Black hole physics in globally hyperbolic space-times

P S JOSHI and J V NARLIKAR
Tata Institute of Fundamental Research, Bombay 400 005, India

Abstract. The usual definition of a black hole is modified to make it applicable in a globally hyperbolic space-time. It is shown that in a closed globally hyperbolic universe the surface area of a black hole must eventually decrease. The implications of this breakdown of the black hole area theorem are discussed in the context of thermodynamics and cosmology. A modified definition of surface gravity is also given for non-stationary universes. The limitations of these concepts are illustrated by the explicit example of the Kerr-Vaidya metric.

1. Introduction

The basic laws of black hole physics are formulated in asymptotically flat space-times. The cosmological considerations on the other hand lead one to believe that the universe may not be asymptotically flat. A realistic discussion of black hole physics must not therefore depend critically on the assumption of an asymptotically flat space-time. Rather it should take account of the global properties found in most of the widely discussed cosmological models like the Friedmann models or the more general Robertson-Walker space times.

Global hyperbolicity is one such important property shared by the above cosmological models. This property is essentially a precise formulation of classical determinism in a space-time and it removes several physically unreasonable pathological space-times from a discussion of what the large scale structure of the universe should be like (Penrose 1972). We therefore propose to reformulate the definition of a black hole so as to make it work in any globally hyperbolic space-time.

Another strong motivation for considering the globally hyperbolic space-times comes from Penrose's strong cosmic censorship hypothesis (Penrose 1974 a, b). It turns out that if this hypothesis is used to rule out the time-like (naked) singularities, the resulting space-time structure must be globally hyperbolic. In the absence of cosmic censorship, signals from space-time singularities would play havoc with the deterministic structure of physics. Global hyperbolicity therefore provides a natural background for the discussion of gravitational collapse and the formation of black holes.

Our approach here will be different from that of Tipler (1977) who also provided a wider definition of a black hole. Tipler defined a black hole for stably causal space-times as an object containing all small trapped surfaces and showed that although the

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local behaviour of such black holes remains unaltered their global behaviour is changed. In particular, the area theorem is no longer obeyed.

In the next section we define a black hole using the notion of trapping of light by the strong gravitational field of a collapsing object in a globally hyperbolic space-time. We will then discuss the conditions under which the area theorem continues to hold.

In § 3, however, we will show that in a closed globally hyperbolic universe in which there is no boundary at infinity and where all future directed curves have finite lengths, the area theorem must break down. A closed Friedmann universe (the Wheeler universe) is an example of this kind.

Next in § 4 we attempt a definition of surface gravity κ in a universe which is not stationary. Stationarity and the existence of an axisymmetric killing vector are needed in the usual definition of κ, neither of which may in fact apply to objects in a real universe. We show, however, that analogy from thermodynamics helps in defining a quantity which resembles κ in local situations but which cannot have a global existence.

The limitations of the laws of black hole physics set up with these modified definitions become apparent in the explicit example discussed in § 5. This is the space-time described by Vaidya (1977). It has a locally Kerr-type object embedded in an asymptotically Friedmann/Robertson-Walker universe. There we see to what extent we can still give a meaning to the various concepts of black hole physics.

The notation and terminology to be followed here are those of Hawking and Ellis (1973) referred to in brief as HE.

2. A black hole redefined

Let $\mathscr{S}$ be any space like hypersurface in a globally hyperbolic spacetime M. Define the set $D^+(\mathscr{S})$ as follows:

$$D^+(\mathscr{S}) = \left\{ p \in M \,\middle|\, \begin{array}{l} \text{all past directed non-spacelike curves} \\ \text{from } p \text{ have past end points in } \mathscr{S}, \end{array} \right\} \tag{1}$$

The set $D^-(\mathscr{S})$ is defined similarly. Since M is globally hyperbolic, we can find $\mathscr{S}$ such that

$$D^+(\mathscr{S}) \cup D^-(\mathscr{S}) = M. \tag{2}$$

In this case $\mathscr{S}$ is a Cauchy surface in M.

