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Algebras of creation and destruction operators

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Abstract. A general analysis of bilinear algebras of creation and destruction operators is performed. Generalizing the earlier work on the single-parameter q-deformation of the Heisenberg algebra, we study two-parameter and four-parameter algebras. Two new forms of quantum statistics called orthofermi and orthobose statistics and a q-deformation interpolating between them have been found. In the Fock representation, quadratic relations among destruction operators, wherever they are allowed, are shown to follow from the bilinear algebra of creation and destruction operators. Postitivity of the Hilbert space for the four-parameter algebra has been studied in the two-particle sector, but for the two-parameter algebra, results are presented up to the four-particle sector.

Keywords. Quantum statistics; q-deformation; Heisenberg algebra; Fock space.

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1. Introduction

Recently, q-deformation of the Heisenberg algebra of the annihilation and creation operators has been used to study the interpolation between Bose and Fermi statistics [1–3]. In this paper, we undertake a more general analysis of the algebras of the annihilation and creation operators (c and $c^\dagger$). We construct the most general bilinear algebra of c and $c^\dagger$ with two types of indices (called space and spin) which is invariant under unitary transformations in these two indices. Such an algebra contains several nontrivial parameters.

One can study various special cases of the general algebra. A two-parameter algebra [4] is analysed in detail and then the analysis is extended to a four-parameter algebra. One of the main results that emerges is the existence of two new forms of statistics called orthofermi and orthobose statistics and a q-deformation which provides an interpolation between them.

Most of our analysis is performed within the framework of Fock space. We find that positivity of the metric in the Fock space provides powerful restrictions on the possible forms of algebras. There exist a number of single-parameter algebras that lead to Fock spaces of positive definite metric, but proof of positivity for two-parameter algebras has not been possible.

Various forms of bilinear algebras are introduced in §2. Section 3 is devoted to the two-parameter algebra. The metric is shown to be positive definite in the two-particle sector inside a square in the two-parameter space, but the region of positivity shrinks as the number of particles increases. However, positivity for arbitrary number of particles is preserved along the two diagonals of the square. One of these diagonals interpolates between Bose and Fermi statistics while the other interpolates between orthobose and orthofermi statistics.

In § 4 we continue with the two parameter algebra and take up the question of the existence of quadratic relations among annihilation operators and show that these exist only at the four corners of the squares which correspond to the familiar Bose and Fermi statistics and the new orthobose and orthofermi statistics.

In § 5, the two-parameter algebra of a system with a single index is taken up for special study. We point out the close connection of this algebra (for a particular point in parameter space) with Greenberg's "infinite statistics" and determine the enlarged region of positivity up to the three-particle sector.

We then return to the two-indexed system and analyze the full four-parameter algebra in § 6 and show that positivity of the metric in the two-particle sector leads to a tetrahedron in the three-parameter projection of the four-parameter space.

A few miscellaneous points deserving brief mention are treated in § 7. Factorization of the two-indexed algebra and its relation to Cuntz algebra [5] are pointed out and number operators are given. The final section is devoted to discussion.

Some of the detailed calculations are presented in Appendices 1–3. In Appendix 1 we show that a nine-parameter algebra can be reduced to a four-parameter algebra in general, apart from a few exceptional cases. Appendices 2 and 3 are devoted to the inner product for the two-parameter algebra. Some general features of the inner product in the multiparticle sector are discussed in Appendix 2 while the determination of the boundary of the region of positivity in the three and four particle sectors is described in Appendix 3.

2. General bilinear algebras of c's and $c^\dagger$'s

We consider the general bilinear algebra

$$c_{k\alpha}c^\dagger_{m\beta} - q_1 c^\dagger_{m\beta}c_{k\alpha} - q_2\delta_{\alpha\beta}\sum_\gamma c^\dagger_{m\gamma}c_{k\gamma} - q_3\delta_{km}\sum_p c^\dagger_{p\beta}c_{p\alpha} - q_0\delta_{km}\delta_{\alpha\beta}\sum_{p,\gamma} c^\dagger_{p\gamma}c_{p\gamma} = \delta_{km}\delta_{\alpha\beta} \tag{1}$$

where c and $c^\dagger$ are destruction and creation operators and q_1, q_2, q_3 and q_0 are real parameters. In a physical problem, the Latin indices $k, m, p\cdots$ and the Greek indices $\alpha, \beta, \gamma\cdots$ may correspond to space (or orbital) and spin indices respectively and so we may call them accordingly. But our results are not in any way dependent on this interpretation and in fact more interesting interpretations will be pointed out in the last section.

The algebra described by eq. (1) is invariant under the unitary transformations on the space indices:

$$d_{k\alpha} = \sum_p U_{kp}c_{p\alpha};\quad U^\dagger U = UU^\dagger = 1 \tag{2}$$

as well as similar unitary transformations on the spin indices:

$$e_{k\alpha} = \sum_\lambda V_{\alpha\lambda}c_{k\lambda};\quad V^\dagger V = VV^\dagger = 1. \tag{3}$$

Invariance under such unitary transformations is an important requirement on a quantum system in the general context; for instance, a system can be described either by position space or momentum space wave function and these are related by the unitary transformation U. Similarly, spin can be described by equivalent sets of orthonormal states related by V. However, it must be noted that for $q_2 \neq 0$, $q_3 \neq 0$, eq. (1) is not invariant under the enlarged unitary transformation involving both the space and spin indices.

It is possible to show under certain conditions, that (1) is the most general bilinear algebra of creation and destruction operators with two indices, which is invariant under the two unitary transformations U and V defined in (2) and (3). First, by taking hermitian conjugate of (1) and comparing, all the coefficients q_1, q_2, q_3 and q_0 are seen to be real. Also, there is no loss of generality in choosing the coefficients of $c_{k\alpha}c^\dagger_{m\beta}$ and $\delta_{km}\delta_{\alpha\beta}$ to be unity (unless one or both of these coefficients vanish and such special cases will be considered later), since (1) can be divided by one of these coefficients and the other can be absorbed by a suitable redefinition of c and $c^\dagger$. Next, one can consider the possibly more general algebra obtained by adding three more terms:

$$\begin{aligned} &c_{k\alpha}c^\dagger_{m\beta} - q_1 c^\dagger_{m\beta}c_{k\alpha} - q_2\delta_{\alpha\beta}\sum_\gamma c^\dagger_{m\gamma}c_{k\gamma} - q_3\delta_{km}\sum_p c^\dagger_{p\beta}c_{p\alpha} \\ &- q_0\delta_{km}\delta_{\alpha\beta}\sum_{p,\gamma} c^\dagger_{p\gamma}c_{p\gamma} + p_2\delta_{\alpha\beta}\sum_\gamma c_{k\gamma}c^\dagger_{m\gamma} + p_3\delta_{km}\sum_p c_{p\alpha}c^\dagger_{p\beta} \\ &+ p_0\delta_{km}\delta_{\alpha\beta}\sum_{p,\gamma} c_{p\gamma}c^\dagger_{p\gamma} = \delta_{km}\delta_{\alpha\beta}. \end{aligned} \tag{4}$$

Although this 7-parameter algebra also is invariant under the unitary transformations (2) and (3), in Appendix 1 we show that eq. (4) can be rewritten in the form of eq. (1) with a redefinition of the 4 parameters q_1, q_2, q_3 and q_0, apart from a few exceptional cases. Thus, under certain conditions, the 4-parameter algebra of (1) can be taken to be the most general bilinear algebra of c and $c^\dagger$ with two indices k and α.

It is easy to see that if the system is characterized by more number of indices, and if we demand invariance under unitary transformations involving each type of index separately, then even more terms can be added to (1). However, in this paper, we shall restrict ourselves to systems with two indices (§§3, 4 and 6) and with a single index (§5).

We shall begin our analysis in the next section with the simpler two-parameter algebra obtained by putting $q_3 = q_0 = 0$ in eq. (1). In fact, this two-parameter algebra turns out to be sufficiently rich in terms of novel features and nontrivial complications so that most of our work (§§3, 4 and 5) will be devoted to it. The pattern obtained in this analysis, will guide us in our study of the more general four-parameter algebra which will be taken up in a later section (§6).

3. The two-parameter algebra

In this section we shall study the two-parameter algebra defined by

$$c_{k\alpha}c^\dagger_{m\beta} - q_1 c^\dagger_{m\beta}c_{k\alpha} - q_2\delta_{\alpha\beta}\sum_\gamma c^\dagger_{m\gamma}c_{k\gamma} = \delta_{km}\delta_{\alpha\beta}. \tag{5}$$

We assume the existence of a vacuum state $|0\rangle$ annihilated by all the annihilators:

$$c_{k\alpha}|0\rangle = 0. \tag{6}$$

The Fock space is constructed in the obvious way. We consider the set of states which are linear combinations with complex coefficients of the monomials $c^{\dagger}_{k\alpha}c^{\dagger}_{m\beta}\cdots c^{\dagger}_{j\rho}|0\rangle$ and their duals $\langle 0|c_{p\gamma}\cdots c_{m\beta}$. Their inner product $\langle 0|c_{p\gamma}\cdots c^{\dagger}_{j\rho}|0\rangle$ or in fact the vacuum matrix element of any polynomial in the c's and $c^{\dagger}$'s arbitrarily ordered, can be calculated using eqs (5) and (6). No commutation rule on cc or $c^{\dagger}c^{\dagger}$ is required for this.

