} - \frac{3c^3 V}{8\pi G} \int_{t_1}^{t_2} (\dot{\phi}^2 - \tfrac{1}{6}\bar{R}\phi^2)\bar{S}^3 \,dt \quad (29)$$

where $\bar{R}$ is R computed for the line element (2) with $S = \bar{S}$ and V is the coordinate 3-volume of V.

Although (29) looks more complicated than (14), we can compute the path integral

$$K[\phi_2, t_2; \phi_1, t_1] = \int \exp(iJ/\hbar) D\phi(t) \quad (30)$$

exactly. We can use the resulting propagator K to compute the dispersion of an initial wavepacket given by

$$\psi(\phi_1, t_1) = (2\pi\Delta_1)^{-1/4} \exp\left(-\frac{\phi_1^2}{4\Delta_1^2}\right). \quad (31)$$

We find that the dispersion Δ_2 at t_2 is given by (with $c = 1$, $G = 1$, $\hbar = 1$)

$$\Delta_2^2 = \left(\frac{2\pi T}{3V\bar{S}_1\bar{S}_2}\right)^2 \left\{1 + \frac{3V}{2\pi T}\Delta_1^2\bar{S}_1^2(1 + T\dot{\bar{S}}_1)^2\right\}, \quad (32)$$

where $\bar{S}_1 = \bar{S}(t_1)$ etc. and T is the integral of $1/\bar{S}(t)$ over the range (t_1, t_2).

What does it all mean? To interpret the above calculations physically let us use the propagation relation

$$\psi(\phi_2, t_2) = \int K(\phi_2, t_2; \phi_1, t_1)\psi(\phi_1, t_1)\,d\phi_1 \quad (33)$$

backwards in time. That is, we ask the following question: "Given the present state of the universe as $\psi(\phi_1, t_1)$, what was the spread of states at earlier epoch t_2 from which it could have evolved to this state?" Naturally we expect the uncertainty implicit in the answer to grow as t_2 goes further back in time. The interesting result from (32) is that $\Delta_2 \to \infty$ as $t_2 \to 0$ since $\bar{S}(t_2) \to 0$. The quantum uncertainty therefore grows indefinitely large as we try to use the propagator arbitrarily close to the classical big bang epoch.

Notice that the wavepacket at t_1 is centred on $\phi_1 = 0$, the classical solution, an assumption that is justified if $t_1 \gg t_p$. However, even though the wavepacket at t_2 continues to be centred on $\phi_2 = 0$ the implied classical average is no longer reliable in view of the divergence of Δ_2.

If the classical solution is not reliable, can we attach a probability measure to the solutions that, though nonclassical, are singular at $t_2 \to 0$? Such solutions are given by those functions $\phi_2(t_2)$ for which

$$\bar{S}(t_2)\{1 + \phi_2(t_2)\} \to 0 \quad \text{as} \quad t_2 \to 0. \quad (34)$$

A probability measure can indeed be attached to such models and it can be shown[9] that it tends to zero as $t_2 \to 0$. Thus it is extremely unlikely that the universe had a singular beginning. Models that do not have the property (34) have probability approaching unity. Such models are nonsingular.

This result has further implications in relation to the socalled *horizon problem.* The particle horizon of an observer P at any time t is the region H of space from where light signals have had time to reach P. If there is a spacetime singularity at $t = 0$ the signals from H cannot have originated prior to $t = 0$. Clearly as $t \to 0$, H itself shrinks to zero volume at P.

In the classical big bang model the horizon places a limit on communication. If such severe limits existed in the past, how did the different parts of the universe achieve large scale homogeneity? In particular, the radiation background in the microwaves today shows remarkable isotropy on large angular scales. If the background were of primordial origin then this isotropy implies a uniformity on a linear scale far exceeding the size of the particle horizon.

If the singularity is avoided, however, the particle horizon need not exist. The past light cone of P could very well extend to $t < 0$, out to large physical distances from P.

GENERALIZATIONS

The above work for homogeneous isotropic models has been generalized to other cosmological models in the following way[10,11].

Take *any* solution of classical Einstein equations (1) for T_{ik} a system of massive noninteracting particles. Suppose that the metric tensor is given by $\bar{g}_{ik}$. Then consider arbitrary conformal fluctuations of this geometry:

$$g_{ik} = (1+\phi)^2 \bar{g}_{ik} \tag{35}$$

where ϕ can now be any function of space and time.

Using $\bar{g}_{ik}$ as the background metric for raising or lowering tensor indices we find that corresponding to (19) and (29)

$$J = \bar{J} + \frac{c^3}{16\pi G} \int_V (\bar{R}\phi^2 - 6\phi_i \phi^i)\sqrt{-\bar{g}}\, d^4x. \tag{36}$$

Note that instead $\dot{\phi}^2$ we now have $\phi_i \phi^i$—a covariant expression. However, the ϕ, ϕ_i dependent integrand is still only quadratic. This means that the path integral

$$\int \exp(iJ/\hbar)\, D\phi \tag{37}$$

can be evaluated *exactly*. It is therefore possible to evaluate the bifunctional propagator K.

The conclusion of §4 with regard to singular solutions being exceptions rather than the rule can also be established in this general case[11]. Thus it is clear that within the framework of conformal fluctuations quantum cosmology eliminates the classical problem of singularity.

To what extent can we take this result as general? Obviously the conformal fluctuations are only one amongst the many types of fluctuations of the metric tensor. So to get the complete answer one must tackle the as yet unsolved problem of quantizing non-conformal fluctuations. There are, however, two conceptual advantages of conformal fluctuations.

First, the question of singularity is linked with the overall volume of the universe which is affected directly by conformal transformations. It is therefore indisputable that of all the metric fluctuations the conformal ones are the most relevant to the singularity problem. In the example of the H-atom we quantized only the radial motion, and still caught the essence of the problem *viz* the stationarity of electron state. Although the angular motions were ignored in (4) the simplification did not alter the correct conclusion. In fact (5) is a solution that corresponds to the S-state of the H-atom. Likewise, although we have simplified the problem by considering conformal fluctuations only, the conclusion of avoidance of singularity may very well be a valid one.

The second property of conformal fluctuations is that they are necessary and sufficient to preserve the causal structure of spacetime[12]. Thus even while the spacetime geometry is fluctuating, the causal relationships between any two spacetime points are unaltered. The essential physics (represented by T_{ik} in Einstein's equations) is therefore unambiguously described. Nonconformal fluctuations do not preserve causality and thus their physical meaning is not clear.

Conformal fluctuations have proved useful in other branches of physics also. For example Padmanabhan[13] has shown that when all geometries conformal to flat Minkowski spacetime are considered together, the self energy problem of quantum electrodynamics is resolved. The various probability integrals of QED have a natural cut off at the Planck energy

$$E = \hbar / t_P, \tag{38}$$

and thus the need to renormalize is gone.

CONCLUSION

If we take the results of quantum conformal cosmology seriously we find that it is extremely unlikely that the universe 'started' with a big bang. A more likely picture is of a universe without a beginning and without an end. In such a universe there may be phases when it was (or will be) highly compressed with radius of curvature comparable to Planck length

$$L_P = c\, t_P = (\hbar G/c^3)^{1/2} \approx 10^{-33}\,\text{cm}. \tag{39}$$

At such stages its dynamics was governed by quantum considerations. Such a picture helps resolve some of the outstanding problems of the big bang cosmology without losing its essential advantages. It also illustrates how much richer and more powerful quantum cosmology is compared to classical cosmology.

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