We evolve this Cauchy surface into the past as well as future. Following Geroch (1970) we write

$$M = \mathscr{S} \times R^1. \tag{3}$$

With $\mathscr{S}_t$ a Cauchy surface for each $t \in R^1$, we have a family of spacelike surfaces $\{\mathscr{S}_t\}$ spanning M. The parameter t serves as the 'cosmic time' for M. We will assume that the future development of the system corresponds to increasing values of t.

In an asymptotically flat spacetime a black hole is the set $M - J^-(\mathcal{J}^+)$ where $\mathcal{J}^+$ is future null infinity, defined by imbedding M into another conformal manifold $\overline{M}$ (HE p. 222). However, independent of the existence of infinities one would like to say that a collapsing object enters the black hole state when its gravitational field becomes so strong as to start ' trapping ' even the light trays. This was formalised by Penrose in the notion of a trapped surface in M (HE p. 262) which is a closed compact spacelike 2-surface T such that the two families of null geodesics orthogonal to it are converging at T, i.e. $_1X_{ab}\, g^{ab} < 0$ and $_2X_{ab}\, g^{ab} < 0$ where $_1X_{ab}$ and $_2X_{ab}$ are the two null second fundamental forms of T. In the spherically symmetric case the event horizon is formed by the marginally trapped surfaces $\tilde{T}$ (*i.e.* for which $_1X_{ab}\, g^{ab} = 0$ and $_2X_{ab}\, g^{ab} < 0$) meaning that one family of null geodesics orthogonal to $\tilde{T}$ has zero convergence whereas the other family converges ' inwards'. In more general cases also one would expect the boundary of the trapped surface region to be formed by marginally trapped surfaces.

Using the above concepts now, we define a black hole in a globally hyperbolic spacetime as a future set generated by all possible trapped surfaces of spacetime. (A set S is called a future set if $S = I^+(S)$. The sets of the form $I^+(x)$ for $x \in M$ are future sets). Thus our definition given below covers *all* possible local collapse situations.

Definition: Let M be a globally hyperbolic space-time and let $\{T_t\}$ be the family of all closed compact spacelike 2-surfaces T_t at the epoch t which are either trapped or marginally trapped. Then the black hole in M is the future set

$$B \equiv I^+\left\{ \cup_t\, T_t \right\}. \tag{4}$$

Further, the boundary of the black hole is identified with the boundary of the above future set.

The above definition, which resembles a similar definition given by Tipler (1977), does not distinguish between local and cosmological trapped surfaces. For example, in closed Friedman models the 2-spheres which are the " equators " of the 3-Cauchy spheres of spatial homogeneity, are trapped during the entire collapsing phase of the universe and the black hole set would consist of the entire universe in the future of such a 3-sphere of time symmetry, giving an unwanted situation. Tipler *et al* (1980) have proposed a detailed method of distinguishing local trapped surfaces from the cosmological ones. There are certain characteristics which differentiate these classes. For example, whereas in the Schwarzschild case the family of null geodesics with $_2X_{ab}\, g^{ab} < 0$ points towards the trapped surfaces, in the Friedman case this family points away from them. Here we shall take for granted that such distinctions between the local and the global case can be made. We are dealing only with the local trapped surfaces arising out of local collapse situations, and not with the global ones which essentially arise from the overall cosmological nature of space-time.

It can be seen that a black hole defined above is a black hole in the usual sense when the spacetime is asymptotically flat. For, a black hole by the above definition implies that M contains a Cauchy surface $\mathcal{S}_t$ with trapped or marginally trapped surfaces T_t. This implies the existence of an event horizon $\partial J^-(\mathcal{J}^+, M)$ containing T_t, provided the null convergence condition $R_{ab}\, K^a\, K^b \nleqslant 0$ is satisfied for all null vectors K^a (HE, p. 320). This ensures a usual black hole in M.