The following properties of the matrix elements can be easily established:

(a) The matrix element $\langle 0|cc\cdots cc^{\dagger}\cdots c^{\dagger}c^{\dagger}|0\rangle$ vanishes if $n \neq m$, where n is the number of $c^{\dagger}$'s and m is the number of c's. Hence, calling $c^{\dagger}c^{\dagger}\cdots c^{\dagger}|0\rangle$ with n number of $c^{\dagger}$'s as the n-particle state, we see that states with different number of particles are orthogonal.
(b) The matrix element $\langle 0|cc\cdots cc^{\dagger}\cdots c^{\dagger}c^{\dagger}|0\rangle$ vanishes if every space index in the c set is not matched by a space index in the $c^{\dagger}$ set. Similarly for the spin index. Thus, for the nonvanishing matrix elements, the indices for the c's are some permutation of those for the $c^{\dagger}$'s.
(c) For general values of q_1 and q_2, state vectors obtained by permuting the indices are independent although non-orthogonal. For example, in the two-particle sector, the four state vectors $c^{\dagger}_{k\alpha}c^{\dagger}_{m\beta}|0\rangle$, $c^{\dagger}_{m\beta}c^{\dagger}_{k\alpha}|0\rangle$, $c^{\dagger}_{m\alpha}c^{\dagger}_{k\beta}|0\rangle$ and $c^{\dagger}_{k\beta}c^{\dagger}_{m\alpha}|0\rangle$ are independent.

We now wish to calculate the inner product in the n-particle sector:

$$\langle 0|\cdots c_{t\gamma}c_{m\beta}c_{k\alpha}c^{\dagger}_{p\lambda}c^{\dagger}_{s\mu}c^{\dagger}_{q\nu}\cdots|0\rangle.$$

As already noted, this inner product is zero unless the space indices $(p, s, q\cdots)$ is some permutation of $(k, m, t\cdots)$ and similarly for the spin indices. The non-vanishing inner products are in general some polynomials of q_1 and q_2 and these polynomials are determined by the relative ordering of the indices in the $c^{\dagger}$ set with respect to those in the c set but are otherwise independent of the indices themselves.

We first consider the simpler algebra obtained by putting $q_2 = 0$ in (5). For this case it is easy to show

$$\langle 0|\cdots c_{t\gamma}c_{m\beta}c_{k\alpha}c^{\dagger}_{p\lambda}c^{\dagger}_{s\mu}c^{\dagger}_{q\nu}\cdots|0\rangle = q_1^J, \tag{7}$$

where J is the number of inversions in the permutation:

$$(p\lambda, s\mu, q\nu\cdots) \to (k\alpha, m\beta, t\gamma\cdots) \tag{8}$$

We assume the indices $(p, s, q\cdots)$ to be distinct and similarly for $(\lambda, \mu, \nu\cdots)$. Number of inversions is defined as the minimum number of transpositions of successive indices required in this permutation (8). Note that in this particular permutation, the space and spin indices go together. For other permutations, namely for those involving independent exchanges of space and spin indices, the matrix element in (7) vanishes.

We next introduce an alternative mode of expressing the above result which turns out to be more convenient for further generalization. Still considering the $q_2 = 0$ case and also dropping the spin index, we write for the n-particle matrix element,

$$\begin{aligned}&\langle 0|\cdots c_t c_m c_k c^{\dagger}_p c^{\dagger}_s c^{\dagger}_q\cdots|0\rangle\\ &= I(I + q_1P_{21})(I + q_1P_{32} + q_1^2P_{32}P_{21})\end{aligned}$$

$$(I+q_1P_{43}+q_1^2P_{43}P_{32}+q_1^3P_{43}P_{32}P_{21})$$
$$\cdots\left(1+q_1P_{n\,n-1}+q_1^2P_{n\,n-1}P_{n-1\,n-2}+\cdots q_1^{n-1}\prod_{i=1}^{n-1}P_{n+1-i\,n-i}\right), \tag{9}$$

where I is the unit operator and P_{ij} stand for exchange operators. The operator P_{21} for instance exchanges the indices in the first and second *position* of the initial state (p and s in the above example) and similarly for the other operators.

In eq. (9), the factors on the right side must be multiplied and expressed as a sum, using $IP_{21}=P_{21}$ etc. Thus, taking the three-particle state for instance,

$$\begin{aligned}\langle 0|c_tc_mc_kc_p^\dagger c_s^\dagger c_q^\dagger|0\rangle &= I(I+q_1P_{21})(I+q_1P_{32}+q_1^2P_{32}P_{21})\\ &= I+q_1P_{21}+q_1P_{32}+q_1^2P_{21}P_{32}+q_1^2P_{32}P_{21}+q_1^3P_{21}P_{32}P_{21}.\end{aligned} \tag{10}$$

Then, each of the terms in (10), must be replaced as indicated below:

$$\begin{aligned}I &= \delta_{kp}\delta_{ms}\delta_{tq}\cdots\\ P_{21} &= \delta_{ks}\delta_{mp}\delta_{tq}\cdots\\ P_{32} &= \delta_{kp}\delta_{mq}\delta_{ts}\cdots\\ P_{32}P_{21} &= \delta_{ks}\delta_{mq}\delta_{tp}\cdots\\ &\text{etc.}\end{aligned} \tag{11}$$

Equation (9) is valid whether the indices $(p,s,q\cdots)$ are distinct or not.

The generalization of (9) to the $q_2\neq 0$ case is obtained by the replacement

$$q_1P\to q_1P^t+q_2P^0. \tag{12}$$

Thus,

$$\begin{aligned}&\langle 0|\cdots c_{t\gamma}c_{m\beta}c_{k\alpha}c_{p\lambda}^\dagger c_{s\mu}^\dagger c_{q\nu}^\dagger\cdots|0\rangle\\ &=\{I+(q_1P_{21}^t+q_2P_{21}^0)\}\{I+(q_1P_{32}^t+q_2P_{32}^0)\\ &\quad+(q_1P_{32}^t+q_2P_{32}^0)(q_1P_{21}^t+q_2P_{21}^0)\}+\cdots\\ &\quad\cdots\Big\{I+(q_1P_{n\,n-1}^t+q_2P_{n\,n-1}^0)+(q_1P_{n\,n-1}^t+q_2P_{n\,n-1}^0)\\ &\quad\times(q_1P_{n-1\,n-2}^t+q_2P_{n-1\,n-2}^0)+\cdots+\prod_{i=1}^{n-1}(q_1P_{n+1-i\,n-i}^t\\ &\quad+q_2P_{n+1-i\,n-i}^0)\Big\}\end{aligned} \tag{13}$$

where the meaning of the symbols on the right hand side are similar to those as explained for the $q_2=0$ case except that P^t denotes total exchange i.e. exchange of both space and spin indices while P^0 denotes exchange of space (also called orbital) index only. For instance,

$$\begin{aligned}I &= \delta_{kp}\delta_{ms}\delta_{tq}\cdots\delta_{\alpha\lambda}\delta_{\beta\mu}\delta_{\gamma\nu}\cdots\\ P_{21}^t &= \delta_{ks}\delta_{mp}\delta_{tq}\cdots\delta_{\alpha\mu}\delta_{\beta\lambda}\delta_{\gamma\nu}\cdots\ .\\ P_{21}^0 &= \delta_{ks}\delta_{mp}\delta_{tq}\cdots\delta_{\alpha\lambda}\delta_{\beta\mu}\delta_{\gamma\nu}\cdots\end{aligned} \tag{14}$$

We note that, for each inversion of both space and spin indices, there is a factor q_1, while each inversion of space indices alone leads to a factor q_2. If the matrix element involves the inversion of spin indices alone, then it vanishes. This is the complete set of rules for the inner product based on the algebra of (5). More details following from the expansion in (13) are given in Appendix 2.

We now enquire into the restrictions imposed by the positivity of the inner product. We first do this for the two-particle sector. There are four states:

$$\left.\begin{aligned} |1\rangle = c^{\dagger}_{k\alpha}c^{\dagger}_{m\beta}|0\rangle, \quad |2\rangle = c^{\dagger}_{m\beta}c^{\dagger}_{k\alpha}|0\rangle \\ |3\rangle = c^{\dagger}_{m\alpha}c^{\dagger}_{k\beta}|0\rangle, \quad |4\rangle = c^{\dagger}_{k\beta}c^{\dagger}_{m\alpha}|0\rangle \end{aligned}\right\} \tag{15}$$

(We assume $k \neq m$ and $\alpha \neq \beta$; if the indices become equal, we get only weaker conditions). The inner product between these four states and their duals written in the form of 4×4 matrix M_2 is

$$M_2 = \begin{array}{c|cccc} & |1\rangle & |2\rangle & |3\rangle & |4\rangle \\ \hline \langle 1| & 1 & q_1 & q_2 & 0 \\ \langle 2| & q_1 & 1 & 0 & q_2 \\ \langle 3| & q_2 & 0 & 1 & q_1 \\ \langle 4| & 0 & q_2 & q_1 & 1. \end{array} \tag{16}$$

The eigenvalues of this matrix are $1+q_1+q_2$, $1-q_1+q_2$, $1+q_1-q_2$, and $1-q_1-q_2$. All these four eigenvalues are positive inside the square BCFG depicted in figure 1. On each of the four sides of this square given by the straight lines: $1 \pm q_1 \pm q_2 = 0$, one of the eigenvalues vanishes. Outside the square, one or more eigenvalues become negative. Hence the square BCFG demarcates the boundaries of the parameter space for which two-particle vector space with positive definite metric exists.

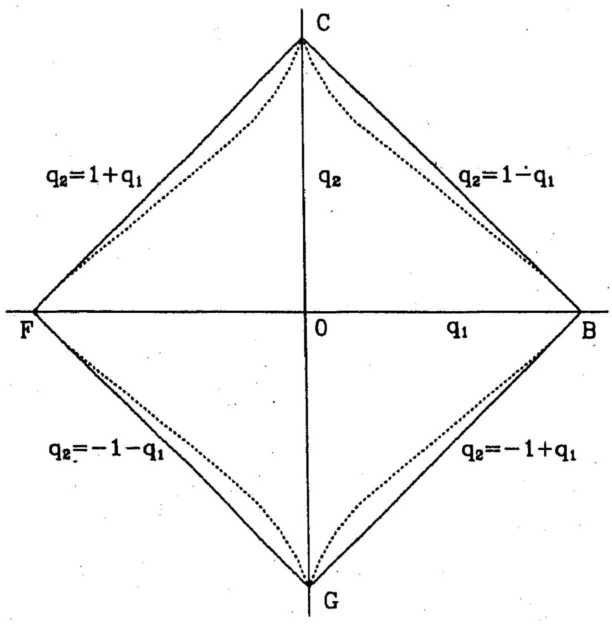

Figure 1. The square FGBC which bounds the allowed region in the (q_1, q_2) parameter space for the two-particle sector. The four corners B F, C and G correspond to Bose-Einstein, Fermi-Dirac, orthobose and orthofermi statistics respectively. The dotted curve indicates the boundary of the allowed region for the three-particle sector with the spin index taking two values.

Since on each of the four sides BC, CF, FG and GB one of the eigenvalues becomes zero, the corresponding eigenvector can be regarded as a null vector since it has zero norm. These null vectors along FG, BC, BG and FC are respectively the following:

$$(c^\dagger_{k\alpha}c^\dagger_{m\beta} + c^\dagger_{m\beta}c^\dagger_{k\alpha} + c^\dagger_{m\alpha}c^\dagger_{k\beta} + c^\dagger_{k\beta}c^\dagger_{m\alpha})|0\rangle = 0, \tag{17}$$

$$(c^\dagger_{k\alpha}c^\dagger_{m\beta} - c^\dagger_{m\beta}c^\dagger_{k\alpha} - c^\dagger_{m\alpha}c^\dagger_{k\beta} + c^\dagger_{k\beta}c^\dagger_{m\alpha})|0\rangle = 0, \tag{18}$$

$$(c^\dagger_{k\alpha}c^\dagger_{m\beta} - c^\dagger_{m\beta}c^\dagger_{k\alpha} + c^\dagger_{m\alpha}c^\dagger_{k\beta} - c^\dagger_{k\beta}c^\dagger_{m\alpha})|0\rangle = 0, \tag{19}$$

$$(c^\dagger_{k\alpha}c^\dagger_{m\beta} + c^\dagger_{m\beta}c^\dagger_{k\alpha} - c^\dagger_{m\alpha}c^\dagger_{k\beta} - c^\dagger_{k\beta}c^\dagger_{m\alpha})|0\rangle = 0. \tag{20}$$

Thus, although there is no commutation rule on cc or $c^\dagger c^\dagger$ for any point inside the square, weaker forms of such rules [eqs (17)–(20)] get generated on the four sides of the square, through the vanishing of the norms.

At each of the four corners B, C, F and G two of the eigenvalues vanish and so two eigenvectors become null vectors. Thus, for instance, at F, both (17) and (20) are simultaneously valid, which can be combined to give

$$(c^\dagger_{k\alpha}c^\dagger_{m\beta} + c^\dagger_{m\beta}c^\dagger_{k\alpha})|0\rangle = 0. \tag{21}$$

Further, as shown in the next section, (21) can be replaced by the operator identity:

$$c^\dagger_{k\alpha}c^\dagger_{m\beta} + c^\dagger_{m\beta}c^\dagger_{k\alpha} = 0. \tag{22}$$

Similar things occur at all the four corners, thus leading to the strong commutation rules which we shall presently write down. However, the replacement of eqs (17)–(20) by the corresponding operator identities is not possible. In fact, we prove in the next section that cc relations exist only at the four corners B, C, F and G.

Let us now present the equations valid at the four corners of the square:

(i) *At the corner F (Fermi-Dirac statistics)*

$$c_{k\alpha}c^\dagger_{m\beta} + c^\dagger_{m\beta}c_{k\alpha} = \delta_{km}\delta_{\alpha\beta}, \tag{23}$$

$$c_{k\alpha}c_{m\beta} + c_{m\beta}c_{k\alpha} = 0. \tag{24}$$

(ii) *At B (Bose-Einstein statistics)*

$$c_{k\alpha}c^\dagger_{m\beta} - c^\dagger_{m\beta}c_{k\alpha} = \delta_{km}\delta_{\alpha\beta}, \tag{25}$$

$$c_{k\alpha}c_{m\beta} - c_{m\beta}c_{k\alpha} = 0. \tag{26}$$

(iii) *At G (orthofermi statistics)*

$$c_{k\alpha}c^\dagger_{m\beta} + \delta_{\alpha\beta}\sum_\gamma c^\dagger_{m\gamma}c_{k\gamma} = \delta_{km}\delta_{\alpha\beta}, \tag{27}$$

$$c_{k\alpha}c_{m\beta} + c_{m\alpha}c_{k\beta} = 0. \tag{28}$$

(iv) *At C (orthobose statistics)*

$$c_{k\alpha}c^\dagger_{m\beta} - \delta_{\alpha\beta}\sum_\gamma c^\dagger_{m\gamma}c_{k\gamma} = \delta_{km}\delta_{\alpha\beta}, \tag{29}$$

$$c_{k\alpha}c_{m\beta} - c_{m\alpha}c_{k\beta} = 0. \tag{30}$$

Orthofermi statistics was studied in an earlier paper [6] and it is characterized by a new exclusion principle which is more "exclusive" than Pauli's exclusion principle: an orbital state shall not contain more than one particle, whatever be the spin direction. Further, the wave function is antisymmetric in spatial indices alone, with the order of the spin-indices frozen. Both these properties follow from (28); the positions of α and β in this equation must be particularly noted. Orthobose statistics is the corresponding Bose-analogue; from (30), it follows that the wave function is symmetric in spatial indices alone, with the order of the spin-indices frozen. Statistical mechanics based on orthostatistics is treated in [6, 8].

Thus, at the four corners of the square, we have four kinds of statistics namely, Fermi-Dirac, Bose-Einstein, orthofermi and orthobose statistics which respectively correspond to total antisymmetry, total symmetry, spatial antisymmetry and spatial symmetry of the wave function.

What about the positivity of the inner product in the $\{q_1, q_2\}$ plane? Do the sides of the square continue to be the boundaries of the region of positivity for three and more particles? We have found that the answer is in the negative. The relevant calculations become much more complex since the total number of states increases rapidly. The details are presented in Appendix 3, but the result for three particles with the spin index taking two values is shown in figure 1. We see that the region of positivity shrinks. How does the boundary of this region move as the number of particles n increases further and what is the limiting boundary for $n \to \infty$? These are open questions for the present.

However, we have definite answers along the two diagonals. Along the diagonal BF ($q_2 = 0$), the algebra (5) reduces to the "q-mutator algebra" of Greenberg [1]:

$$c_{k\alpha} c^{\dagger}_{m\beta} - q_1 c^{\dagger}_{m\beta} c_{k\alpha} = \delta_{km} \delta_{\alpha\beta}, \tag{31}$$

for which the inner product is given in (7) and in this case Fivel [2] and Zagier [9] have proved the positive-definiteness of the metric for the n-particle sector in the interval $-1 \leqslant q_1 \leqslant 1$. These are "$q$-ons" having orbital and spin indices.

Along the other diagonal CG ($q_1 = 0$), we have the algebra:

$$c_{k\alpha} c^{\dagger}_{m\beta} - q_2 \delta_{\alpha\beta} \sum_{\gamma} c^{\dagger}_{m\gamma} c_{k\gamma} = \delta_{km} \delta_{\alpha\beta}. \tag{32}$$

In this case, the inner product becomes (from eq. 13)

$$\begin{aligned} &\langle 0| \cdots c_{t\gamma} c_{m\beta} c_{k\alpha} c^{\dagger}_{p\lambda} c^{\dagger}_{s\mu} c^{\dagger}_{q\nu} \cdots |0\rangle \\ &\quad = q_2^k \delta_{\alpha\lambda} \delta_{\beta\mu} \delta_{\gamma\nu} \cdots \end{aligned} \tag{33}$$

where K is the number of inversions in the permutation of the space indices. We see that the spin-ordering is frozen all along this diagonal CG and as for the space indices the behaviour is exactly that of q-ons possessing only space index. Hence, the Fivel-Zagier proof of positivity of the metric for arbitrary n-particle states is applicable along this diagonal also, in the interval $-1 \leqslant q_2 \leqslant 1$. This algebra of (32) interpolates between orthofermi and orthobose statistics.

4. Quadratic relations for destruction operators

It is an important fact which does not seem to be well recognized, that within the framework of Fock space, quadratic relations among c's are not required. For, given the vacuum state defined by eq. (6) and a rule for the commutation of c and $c^\dagger$ such as (5), *all* matrix elements in Fock space can be computed. Hence, any cc relation which is imposed will be either inconsistent with the $cc^\dagger$ relation, or superfluous if consistent. Nevertheless in the latter case, it is useful to know the explicit form of the relation.

In this section, we shall show that appropriate cc relations follow from the corresponding $cc^\dagger$ relations at the four corners of the square BCFG, but no such relations are possible anywhere else in the $\{q_1, q_2\}$ plane.