Conversely, let M which is asymptotically flat, contain a non-empty black hole $B = M - J^-(\mathcal{J}^+)$. Now consider $I^+(B)$ and suppose there is $p \in M$ such that $p \in I^+(B)$ but $p \notin B$. So there is some $q \in B$ such that there is a timelike curve from q to p. Then since $p \in J^-(\mathcal{J}^+)$, there is a timelike curve from q to $\mathcal{J}^+$ which is not possible. Hence $B = I^+(B)$ is a future set. However, as demonstrated by the spherically symmetric case, $B \cap \mathscr{S}_t$ need not necessarily contain a trapped surface during the time dependent evolving stage when the event horizon is continuously moving outwards, increasing in area. It may be that in the final time independent limit every point of B is in the future of a trapped or marginally trapped surface as happens in the Schwarzchild picture, in which case both the definitions would become equivalent.

It is necessary here to make a distinction between the apparent horizon and the black hole boundary defined above. The apparent horizon is the outer boundary of a connected component of a trapped region on a partial Cauchy surface. The example of the spherically symmetric collapse to a Schwarzschild black hole followed by the spherically symmetric collapse of a shell of small mass shows that the apparent horizon moves discontinuously. By contrast, the black hole boundary considered here is a future set which not only varies continuously but also has a differentiable structure. In fact it is an achronal 3-manifold (HE p. 187).

Next, B being a future set, ∂B will be generated by null geodesics which are either past endless or have past end points on one of the trapped surfaces (Penrose 1972). In our case M being globally hyperbolic the generators will have past end points on a trapped or a marginally trapped surface $T \in T_t$ for some epoch t. Now the weak energy condition $T_{ab}K^aK^b \geqslant 0$ for all null vectors K^a and Einstein's equations imply $R_{ab}K^aK^b \leqslant 0$ and then the Raychaudhuri equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}v} = R_{ab}K^aK^b - 2\sigma^2 - \frac{1}{2}\theta^2, \tag{5}$$

(where v is the affine parameter along the null geodesic generators) gives the result that the expansion θ of these generators will be non-positive all along since it is non-positive at T.

Again, a generator with a past end point on some $T \in T_t$ cannot have negative expansion at T because then one would be able to deform T outwards in $\mathscr{S}_t$ to obtain another trapped surface, giving points of B outside ∂B which is not possible. This shows that the generators of the black hole boundary begin with zero expansion.

However, if they encounter some matter or radiation in future, the expansion would become negative and they will have future end points in ∂B. Then these generators enter the black hole and the black hole boundary will move outside. But in the final time independent stationary state of the collapse one would rule out such possibilities and the generators should continue to move in future with constant zero expansion without future end points in ∂B provided they do not encounter singularities in future, in which case expansion can again go negative.

If one rules out this also by a suitable form of cosmic censorship, ensuring future completeness of the generators then the well-known area theorem will hold good for general globally hyperbolic space-time also in the sense that the black hole boundary does not decrease in area.

Of course the above argument also indicates that if we were in a closed universe where all the material and light signals had finite lengths in future, then there is a possibility of the violation of the area principle. We will consider this possibility next.

3. Violation of the area theorem

We shall now consider black holes in the cosmological context of a closed universe *i.e.* of a space-time wherein there are no points at infinity and in which all non-space-like curves have finite proper lengths in future (Penrose 1974a). Tipler (1977) considered a similar situation in a Wheeler Universe where all the non-spacelike curves fall into a universal curvature singularity in future and he showed that the area of spherically symmetric black holes must ultimately decrease in such space-times.

We would now appeal to the strong cosmic censor to bring in global hyperbolicity into our picture and use the prescription given by Geroch *et al* (1972) to classify the space-time boundary. Since there are no points at infinity in our case, all boundary points are space-time singularities which are the singular end points of otherwise endless future or past directed timelike curves. The timelike boundary points (*i.e.* naked singularities) are classified as follows (cf. Penrose 1974a): Consider a past endless timelike curve γ with a past singular end point q. Then q can be said to be lying in the future of some event p if $I^+ (p) \supset I^+ (\gamma)$. Then any $r \in \gamma$ lies in $I^+ (p)$ and there are timelike curves from p to r. An observer following one of these curves will have q to his future when he is at p and to his past when he is at r. Such a situation has been termed as a naked singularity which is not admitted by the normal cosmological models and which is ruled out by the strong cosmic censorship, which in turn is equivalent to the global hyperbolicity of the space-time.

In physically meaningful situations one would be interested in an initial non-singular state from which a black hole develops as a result of gravitational collapse. To consider such a case, let S be a compact collapsing object in M and let

$$S_t = S \cap \mathscr{S}_t, \tag{6}$$

describe the evolution of the object in time. Let S_t be an initial non-singular state. All the trapped and marginally trapped surfaces which are going to be formed in the future stages of collapse are to be contained in $I^+ (S_t)$. We shall consider here the smallest set B' which contains all the trapped surfaces to be formed and which can be obtained by intersecting all the future sets $I^+ (S_t)$ which contain all the trapped surfaces. Then it is easy to see that B' is also a future set and the black hole $B \subseteq B'$.

Though we provided argument in § 2 for the area theorem to hold in general globally hyperbolic spacetimes, there is an important difference in the situation being considered here. Whereas the boundary generators were assumed to be future-complete earlier, now they all have finite lengths in future after which they end in future singularities. We shall consider here $\partial B'$ and it will be shown that its cross-sectional area will decrease for all times after a certain finite epoch $\bar{t}$ in future.

For that we first note that by a result of Clarke (1975), except for certain very specialised situations all the finite boundary points of a globally hyperbolic space-

time must be curvature singularities, and so in our case all the generators of $\partial B'$ end up in future curvature singularities. Now one would like to believe that for a curvature singularity the objects falling into it are crushed to zero proper volume. Following Tipler (1977) we shall assume that along all future directed paths λ the volume or area elements defined by linearly independent spacelike Jacobi fields along λ vanish as λ approaches the future curvature singularity.

Now we prove the following:

Theorem: Let M be a closed globally hyperbolic space-time foliated by Cauchy surfaces $\mathscr{S}_t$, satisfying the following conditions:

(a) the volume or area elements along all future directed paths λ vanish as λ approaches the corresponding future curvature singularity.

(b) the weak energy condition, *viz* $T_{ab} V^a V^b \geqslant 0$ for all timelike vectors V^a and Einstein's equations hold in M.

Then there exists an epoch t on which the area of $\partial B'$ decreases everywhere.

Proof: The area $\mathbf{a}$ of $\partial B'$ at an epoch t is measured by summing up the two-dimensional cross-sectional areas of the infinitesimal generator bundles in $\partial B_t = \partial B' \cap \mathscr{S}_t$. Suppose t_0 is an epoch when the black hole area theorem is obeyed and for all generator bundles we have

$$\frac{d\mathbf{a}}{dt} \geqslant 0 \text{ at } t = t_0. \tag{7}$$

Let λ be a typical generator $\partial B'$ with a past end-point on $\mathscr{S}_{t_0}$ and let B'_λ be the corresponding bundle with area $\mathbf{a}_\lambda$. Since for all null generators $\mathbf{a}_\lambda \to 0$ at the corresponding future singularity, there will be a finite parameter value t_λ such that

$$\frac{d\mathbf{a}_\lambda}{dt} < 0 \quad \text{for } t = t_\lambda. \tag{8}$$

In view of (7) of course, $t_t > t_0$. We, therefore, have a real valued function $\tilde{t}_\lambda$ defined on $\partial B'_{t_0} = \partial B' \cap \mathscr{S}_{t_0}$ as follows. From the weak energy condition and Einstein's equations it follows that for any null vector K^a, $R_{ab} K^a K^b \leqslant 0$. The focussing theorem (Misner *et al* 1973 p. 932)