(i) *Fermi-Dirac*: From eq. (23), it is easy to show that the anticommutator $\{c_{k\alpha}, c_{m\beta}\}$ commutes with any product of a string of creation operators:

$$\{c_{k\alpha}, c_{m\beta}\}(c^\dagger_{p\gamma} c^\dagger_{q\sigma} c^\dagger_{s\tau} \cdots) = (c^\dagger_{p\gamma} c^\dagger_{q\sigma} c^\dagger_{s\tau} \cdots)\{c_{k\alpha}, c_{m\beta}\}. \tag{34}$$

Applying this equation on $|0\rangle$, we see that $\{c_{k\alpha}, c_{m\beta}\}$ acting on *any* Fock state gives zero. Hence, we may write

$$\{c_{k\alpha}, c_{m\beta}\} = 0. \tag{35}$$

(ii) *Bose-Einstein*: This is similar to the above. From (25), we have

$$[c_{k\alpha}, c_{m\beta}](c^\dagger_{p\gamma} c^\dagger_{q\sigma} c^\dagger_{s\tau} \cdots) = (c^\dagger_{p\gamma} c^\dagger_{q\sigma} c^\dagger_{s\tau} \cdots)[c_{k\alpha}, c_{m\beta}]. \tag{36}$$

and hence

$$[c_{k\alpha}, c_{m\beta}] = 0. \tag{37}$$

(iii) *Orthofermi*: In this case, the relevant combination of c's does not commute with $c^\dagger$. Instead, from (27) we get

$$(c_{k\alpha} c_{m\beta} + c_{m\alpha} c_{k\beta}) c^\dagger_{p\gamma} = \delta_{\beta\gamma} \sum_\rho c^\dagger_{p\rho}(c_{k\rho} c_{m\alpha} + c_{m\rho} c_{k\alpha}). \tag{38}$$

The form-invariance of the expression $(c_{k\alpha} c_{m\beta} + c_{m\alpha} c_{k\beta})$ on being pushed to the right of $c^\dagger_{p\gamma}$ is sufficient for our purpose. With one more $c^\dagger$, we get

$$(c_{k\alpha} c_{m\beta} + c_{m\alpha} c_{k\beta}) c^\dagger_{p\gamma} c^\dagger_{q\sigma} = \delta_{\beta\gamma} \delta_{\alpha\sigma} \sum_{\rho,\tau} c^\dagger_{p\rho} c^\dagger_{q\tau}(c_{k\tau} c_{m\rho} + c_{m\tau} c_{k\rho}) \tag{39}$$

Applying eqs (38) and (39) on $|0\rangle$, we see that $(c_{k\alpha} c_{m\beta} + c_{m\alpha} c_{k\beta})$ acting on one and two-particle states gives zero and this procedure can be extended to arbitrary number of particles. Therefore we have

$$(c_{k\alpha} c_{m\beta} + c_{m\alpha} c_{k\beta}) = 0. \tag{40}$$

(iv) *Orthobose*: From eq. (29) we get

$$(c_{k\alpha} c_{m\beta} - c_{m\alpha} c_{k\beta}) c^\dagger_{p\gamma} = \delta_{\beta\gamma} \sum_\rho c^\dagger_{p\rho}(c_{k\rho} c_{m\alpha} - c_{m\rho} c_{k\alpha}). \tag{41}$$

By the same argument as in the orthofermi case, we get

$$(c_{k\alpha} c_{m\beta} - c_{m\alpha} c_{k\beta}) = 0. \tag{42}$$

Thus, in the cases of all the four statistics including the familiar Fermi-Dirac and

Bose-Einstein statistics, the cc relations follow from the corresponding $cc^\dagger$ relations and the existence of the vacuum state.

We shall now prove that cc relations do not exist anywhere in the $\{q_1, q_2\}$ plane except at B, C, F and G. We assume the algebra of (5) and try to commute a general quadratic in c's with $c^\dagger_{p\tau}$. Defining

$$Q_{k\alpha,m\beta} \equiv w c_{k\alpha} c_{m\beta} + x c_{m\beta} c_{k\alpha} + y c_{m\alpha} c_{k\beta} + z c_{k\beta} c_{m\alpha}, \tag{43}$$

where w, x, y and z are arbitrary constants, we get

$$Q_{k\alpha,m\beta} c^\dagger_{p\tau} = q_1^2 c^\dagger_{p\tau} Q_{k\alpha,m\beta} + q_1 q_2 \delta_{\alpha\tau} \sum_\gamma c^\dagger_{p\gamma} Q_{k\gamma,m\beta}$$
$$+ q_1 q_2 \delta_{\beta\tau} \sum_\gamma c^\dagger_{p\gamma} Q_{k\alpha,m\gamma} + A + B \tag{44}$$

where

$$A = q_2^2 \sum_\gamma c^\dagger_{p\gamma} [\delta_{\alpha\tau}(x c_{m\gamma} c_{k\beta} + z c_{k\gamma} c_{m\beta})$$
$$+ \delta_{\beta\tau}(w c_{k\gamma} c_{m\alpha} + y c_{m\gamma} c_{k\alpha})] \tag{45}$$
$$B = (w + y q_2 + x q_1) \delta_{mp} \delta_{\beta\tau} c_{k\alpha} + (x + z q_2 + w q_1) \delta_{kp} \delta_{\alpha\tau} c_{m\beta}$$
$$+ (y + w q_2 + z q_1) \delta_{kp} \delta_{\beta\tau} c_{m\alpha} + (z + y q_1 + x q_2) \delta_{mp} \delta_{\alpha\tau} c_{k\beta}. \tag{46}$$

The same form of Q is reproduced on the first three terms on the right side of (44), but the A and B terms are different. In the absence of A and B, by applying the same argument used earlier in this section, we can see that Q acting on any Fock state would be zero.

There are two possibilities, the first being that both A and B vanish which implies

$$q_2 = 0, \tag{47}$$

$$\begin{aligned} w + y q_2 + x q_1 &= 0 \\ x + z q_2 + w q_1 &= 0 \\ y + w q_2 + z q_1 &= 0 \\ z + x q_2 + y q_1 &= 0\ . \end{aligned} \tag{48}$$

The set of equations (48) can be recast in the form:

$$M_2 \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = 0, \tag{49}$$

where M_2 is precisely the same 4×4 matrix (eq. (16)) encountered earlier, namely the inner product matrix in the 2-particle sector. For nontrivial solutions of (49) the determinant of M_2 must vanish, which, as we know already, occurs only along the four sides of the square BCFG. But, in addition, we want (47) to be satisfied, and thus, the corners B and F alone are picked out.

The other possibility requires

$$\text{(a)} \quad x = z = 0 \tag{50}$$

(or)

$$\text{(b)} \quad w = y = 0. \tag{51}$$

It can be shown that either (a) or (b) leads to same consequences. Let us choose (a) and also put

$$\begin{aligned} w + yq_2 &= 0, \\ wq_1 &= 0, \\ y + wq_2 &= 0, \\ yq_1 &= 0. \end{aligned} \tag{52}$$

Equations (50) and (52) together make B vanish, but A does not vanish. However, in view of eqs (43) and (50), A takes the form

$$q_2^2 \sum_\gamma c^\dagger_{p\gamma} \delta_{\beta\tau} Q_{k\gamma,m\alpha} \tag{53}$$

which implies that the commutation of Q by $c^\dagger$ preserves the form of Q and consequently, Q as an operator in Fock space becomes a null operator. For nontrivial solution of w and y, eq. (52) requires $q_1 = 0$, $q_2 = \pm 1$. These are the corners C and G. The proof is complete.

5. Systems with single index

Consider the two-parameter algebra (5) for the same latin index ($k = m$) which is then suppressed. We get

$$c_\alpha c_\beta^\dagger - q_1 c_\beta^\dagger c_\alpha - q_2 \delta_{\alpha\beta} \sum_\gamma c_\gamma^\dagger c_\gamma = \delta_{\alpha\beta}. \tag{54}$$

In fact, this is the most general bilinear algebra of creation and destruction operators with a single index, which is invariant under the unitary transformations V defined in § 2. This single-indexed system, being a simpler one, deserves special study.

Referring to the (q_1, q_2) plane represented by figure 1, let us first consider the four points B, F, C and G. The points B and F respectively correspond to Bose or Fermi statistics in the Greek indices. What about C and G? At C, (54) becomes

$$c_\alpha c_\beta^\dagger - \delta_{\alpha\beta} \sum_\gamma c_\gamma^\dagger c_\gamma = \delta_{\alpha\beta}. \tag{55}$$

Let us define the total number operator

$$N = \sum_\gamma c_\gamma^\dagger c_\gamma \tag{56}$$

which satisfies

$$[N, c_\alpha] = -c_\alpha \tag{57}$$

as can be verified using (55). In Fock space, $(1+N)$ is a positive-definite diagonal operator. So, one is allowed to define new annihilation and creation operators by

$$c_\alpha = (1+N)^{1/2} a_\alpha; \qquad c_\alpha^\dagger = a_\alpha^\dagger (1+N)^{1/2} \tag{58a}$$

or,

$$a_\alpha = (1+N)^{-1/2} c_\alpha; \quad a_\alpha^\dagger = c_\alpha^\dagger (1+N)^{-1/2} \tag{58b}$$

so that (55) can be cast into the form:

$$a_\alpha a_\beta^\dagger = \delta_{\alpha\beta}. \tag{59}$$

Equation (59) is Greenberg's algebra [10] leading to "infinite statistics". Thus, the point C corresponds to infinite statistics and (55) provides merely a different algebraic representation for the same. One also notes that N is a simple quadratic expression of c and $c^\dagger$ whereas, in terms of a and $a^\dagger$, it is known to be an infinite series:

$$N = \sum_\gamma a_\gamma^\dagger a_\gamma + \sum_{\gamma,\beta} a_\beta^\dagger a_\gamma^\dagger a_\gamma a_\beta + \cdots \tag{60}$$

We shall say more on the number operator in §7.

At the point G, we have

$$c_\alpha c_\beta^\dagger + \delta_{\alpha\beta} \sum_\gamma c_\gamma^\dagger c_\gamma = \delta_{\alpha\beta}. \tag{61}$$

Although N can still be defined by (56), the transformation corresponding to (58a) will now involve the operator $(1-N)$ which is not positive definite and hence one cannot obtain a and $a^\dagger$ from c and $c^\dagger$. So, it is best to leave (61) as it stands. Further, there exists a relation at G:

$$c_\alpha c_\beta = 0 \tag{62}$$

which follows from eq. (28) by putting $k=m$ and then suppressing the Latin index. (Note that such a relation does not exist at the point C; the corresponding eq. (30) is trivially satisfied for $k=m$.) As a consequence of eq. (62), the eigenvalues of N are restricted to 0 and 1 only. The algebra defined by (61) and (62) has been used to construct a generalization of supersymmetric quantum mechanics named orthosupersymmetric quantum mechanics [11].