$$\frac{d^2\sqrt{\mathbf{a}_\lambda}}{dt^2} = R_{ab} K^a K^b - \sigma^2, \tag{9}$$

where σ is shear and $\sigma^2 > 0$ tells us that

$$\frac{d^2\sqrt{\mathbf{a}}}{dt^2} \leqslant 0. \tag{10}$$

Therefore, from (8) we get

$$\frac{d\sqrt{\mathbf{a}}}{dt} < 0 \text{ for } t > t_\lambda. \tag{11}$$

Then one can define $\tilde{t}_\lambda$ for the bundle by

$$\tilde{t}_\lambda = \inf \left\{ t_\lambda \;\middle|\; \frac{da_\lambda}{dt} < 0 \text{ for } t = t_\lambda \right\}. \tag{12}$$

Clearly $\tilde{t}_\lambda$ occurs well before the final singularity for λ,

$$t_0 < \tilde{t}_\lambda < t_s. \tag{13}$$

Again the area a_λ is defined by the Jacobi fields V_λ^a which are solutions of the Jacobi equation

$$D^2 V_\lambda^a = R^a_{bcd} T_\lambda^b V_\lambda^c T_\lambda^d, \tag{14}$$

where T_λ^b is the tangent to the geodesic λ and the fields V_λ^a measure the geodesic deviation. The V_λ^a are continuous functions of their intial values on $\partial B'_{t_0}$ and hence $\tilde{t}_\lambda$ is continuous on it. Now $\partial B'_{t_0}$ is closed being the intersection of two closed sets and again if S_t is some earlier non-singular state as mentioned before then $\partial B'_{t_0} \subset J^+(S_t) \cap J^-(\mathscr{S}_{t_0})$ which is compact (prop. 6.6.6 HE, 1973). Thus $\partial B'_t$ is compact, being a closed subset of a compact set. Then $\tilde{t}_\lambda$ will attain its maximum on $\partial B'_{t_0}$ say $\tilde{t}_m$. Then for all Cauchy surfaces $\mathscr{S}_t$ with $t > \tilde{t}_m$

$$da_\lambda/dt < 0, \tag{15}$$

for all generator bundles on $\partial B'$ which implies the decrease of its cross-sectional areas at these epochs, proving the result.

In fact one would like to identify $\partial B'$ with the black hole boundary ∂B. Since B' is the smallest future set containing all trapped surfaces one expects $\partial B'$ to contain marginally trapped surfaces. The above result then shows the violation of the area theorem in closed globally hyperbolic universes. (Note that there is no assumption in our proof that the generators of $\partial B'$ remain forever in $\partial B'$. Indeed generators of $\partial B'$ can surely fall into the black hole; however as soon as they leave $\partial B'$ they no longer contribute to the area of the boundary.)

In any case since the set B is contained in the set B', it is easy to see that the area of ∂B must decrease at some stage in the future. For, if it did not, it would remain non-zero and hence exceed the area of $\partial B'$ (which tends to zero) at some future epoch, thus contradicting our earlier conclusion $B \subseteq B'$.

The existence of an epoch $\tilde{t}_m$ proved above raises an interesting issue which goes beyond the physics of black holes. In the conventional treatment of black holes in asymptotically flat space-times considerable emphasis is laid on the analogy between black hole physics and thermodynamics. In particular the second law of black hole physics implies a behaviour of the surface area which is similar to that of entropy in thermodynamics. What happens to this similarity in the type of universes discussed above?

It is true that for $t > \tilde{t}_m$ the area of a black hole decreases with time and so the similarity with entropy appears to be lost. However, it is interesting to speculate

what happens to entropy in such a universe. Clearly the Olbers paradox will be important in a contracting universe and if the thermodynamics went in the usual way, we would be faced with inordinately high background radiation. This suggests that in a contracting universe thermodynamics may also go in the reverse direction making entropy a non-increasing quantity. Arguments of this kind relating cosmology to thermodynamics have been discussed elsewhere in detail (Gold 1967). In the present context we can appeal to such arguments to re-establish the analogy between thermo-dynamics and black hole physics.

In particular, if we speculate that thermodynamics goes the 'wrong' way in the contracting phase of the universe we might also have biological time arrows reversed, although so far this has not been explicitly demonstrated. In that event a living observer would in fact see such a universe expand and the contraction to a singularity as an emergence from a singularity. The surface area of our black hole would there-fore appear to increase. In other words, the cosmological arrow of time (in the sense of the expanding universe) will be aligned with the thermodynamic arrow of time (in the sense of increasing entropy) and the arrow of time of black hole physics (in the sense of increasing area). Under these circumstances the second law of black hole physics may be reinstated.

4. Surface gravity

The concept of surface gravity was introduced by Bardeen *et al* (1973) as an entity playing a role in black hole physics analogous to that of temperature in thermo-dynamics. Surface gravity can be defined for stationary axisymmetric black holes in asymptotically flat space-times. In this section we will examine the extent to which these restrictions can be relaxed.

In a stationary axisymmetric spacetime there are two killing vectors K^a and $\tilde{K}^a$ representing stationarity and axisymmetry. The null vector tangent to the generators of the horizon is then expressed as

$$l^a = K^a + \Omega \, \tilde{K}^a, \tag{16}$$

where Ω, the angular velocity of the black hole is constant over the horizon. A space-like hypersurface tangent to K^a intersects the event horizon in a 2-surface ∂B. To define surface gravity K we need a null vector n_a, orthogonal to ∂B and satisfying the condition $l^a n_a = 1$ (We are using the space-time signature as $(-, -, -, +)$. We then have

$$\kappa = l_{a\,b} \, n^a \, l^b. \tag{17}$$

Bardeen *et al* (1973) have then derived two important results. The first is

$$\kappa = \text{constant}, \tag{18}$$

over the horizon, while the second is the differential mass formula

$$\delta m = \frac{\kappa}{8\pi} \delta a + \Omega \, \delta J, \tag{19}$$

where m = mass of the black hole, a = surface area of the black hole and J = total angular momentum of the black hole.

Clearly, in a general globally hyperbolic spacetime there can be no *a priori* timelike killing vectors like K^a. It is therefore impossible to apply the above definition of surface gravity as it stands. Nevertheless we can attempt a modified but approximate prescription along the following lines:

Let M be a globally hyperbolic spacetime containing a sequence of Cauchy surfaces $\mathscr{S}_t$. Suppose further that M contains a black hole as mentioned earlier such that the space-time is axisymmetric. Let g_t denote the space-time metric on M. Then to calculate the surface gravity $K(\tau)$ of the black hole for any $\tau > t_0$ construct another space-time metric $\tilde{g}_t$ on M with the following property:

$$\tilde{g}_t = g_t \text{ for } t \leqslant \tau,$$

$$\tilde{g}_t = g_\tau \text{ for } t > \tau. \tag{20}$$

Then $\kappa(\tau)$ is equal to the surface gravity of the black hole as computed with the metric g_t.

The above prescription is based on intuitive analogy with thermodynamics. If a system is in thermal equillibrium we can properly assign a temperature to it. In practice, however, the system is never in perfect equilibrium; it is always subject to heat loss or heat gain from the surroundings. Nevertheless we can still define a temperature for the system provided the departure from equilibrium is not too great. Thus a cooling system passes through successive states of quasi equilibrium with lower and lower temperatures. In the case of black holes, we may argue that it is still possible to ascribe a surface gravity to the system provided the cosmological changes in the space-time are slow. For example, the cosmological time scale at the present epoch is of the order of 10^{10} years. The astrophysical time scales associated with stellar or supermassive black holes are considerably shorter than this value. Therefore, one can meaningfully apply the above definition to rotating black holes at the present epoch. We will refer to this approximation as the quasistationary approximation.