Let us now consider the algebra on the line GC$(q_1=0)$:

$$c_\alpha c_\beta^\dagger - q_2 \delta_{\alpha\beta} \sum_\gamma c_\gamma^\dagger c_\gamma = \delta_{\alpha\beta}. \tag{63}$$

The n-particle inner product for this case is given by

$$\begin{aligned} &\langle 0| \cdots c_\gamma c_\beta c_\alpha c_\lambda^\dagger c_\mu^\dagger c_\nu^\dagger \cdots |0\rangle \\ &= [1(1+q_2)(1+q_2+q_2^2)\cdots(1+q_2+q_2^2+\cdots+q_2^{n-1})] \times \delta_{\alpha\lambda}\delta_{\beta\mu}\delta_{\gamma\nu}\cdots \end{aligned} \tag{64}$$

The q_2-dependent factor on the right side of (64) is positive for $q_2 \geqslant -1$. We thus see that Fock space with positive-definite metric exists all along the line $q_1=0$, $q_2 \geqslant -1$, i.e. the infinite line GC starting with G and extending beyond C to $q_2 \to +\infty$ (figure 1).

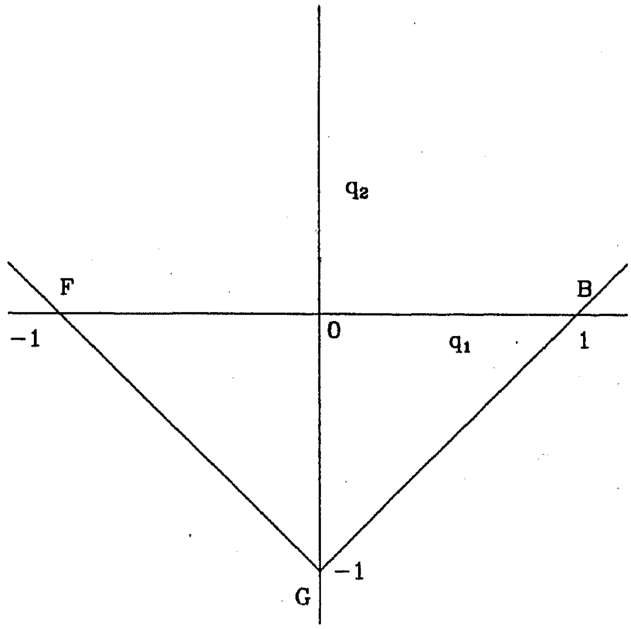

Figure 2. The region of positivity in the two and three particle sectors for the single-indexed system with the index taking only two values. This is the triangular region lying above and bounded by the straight lines GF and GB both extended to infinity.

For the discussion of positivity in the full $\{q_1, q_2\}$ plane, we shall consider the particular case when the Greek index ranges over only two values. Straightforward calculations lead to the following results for this single-indexed algebra. The metric in the two particle sector is positive in the infinite triangular region bounded by the straight lines GF$(1 + q_1 + q_2 = 0)$ and GB$(1 + q_2 - q_1 = 0)$ (see figure 2). For the three particle sector, we have checked the positivity in the same region numerically. Thus in the contrast to the case of the two-indexed algebra, the positivity of the metric in the three-particle sector is preserved over the whole region for which the metric in the two-particle sector was positive.

We also see that positivity extends to a much larger region than for the two-indexed algebra studied in § 3. Here we have an example of a general result that the region of positivity is enlarged if the range of indices is reduced. Note that the single-indexed algebra can be obtained from the two-indexed algebra by reducing the range of one of the indices to unity.

Finally we would like to make a remark on the dimension of the Fock space. In general, c and $c^\dagger$ can be expressed as infinite-dimensional matrices. However, for two special cases c and $c^\dagger$ become finite-dimensional. These correspond to the Fermi-Dirac and orthofermi points, F and G respectively. For the single-indexed algebra with the index α ranging over 1 and 2 only, c_1, c_2, $c_1^\dagger$ and $c_2^\dagger$ can be represented by 4×4 and 3×3 matrices at F and G respectively. However, it is interesting to note that along the line FG connecting these two points, finite-dimensional representation of the algebra is not possible. One may ask whether there exists an algebra which can interpolate between the Fermi-Dirac and orthofermi points via finite-dimensional matrices. The answer turns out to be in the affirmative, but one has to go beyond the bilinear algebra and include biquadratic terms in c and $c^\dagger$ [12].

6. The four parameter algebra and the tetrahedron

We now take up the four-parameter algebra of eq. (1) and consider the two-particle sector comprising the same four states given in (15). The 4×4 matrix of inner products of eq. (16) is now replaced by

$$M'_2 = \begin{pmatrix} 1+q_0 & q_1 & q_2 & q_3 \\ q_1 & 1+q_0 & q_3 & q_2 \\ q_2 & q_3 & 1+q_0 & q_1 \\ q_3 & q_2 & q_1 & 1+q_0 \end{pmatrix}. \tag{65}$$

We see that at the two-particle level, the rules for the inner product are the following: (a) Diagonal term $= 1 + q_0$ (b) Inversion of space and spin indices: q_1 (c) Inversion of space indices alone: q_2 (d) Inversion of spin indices alone: q_3.

The matrix M'_2 has the following eigenvectors and corresponding eigenvalues:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

with their respective eigenvalues $1 + q_0 + q_1 + q_2 + q_3$, $1 + q_0 + q_1 - q_2 - q_3$, $1 + q_0 - q_1 + q_2 - q_3$, $1 + q_0 - q_1 - q_2 + q_3$. As before, equating these eigenvalues to zero, we get the equations of the manifolds which will be the boundaries of the region of positivity for the matrix of inner products:

$$\begin{aligned} 1+q_0+q_1+q_2+q_3 &= 0 \\ 1+q_0+q_1-q_2-q_3 &= 0 \\ 1+q_0-q_1+q_2-q_3 &= 0 \\ 1+q_0-q_1-q_2+q_3 &= 0\ . \end{aligned} \tag{66}$$

It is convenient to consider a fixed value of q_0 first. Defining

$$\xi = 1 + q_0, \tag{67}$$

we have

$$\begin{aligned} q_1+q_2+q_3+\xi &= 0 \\ q_1-q_2-q_3+\xi &= 0 \\ -q_1+q_2-q_3+\xi &= 0 \\ -q_1-q_2+q_3+\xi &= 0\ . \end{aligned} \tag{68}$$

For fixed q_0, ξ is a constant. In the three-dimensional $\{q_1, q_2, q_3\}$ space, the four equations in (68) denote the four faces of a regular tetrahedron STUV (see figure 3). The region of positivity of the inner products in the two-particle sector is thus identified to be inside the tetrahedron STUV.

Consider the points F, C, B, G, C′ and G′ which are the mid points of the six edges of the tetrahedron STUV in figure 3. Then the three mutually orthogonal planes FCBG, FC′BG′ and GC′CG′ shown in figure 3 as cutting each other and cutting the tetrahedron are respectively the planes described by the equations $q_3 = 0$, $q_2 = 0$ and $q_1 = 0$. Further, each of these quadrilaterals FCBG, FC′BG′ and GC′CG′ is

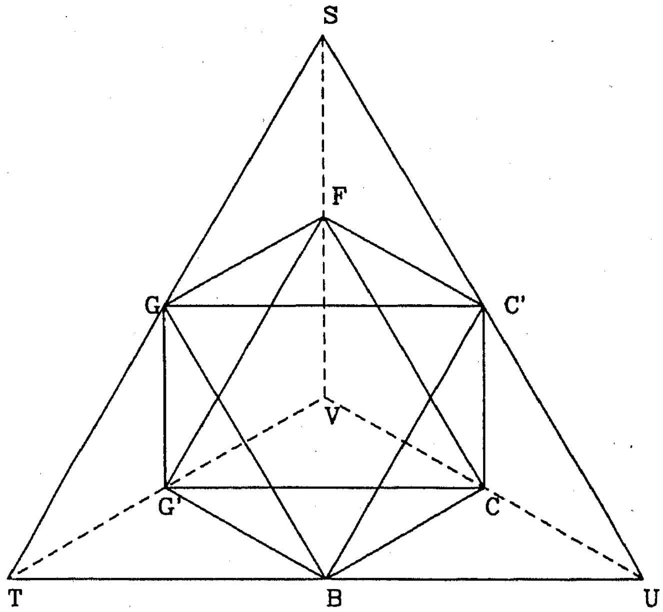

Figure 3. The regular tetrahedron STUV which bounds the allowed region in the (q_1, q_2, q_3) parameter space, at fixed q_0, for the two-particle sector. The centre of the tetrahedron corresponds to $q_1 = q_2 = q_3 = 0$. The three mutually orthogonal planes FCBG, FC′BG′ and GC′CG′ correspond to $q_3 = 0$, $q_2 = 0$ and $q_1 = 0$ respectively. The three cartesian coordinate axes for q_3, q_2 and q_1 which are not shown in the figure, are perpendicular to these three planes. The coordinates of the four vertices U, T, V, S of the tetrahedron are respectively $q_1 = q_2 = q_3 = (1 + q_0)$; $q_1 = -q_2 = -q_3 = (1 + q_0)$; $-q_1 = q_2 = -q_3 = (1 + q_0)$; $-q_1 = -q_2 = q_3 = (1 + q_0)$.

actually a square. The three cartesian coordinate axes for q_3, q_2 and q_1 which are not shown in the figure, are perpendicular to these three planes.

Let us now consider the algebras valid in these three planes. For simplicity, let us first put $q_0 = 0$. Then, in eqs (68), ξ is replaced by unity. In this case, the algebra of c and $c^\dagger$ in the plane FCBG ($q_3 = 0$) of figure 3 is our two-parameter algebra described by (5) and treated in detail in § 3 and FCBG is precisely the same boundary shown in figure 1.

Similarly, in the plane FC′BG′, the two-parameter algebra is

$$c_{k\alpha} c^\dagger_{m\beta} - q_1 c^\dagger_{m\beta} c_{k\alpha} - q_3 \delta_{km} \sum_p c^\dagger_{p\beta} c_{p\alpha} = \delta_{km} \delta_{\alpha\beta} \tag{69}$$

and all the conclusions drawn in § 3 are valid for this algebra, but with the roles of space and spin indices interchanged. The square FC′BG′ (see figure 3) is the boundary of the positivity of the norm in the two-particle sector. At G′ and C′ we have another version of orthofermi and orthobose statistics:

At G′ (orthofermi statistics):

$$c_{k\alpha} c^\dagger_{m\beta} + \delta_{km} \sum_p c^\dagger_{p\beta} c_{p\alpha} = \delta_{km} \delta_{\alpha\beta}, \tag{70}$$

$$c_{k\alpha} c_{m\beta} + c_{k\beta} c_{m\alpha} = 0. \tag{71}$$

At C′ (orthobose statistics):

$$c_{k\alpha}c^{\dagger}_{m\beta} - \delta_{km}\sum_{p} c^{\dagger}_{p\beta}c_{p\alpha} = \delta_{km}\delta_{\alpha\beta}, \tag{72}$$

$$c_{k\alpha}c_{m\beta} - c_{k\beta}c_{m\alpha} = 0. \tag{73}$$

A description of these versions of statistics analogous to that of the earlier orthostatistics can be given. Along the diagonal C′G′ (see figure 3), we have

$$c_{k\alpha}c^{\dagger}_{m\beta} - q_3\delta_{km}\sum_{p} c^{\dagger}_{p\beta}c_{p\alpha} = \delta_{km}\delta_{\alpha\beta}. \tag{74}$$

Along the two diagonals FB and C′G′, positivity of the norm can be proved for arbitrary number of particles. Elsewhere, the boundary shrinks for larger number of particles. Exactly the same dotted curve as in figure 1, but now drawn inside the square FC′ BG′ will be the boundary for three particle sector.

Finally, the algebra valid in the plane GC′CG′ is

$$c_{k\alpha}c^{\dagger}_{m\beta} - q_2\delta_{\alpha\beta}\sum_{\gamma} c^{\dagger}_{m\gamma}c_{k\gamma} - q_3\delta_{km}\sum_{p} c^{\dagger}_{p\beta}c_{p\alpha} = \delta_{km}\delta_{\alpha\beta}.$$

The square GC′CG′ shown in figure 3 is the boundary of positivity of the norm in the two-particle sector. The boundary for larger number of particles is not known. However, along the two diagonals CG and G′C′ complete Fock spaces with positive definite metric exist.

Let us now come back to the tetrahedron (figure 3). The six edges of the tetrahedron are given by the following six pairs of equations:

$$q_1 + \xi = 0; \quad q_2 + q_3 = 0, \tag{75}$$

$$q_2 + \xi = 0; \quad q_3 + q_1 = 0, \tag{76}$$

$$q_3 + \xi = 0; \quad q_1 + q_2 = 0, \tag{77}$$

$$q_1 - \xi = 0; \quad q_2 - q_3 = 0, \tag{78}$$

$$q_2 - \xi = 0; \quad q_3 - q_1 = 0, \tag{79}$$

$$q_3 - \xi = 0; \quad q_1 - q_2 = 0, \tag{80}$$

while the four vertices are given by the points:

$$q_1 = -\xi, \quad q_2 = -\xi, \quad q_3 = +\xi, \tag{81}$$

$$q_1 = +\xi, \quad q_2 = -\xi, \quad q_3 = -\xi, \tag{82}$$

$$q_1 = -\xi, \quad q_2 = \xi, \quad q_3 = -\xi, \tag{83}$$

$$q_1 = \xi, \quad q_2 = \xi, \quad q_3 = \xi. \tag{84}$$

The full extent of the region of positivity of norm (at least for the two-particle sector) is revealed by the coordinates in (81–84). Again putting $q_0 = 0$, note for instance that when q_3 was zero the point $q_1 = q_2 = 1$ was not in the allowed region (see figure 1), but for $q_3 = 1$ this becomes an allowed point.

Finally, consider the four-dimensional parameter space $\{q_0, q_1, q_2, q_3\}$. The region of positivity of the norm in the two-particle sector is given by a four-dimensional object whose cross section at fixed q_0 is the three dimensional tetrahedron STUV considered so far. Since the size of the tetrahedron is determined by $\xi = 1 + q_0$, we see that the size of the tetrahedron contracts to zero for $q_0 = -1$ but expands to infinite extent for $q_0 \to +\infty$. The four-dimensional object is of infinite length in the q_0 direction extending between $q_0 = -1$ and $q_0 = +\infty$.

It is interesting to note that by taking q_0 positive and sufficiently large, the region of positivity of the norm can be enlarged to an arbitrarily large extent. The results presented in this section are for two-particle sector, except for the three-particle boundaries given for the algebras with two parameters $\{q_1, q_2\}$ or $\{q_1, q_3\}$. It appears that positivity of the metric for three and more particles in the cases of the three and four parameter algebras will lead to complex shapes and boundaries. However, all of these will be enclosed within the above four-dimensional object.

The particular case of the (q_0, q_1) algebra:

$$c_{k\alpha}c^{\dagger}_{m\beta} - q_1 c^{\dagger}_{m\beta}c_{k\alpha} - q_0 \delta_{km}\delta_{\alpha\beta}\sum_{p,\gamma} c^{\dagger}_{p\gamma}c_{p\gamma} = \delta_{km}\delta_{\alpha\beta} \tag{85}$$

is of special interest. For, this reduces to the single-indexed algebra studied in § 5, namely eq. (54), by the mapping

$$(k\alpha) \to A; \quad q_1 \to q_1; \quad q_0 \to q_2 \tag{86}$$

where A stands for a collection of indices and hence all the results of §5 are applicable for the algebra of (85). In particular, we see that the complete Fock space with positive definite metric exists all along the line:

$$-1 < q_0 < \infty; \quad q_1 = q_2 = q_3 = 0. \tag{87}$$

In the two-parameter plane (q_0, q_1) as well as for the other two-parameter algebras characterized by the pairs (q_0, q_2) and (q_0, q_3), we have only the two-particle results which are contained in the two-dimensional sections of the four-dimensional figure already described. For the sake of clarity, we may restate the results explicitly. For (q_0, q_1), (q_0, q_2) and (q_0, q_3) algebras, positivity of the metric in the two-particle sector allows the triangular region depicted in figure 2, but with the y-axis replaced by q_0 and the x-axis replaced by q_1, q_2 and q_3 respectively.

For the sake of completeness and clarity, we may here write down all the single parameter algebras with the corresponding domain of the parameter space for which complete Fock spaces of positive definite metric exist:

(a) $$c_{k\alpha}c^{\dagger}_{m\beta} - q_1 c^{\dagger}_{m\beta}c_{k\alpha} = \delta_{km}\delta_{\alpha\beta}, \tag{88}$$

$$-1 \leqslant q_1 \leqslant 1, \tag{89}$$

(b) $$c_{k\alpha}c^{\dagger}_{m\beta} - q_2 \delta_{\alpha\beta}\sum_{\gamma} c^{\dagger}_{m\gamma}c_{k\gamma} = \delta_{km}\delta_{\alpha\beta}, \tag{90}$$

$$-1 \leqslant q_2 \leqslant 1, \tag{91}$$

(c) $$c_{k\alpha}c^{\dagger}_{m\beta} - q_3 \delta_{km}\sum_{p} c^{\dagger}_{p\beta}c_{p\alpha} = \delta_{km}\delta_{\alpha\beta}, \tag{92}$$

$$-1 \leqslant q_3 \leqslant 1, \tag{93}$$

(d) $$c_{k\alpha}c^{\dagger}_{m\beta} - q_0\delta_{km}\delta_{\alpha\beta}\sum_{p,\gamma} c^{\dagger}_{p\gamma}c_{p\gamma} = \delta_{km}\delta_{\alpha\beta}, \tag{94}$$

$$-1 \leqslant q_0 < \infty. \tag{95}$$

cc relations: What are the allowed cc relations, corresponding to the 4-parameter $cc^{\dagger}$ algebra of (1)? To answer this, one can again take Q, the general quadratic in c given by (43) and try to commute it with $c^{\dagger}_{l\tau}$ using (1). We do not give here the resulting equation which is much longer than (44). But pursuing the same kind of arguments as in § 4, we find that apart from the already identified points B, F, C, G, C′ and G′ (see figure 3) where the cc relations already discussed exist, only one more point in the $\{q_0, q_1, q_2, q_3\}$ space is allowed, namely

$$q_0 = -1; \quad q_1 = q_2 = q_3 = 0 \tag{96}$$

and the corresponding cc relation is

$$c_{k\alpha}c_{m\beta} = 0. \tag{97}$$

At this point the $cc^{\dagger}$ algebra (1) becomes

$$c_{k\alpha}c^{\dagger}_{m\beta} + \delta_{km}\delta_{\alpha\beta}\sum_{p,\gamma} c^{\dagger}_{p\gamma}c_{p\gamma} = \delta_{km}\delta_{\alpha\beta}. \tag{98}$$

Equations (98) and (97) can be respectively mapped into (61) and (62) of § 5. This algebra does not allow more than a single particle in the system and hence does not lead to any new statistics.