Of course we presuppose here that the metric $\tilde{g}_t$ will give a reasonable space-time, *i.e.* one in which the energy condition is satisfied. Keeping in mind our intuitive criterion mentioned above that the changes in space-time are slow, we expect that the freezing of the metric to its value at $t = \tau$ will still keep the space-time reasonable in the above sense. Since the definition of κ is a local one, *i.e.* it involves quantities evaluated at τ, we expect the above prescription to work except in pathological cases.

In the most general case the surface gravity evaluated as above will depend upon the slicing chosen, and this might lead to ambiguity in the definition of κ. However in the axisymmetric case considered here we can fix up the slicing as tangential to the rotational Killing field K^a following Bardeen *et al* (1973), thereby making our prescription unique. Of course in special cosmological spaces like the Robertson-Walker spaces, one already has a prefered slicing like the constant cosmic time hypersurfaces; thereby providing a canonical framework for the evaluation of κ by the above prescription.

Though the above prescription can be used for general globally hyperbolic spacetimes, it is in the case of closed universes that we notice the limitations of this definition in a striking way.

For example, as $t \to t_s$, where t_s is the epoch of the universal curvature singularity, the cosmological time scale shortens dramatically and it becomes meaningless to talk of surface gravity—just as it is not possible to ascribe a temperature to a rapidly changing thermal system.

Secondly, (19) makes use of asymptotic flatness and it cannot be derived within our framework. In particular, the meaning attached to m, as the mass measured from infinity becomes ambiguous, since in a non-asymptotically flat universe matter is not localized in a bounded region.

These general points are brought out clearly in an explicit example to be considered below.

5. The Kerr-Vaidya metric

Consider the metric given by Vaidya (1977)

$$ds^2 = \exp\left[2\,F\,(t)\right] \left[2\,(du + a\,\sin^2 \alpha\,d\beta)\,dt - (1 + 2\,m\,\mu\,\exp\,(-2\,F)) \right.$$

$$\left. \times\,(du + a\,\sin^2 \alpha\,d\beta)^2 - M^2 \left\{ \frac{d\,\alpha^2}{1 - \dfrac{a^2\,\sin^2 \alpha}{R^2}} + \sin^2 \alpha\,d\,\beta^2 \right\} \right], \quad (21)$$

where $u = t - r$, (r, α, β) being the rotating ellipsoidal coordinates and

$$\mu = R\,\sin\left(\frac{r}{R}\right)\cos\left(\frac{r}{R}\right) M^{-2}, \quad (22)$$

$$M^2 = (R^2 - a^2)\,\sin^2\left(\frac{r}{R}\right) + a^2\,\cos^2 \alpha. \quad (23)$$

The constant R in (21)–(22) represents the input from cosmology: it is the characteristic coordinate radius of the closed universe. Thus the r-coordinate is limited by

$$r \leqslant \frac{\pi}{2}\,R. \quad (24)$$

for $r \ll R$, the metric (21) reduces to the Kerr metric with mass m and angular momentum ma. The function $F(t)$ is also of cosmological nature; $\exp(F)$ denotes the scale factor in the expanding universe. In a closed universe obeying Einstein's equations, $\exp(F)$ vanishes at $t = 0$ (the origin of the universe) and at $t = t_s$ (the end of the universe), both instants representing universal curvature singularities.

The event horizon is given by the larger root of the equation

$$R^2\,\tan^2\left(\frac{r}{R}\right) - 2\,m\,R\,\tan\left(\frac{r}{R}\right)\exp\,(-2\,F) + a^2 = 0. \quad (25)$$

As pointed out by Vaidya, (25) does not have roots if $\exp(-2F)$ is sufficiently large, a circumstance inevitable in an open Friedmann universe. In a closed universe, however $\exp(F)$ is bounded above and the reality condition

$$\exp(2F) \leqslant \frac{m}{a}, \tag{26}$$

could be satisfied for black holes with sufficiently small rotation. We will assume this to be the case, so that the event horizon will be presumed to exist always. The surface area of the horizon is given by

$$\mathbf{a} = 4\pi \exp(2F)\left(\frac{R}{a}\right)\sinh^{-1}\frac{a}{(R^2-a^2)^{1/2}}\left\{R^2\sin^2\left(\frac{r}{R}\right) + a^2\cos^2\left(\frac{r}{R}\right)\right\}, \tag{27}$$

where r is given by (25) as mentioned before. It is clear from (27) that as the singularity is approached in the future, $\exp(F) \to 0$, the solution of (25) gives $r \to \pi R/2$. In other words, the cosmological boundary conditions increasingly dominate the local behaviour of the black hole.

To obtain the surface gravity of the black hole at any epoch we use our prescription and 'freeze' the function $F(t)$ to its value $F(\tau)$ at τ. The corresponding space-time is then stationary and axisymmetric and (17) can be applied to calculate κ. A straightforward but tedious calculation gives

$$\kappa = \frac{\left\{(R^2-a^2)e^{2F}) + (2m^2+2m\,(m^2-a^2\,e^{4F})^{1/2})\,e^{-2F}\right\}\,(m^2\,e^{-2F}-a^2\,e^{2F})^{1/2}}{R^2\,[2m^2+2m\,(m^2-a^2\,e^{4F})^{1/2}]}. \tag{28}$$

As discussed in § 4, this value of κ is to be treated with caution. It is expected to give reasonable behaviour at epochs away from t_s. As $t \to t_s$, $\exp(F) \to 0$ and $\kappa \to \infty$; but the quasistationary approximation breaks down in this limit.

Also note that the third law of black hole physics is violated, and $\kappa \to 0$ if (26) is violated.

Finally the differential relation (19) does not hold, as can be seen from (27) and (28). Keeping $J \equiv ma = $ constant, a variation of m in (27) gives a variation of $\mathbf{a}$. However, the ratio

$$\frac{\delta m}{\delta \mathbf{a}} \neq \frac{\kappa}{8\pi}, \tag{29}$$

where κ is given by (28). The reason for the discrepancy lies in the fact that the universe outside the black hole is not empty and the mass 'seen from infinity' is greater than m.

6. Conclusion

We have attempted to generalize the basic concepts of black hole physics to non-asymptotically flat universes. It is possible to give a local meaning to what is meant

by a black hole in a globally hyperbolic universe. However, if such a space-time has a universal curvature singularity in the future, the second law of black hole physics is shown to break down eventually but well before the singular epoch.

Under a somewhat limited set of assumptions it is possible to give a meaning to the surface gravity of a black hole. The quasi-stationary approximation under which the surface gravity can be defined, however breaks down if the changes in the large scale structure of the universe occur sufficiently rapidly. In particular, near the singularity rapid changes are expected and hence κ cannot be defined.

In spite of these difficulties one can still see the analogy between black hole physics and thermodynamics holding up. In particular, one can see the area decrease result as analogous to a reversal of the thermodynamic arrow of time and the role of surface gravity as that of temperature in a slowly evolving thermal system.

Our discussion throughout has been at the classical level. The injection of quantum mechanics *via* the Hawking process (Hawking 1975) makes the surface gravity as equivalent (rather than analogous) to temperature. However, the Hawking process also has been studied under the assumptions of stationarity and asymptotic flatness. It would be interesting to investigate how far the relationship between thermodynamics and black hole physics holds up when these assumptions are dropped.

Acknowledgements

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