7. Miscellaneous

7.1 *Factorization*

It is possible to factorize the algebra of (5) into two independent algebras, one involving the space and the other, the spin indices:

$$c_{k\alpha} = f_k b_\alpha, \tag{99}$$

$$f_k b_\alpha - b_\alpha f_k = f^{\dagger}_k b_\alpha - b_\alpha f^{\dagger}_k = 0, \tag{100}$$

$$f_k f^{\dagger}_m - q f^{\dagger}_m f_k = \delta_{km}, \tag{101}$$

$$b_\alpha b^{\dagger}_\beta = \delta_{\alpha\beta}, \tag{102}$$

$$q_1 b^{\dagger}_\beta b_\alpha + q_2\delta_{\alpha\beta}\sum_{\gamma} b^{\dagger}_\gamma b_\gamma = q\delta_{\alpha\beta}, \tag{103}$$

where q is another arbitrary real parameter. It is easy to show that by substituting (99) into (5) and using (100) and (101), the eqs (102) and (103) follow. Thus, we have two commuting sets of operators f_k and b_α, the former satisfying the q-mutator algebra of (101) and the latter satisfies the algebra defined by (102) and (103).

The above is valid for arbitrary values of q. If we put $q = q_2$ and also restrict ourselves to the diagonal GC($q_1 = 0$), eq. (103) becomes

$$\sum_{\gamma} b^{\dagger}_\gamma b_\gamma = 1. \tag{104}$$

Equations (102) and (104) together constitute Cuntz algebra [5]. Our general factorization thus involves a q-generalization [(102) and (103)] of Cuntz algebra.

In spite of the elegance of the above result, it must be pointed out that Cuntz algebra and factorization are inconsistent with Fock representation as can be seen by applying the vacuum state $|0\rangle$ on (103) or (104) on the right.

7.2 *Number operator*

The properties of the matrix elements following from the algebra of (5) were stated at the beginning of § 3. These already lead to the concept of particle number and in fact we have implicitly used this in our discussion so far. Nevertheless it is useful to have an explicit expression for the number operator.

A special case of the unitary transformations of (2) and (3) is the phase transformation. Equation (5) is invariant under the following phase transformations:

$$c_{k\alpha} \to e^{i\phi} c_{k\alpha}, \tag{105}$$

$$c_{k\alpha} \to e^{i\phi_k} c_{k\alpha}, \tag{106}$$

$$c_{k\alpha} \to e^{i\phi_\alpha} c_{k\alpha}, \tag{107}$$

where ϕ is independent of k and α, while ϕ_k and ϕ_α respectively depend on k and α. As a consequence, the total number operator N, as well as the number operators for a definite space index or spin index N_k and N_α exist. However, the algebra of (5) is not invariant under the phase transformation:

$$c_{k\alpha} \to e^{i\phi_{k\alpha}} c_{k\alpha}, \tag{108}$$

where ϕ depends on both k and α. Hence, the number operator $N_{k\alpha}$ for a definite k and α does not exist. The difficulty of defining $N_{k\alpha}$ with usual commutation relations was already pointed out earlier [6,7].

For the case of orthofermi and orthobose statistics, explicit expressions for N and N_k are easy to write down:

$$N = \sum_{k,\alpha} c^\dagger_{k\alpha} c_{k\alpha}, \tag{109}$$

$$N_k = \sum_{\alpha} c^\dagger_{k\alpha} c_{k\alpha}. \tag{110}$$

One can verify

$$[N, c_{k\alpha}] = -c_{k\alpha}, \tag{111}$$

$$[N_k, c_{m\alpha}] = -\delta_{km} c_{m\alpha}. \tag{112}$$

On the other hand, N_α is generally an infinite series in $c^\dagger$ and c just as in the case of Greenberg's algebra for "infinite statistics" [1, 13, 14]. Consider the case of the orthofermi statistics with the spin index taking only two values denoted by σ and $\bar{\sigma}$. Then, the first few terms (up to biquadratic terms in $c^\dagger$ and c) for N_σ are the following:

$$N_\sigma = \sum_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{k,m} c^\dagger_{k\bar{\sigma}} c^\dagger_{m\sigma} c_{m\sigma} c_{k\bar{\sigma}} - \sum_{k,m} c^\dagger_{k\sigma} c^\dagger_{m\bar{\sigma}} c_{m\bar{\sigma}} c_{k\sigma} + \cdots \tag{113}$$

$N_{\bar{\sigma}}$ is obtained by the replacement $\sigma \leftrightarrow \bar{\sigma}$. We have not obtained the complete series.

8. Discussion

Annihilation and creation operators are the basic elements out of which, quantum fields are built, at least at the perturbative level. Thus, a general study of c and $c^\dagger$ throws light on the enlarged framework within which the familiar quantum field theory and statistical mechanics reside. This is the main motivation behind our work.

Our study encompasses the general bilinear algebras of annihilation and creation operators which are possible under the sole constraint of invariance under unitary transformations of the indices. Our main departure from earlier studies is the distinction between the two types of indices which we have introduced and the requirement of invariance under separate unitary transformations on these two types of indices. We have analyzed the general algebraic constraints that result from consistency requirements as well as the constraints arising from the positivity of the norm of the corresponding Fock spaces. The positivity constraints prove to be very strong. Although we do not have the complete solution of the problem of defining the regions of positivity in the multiparameter-space, we have already found some restricted regions of the parameter space for which nontrivial algebras with positive definite Fock spaces exist. These algebras lead to interesting consequences such as orthostatistics.

Considering orthostatistics, we may ask whether there exist examples of this exotic form of statistics in nature. Condensed matter physics is a rich field where such possibilities may be relevant. In an earlier paper [6] we have already pointed out the connection of orthofermi statistics to the Hubbard model of strongly coupled electrons in condensed matter systems. In fact, this was our original motivation for the construction of orthofermi statistics. Even more exciting possibilities may be contemplated in high energy physics. Let the Latin index in our algebra of (5) be reinterpreted to denote collectively all the indices (space, spin and other internal quantum numbers) which are known to characterize a particle and let the Greek index denote a new degree of freedom which may be excited at some higher energy scale. Then orthofermions and orthobosons will obey usual Fermi-Dirac and Bose-Einstein statistics as far as all the conventional degrees of freedom are concerned, but the new statistics will manifest itself in the new degree of freedom.

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Appendix 1. More general algebras

The nine-parameter algebra: We consider the nine-parameter algebra

$$p_1 c_{k\alpha} c^\dagger_{m\beta} + p_2 \delta_{\alpha\beta} R_{km} + p_3 \delta_{km} R_{\alpha\beta} + p_0 \delta_{km} \delta_{\alpha\beta} R - q_1 c^\dagger_{m\beta} c_{k\alpha} - q_2 \delta_{\alpha\beta} T_{mk} - q_3 \delta_{km} T_{\beta\alpha} - q_0 \delta_{km} \delta_{\alpha\beta} T = s \delta_{km} \delta_{\alpha\beta} \tag{A1}$$

where

$$\begin{aligned}
&\sum_{k,\alpha} c_{k\alpha} c^\dagger_{k\alpha} = R \quad ; \quad \sum_{k,\alpha} c^\dagger_{k\alpha} c_{k\alpha} = T \\
&\sum_{\alpha} c_{k\alpha} c^\dagger_{m\alpha} = R_{km} \quad ; \quad \sum_{\alpha} c^\dagger_{m\alpha} c_{k\alpha} = T_{mk} \\
&\sum_{k} c_{k\alpha} c^\dagger_{k\beta} = R_{\alpha\beta} \quad ; \quad \sum_{k} c^\dagger_{k\beta} c_{k\alpha} = T_{\beta\alpha}
\end{aligned} \tag{A2}$$

As mentioned in § 2, the parameters p_1 and s can be easily eliminated by suitable redefinition of c and $c^\dagger$, in which case (A1) will reduce to the 7-parameter algebra of (4), but we start with (A1) for the sake of greater generality. Let n_1 and n_2 be the number of spin (Greek) and space (Latin) indices respectively, both taken to be finite numbers at first and s, p_i and q_i are real parameters. We shall show that for general values of the parameters the algebra (A1) can be reduced to the four-parameter algebra of § 2. There will be a few exceptional cases which we shall consider separately. The limiting case of infinite n_2 also will be taken up subsequently.

Put $\alpha = \beta$ in (A1) and sum over α. We get

$$(p_1 + p_2 n_1) R_{km} + (p_3 + p_0 n_1)\delta_{km} R - (q_1 + q_2 n_1)\, T_{mk} - (q_3 + q_0 n_1)\delta_{km}\, T = s n_1 \delta_{km}. \tag{A3}$$

Next, putting $k = m$ in (A1) and summing over k, we get

$$(p_1 + p_3 n_2) R_{\alpha\beta} + (p_2 + p_0 n_2)\delta_{\alpha\beta} R - (q_1 + q_3 n_2)\, T_{\beta\alpha} - (q_2 + q_0 n_2)\delta_{\alpha\beta}\, T = s n_2 \delta_{\alpha\beta}. \tag{A4}$$

Finally, we put $k = m$ in (A3) and sum over k. This leads to

$$(p_1 + p_2 n_1 + p_3 n_2 + p_0 n_1 n_2) R - (q_1 + q_2 n_1 + q_3 n_2 + q_0 n_1 n_2)\, T = s n_1 n_2. \tag{A5}$$

From these three equations (A3), (A4) and (A5), R, $R_{\alpha\beta}$ and R_{km} can be obtained in terms of T, $T_{\beta\alpha}$ and T_{mk}:

$$R = \frac{s n_1 n_2 + (q_1 + q_2 n_1 + q_3 n_2 + q_0 n_1 n_2)\, T}{p_1 + p_2 n_1 + p_3 n_2 + p_0 n_1 n_2}, \tag{A6}$$

$$R_{km} = \frac{1}{(p_1 + p_2 n_1)} \left[\frac{s n_1 (p_1 + p_2 n_1) + \{(q_3 + q_0 n_1)(p_1 + p_2 n_1) - (p_3 + p_0 n_1)(q_1 + q_2 n_1)\}\, T}{p_1 + p_2 n_1 + p_3 n_2 + p_0 n_1 n_2} \delta_{km} + (q_1 + q_2 n_1)\, T_{mk} \right], \tag{A7}$$

$$R_{\alpha\beta} = \frac{1}{(p_1 + p_3 n_2)} \left[\frac{s n_2 (p_1 + p_3 n_2) + \{(q_2 + q_0 n_2)(p_1 + p_3 n_2) - (p_2 + p_0 n_2)(q_1 + q_3 n_2)\}\, T}{p_1 + p_2 n_1 + p_3 n_2 + p_0 n_1 n_2} \delta_{\alpha\beta} + (q_1 + q_3 n_2)\, T_{\beta\alpha} \right]. \tag{A8}$$

Substitution of these results into (A1) and some straightforward manipulations yield:

$$c'_{k\alpha}c'^{\dagger}_{m\beta} - q'_1 c'^{\dagger}_{m\beta}c'_{k\alpha} - q'_2\delta_{\alpha\beta}\sum_{\gamma} c'^{\dagger}_{m\gamma}c'_{k\gamma} - q'_3\delta_{km}\sum_{p} c'^{\dagger}_{p\beta}c'_{p\alpha}$$

$$- q'_0\delta_{km}\delta_{\alpha\beta}\sum_{p,\gamma} c'^{\dagger}_{p\gamma}c'_{p\gamma} = \delta_{km}\delta_{\alpha\beta}. \tag{A9}$$

where we have defined

$$c'_{k\alpha} = [s^{-1}(p_1 + p_2 n_1 + p_3 n_2 + p_0 n_1 n_2)]^{1/2} c_{k\alpha} \tag{A10}$$

$$q'_1 = \frac{q_1}{p_1};\quad q'_2 = \frac{q_2 p_1 - p_2 q_1}{p_1(p_1 + p_2 n_1)};\quad q'_3 = \frac{q_3 p_1 - p_3 q_1}{p_1(p_1 + p_3 n_2)}$$

$$q'_0 = \left[q_0 p_1 + q_1\left\{\frac{p_2 p_3 - p_0 p_1}{p_1 + p_2 n_1} + \frac{p_3(p_2 + p_0 n_2)}{p_1 + p_3 n_2}\right\} - \frac{q_2 p_1(p_3 + p_0 n_1)}{p_1 + p_2 n_1}\right.$$

$$\left. - \frac{q_3 p_1(p_2 + p_0 n_2)}{p_1 + p_3 n_2}\right]\{p_1(p_1 + p_2 n_1 + p_3 n_2 + p_0 n_1 n_2)\}^{-1}. \tag{A11}$$

Thus, in general, the algebra defined by (A1) is identical to the four-parameter algebra (1) considered in § 2.

The exceptional cases: The exceptional cases are given by some special values of the parameters which correspond to the vanishing of the denominators encountered in the above manipulations. We consider a few of these cases:

(i) $$p_1 + p_2 n_1 + p_3 n_2 + p_0 n_1 n_2 = 0. \tag{A12}$$

In this case, if R in (A6) is to be finite, we find

$$(q_1 + q_2 n_1 + q_3 n_2 + q_0 n_1 n_2)\, T = -s n_1 n_2. \tag{A13}$$

This equation is not consistent with the existence of the vacuum state $|0\rangle$ and hence is not possible in Fock representation. But as general algebras, such possibilities exist. In fact, the Cuntz algebra referred to in § 7 belongs to this category.

(ii) $$p_1 + p_2 n_1 = 0. \tag{A14}$$

Substitution of this into (A7) and the requirement of finiteness of R_{km} leads to the condition:

$$T_{mk} = \frac{1}{n_2} T\delta_{mk}. \tag{A15}$$

(iii) $$p_1 + p_3 n_2 = 0. \tag{A16}$$

This, along with the assumption of finiteness of $R_{\alpha\beta}$ in (A8) gives

$$T_{\beta\alpha} = \frac{1}{n_1} T\delta_{\beta\alpha}. \tag{A17}$$

Both (A15) and (A17) may lead to interesting constraints on algebras. Finally we consider two more exceptional cases:

(iv) $p_1 = 0.$ (A18)

If, in addition to this condition, p_2, p_3 and p_0 also vanish, then (A1) will be inconsistent with the existence of $|0\rangle$; otherwise, it is a possibility.

(v) $s = 0.$ (A19)

In this case, one can verify that all matrix elements in Fock space are zero. So, Fock representation does not exist.

Infinite sums: So far we assumed n_1 and n_2 to be finite; we now consider the limit of infinite n_2. The results will depend on whether R, T, $R_{\alpha\beta}$ and $T_{\beta\alpha}$ (defined in (A2)) which now involve infinite sums are finite or not. Let us first prove that all these four quantities cannot be finite. This is proved by first assuming that they are finite and obtaining a contradiction. Thus, for $n_2 \to \infty$, with R, T, $R_{\alpha\beta}$ and $T_{\beta\alpha}$ assumed finite, eqs (A6–A8) become

$$R = \frac{sn_1 + (q_3 + q_0 n_1)\,T}{p_3 + p_0 n_1}, \tag{A20}$$

$$R_{km} = \frac{q_1 + q_2 n_1}{p_1 + p_2 n_1}\,T_{mk}, \tag{A21}$$

$$R_{\alpha\beta} = \frac{sp_3 + (q_0 p_3 - p_0 q_3)\,T}{p_3(p_3 + p_0 n_1)}\,\delta_{\alpha\beta} + \frac{q_3}{p_3}\,T_{\beta\alpha}. \tag{A22}$$

Substituting these into (A1), we get

$$c_{k\alpha} c^{\dagger}_{m\beta} - q'_1 c^{\dagger}_{m\beta} c_{k\alpha} - q'_2 \delta_{\alpha\beta} \sum_{\gamma} c^{\dagger}_{m\gamma} c_{k\gamma} = 0, \tag{A23}$$

where q'_1 and q'_2 are the same as defined in (A11). This algebra (A23) is inconsistent with the existence of the vacuum state. Hence within Fock space, the algebra of (A1) with infinite sums over latin indices but finite R, T, $R_{\alpha\beta}$ and $T_{\beta\alpha}$, is not possible.

However other possibilities exist, if some of these quantities are allowed to become infinite. The simplest one is to take the algebra of (A1) without R and $R_{\alpha\beta}$. This situation can be analyzed by putting $p_3 = p_0 = 0$ in (A6)–(A8). One finds that these equations are consistent for infinite n_2 if R and $R_{\alpha\beta}$ are infinite, but T and $T_{\beta\alpha}$ finite. In this case, R_{km} can still be expressed in terms of T_{mk} and hence finally one gets back the four-parameter algebra of (1) again.

Algebras defined by two equations: Let us now enquire whether an algebra can be defined by more than one equation. Consider the algebra defined by the following two equations which are taken to be simultaneously valid:

$$c_{k\alpha} c^{\dagger}_{m\beta} - q_1 c^{\dagger}_{m\beta} c_{k\alpha} - q_2 \delta_{\alpha\beta}\,T_{mk} - q_3 \delta_{km}\,T_{\beta\alpha} - q_0 \delta_{km} \delta_{\alpha\beta}\,T = \delta_{km} \delta_{\alpha\beta} \tag{A24}$$

$$c_{k\alpha} c^{\dagger}_{m\beta} - t_1 c^{\dagger}_{m\beta} c_{k\alpha} - t_2 \delta_{\alpha\beta}\,T_{mk} - t_3 \delta_{km}\,T_{\beta\alpha} - t_0 \delta_{km} \delta_{\alpha\beta}\,T = t_4 \delta_{km} \delta_{\alpha\beta} \tag{A25}$$

where $q_1 \neq t_1$; $q_2 \neq t_2$; $q_3 \neq t_3$; $q_0 \neq t_0$; $t_4 \neq 1$ and T, T_{mk} and $T_{\beta\alpha}$ are as defined in (A2). Let n_1 and n_2 be the number of spin and space indices respectively. By straightforward manipulations on these two equations, it is possible to show

$$c_{k\alpha}c^{\dagger}_{m\beta} = f\,\delta_{km}\delta_{\alpha\beta}, \tag{A26}$$

$$c^{\dagger}_{m\beta}c_{k\alpha} = g\,\delta_{km}\delta_{\alpha\beta}, \tag{A27}$$

where f and g are known functions of q_i, t_i, n_1 and n_2. Thus, in general, the two equations (A24) and (A25) reduce both the operator products $c_{k\alpha}\,c^{\dagger}_{m\beta}$ and $c^{\dagger}_{m\beta}\,c_{k\alpha}$ to trivial forms and further (A27) is inconsistent with vacuum state. Hence, we conclude that, nontrivial algebras leading to Fock representation cannot be defined by giving two general equations. However, for special values of the parameters, nontrivial algebras are possible and these belong to the general class of algebras which include the Cuntz algebra [5].

Appendix 2. Inner product in the multiparticle sector

Multiplying the products of the operators on the right side of (13) explicitly, we have the 3-particle matrix element:

$$\begin{aligned}
&\langle 0|\, c_{t\gamma}c_{m\beta}c_{k\alpha}c^{\dagger}_{p\lambda}c^{\dagger}_{s\mu}c^{\dagger}_{q\nu}|0\rangle \\
&= I + q_1P^t_{21} + q_2P^0_{21} + q_1P^t_{32} + q_2P^0_{32} + q_1^2P^t_{21}P^t_{32} + q_2^2P^0_{21}P^0_{32} \\
&\quad + q_1q_2P^t_{21}P^0_{32} + q_1q_2P^0_{21}P^t_{32} + q_1^2P^t_{32}P^t_{21} + q_2^2P^0_{32}P^0_{21} + q_1q_2P^t_{32}P^0_{21} \\
&\quad + q_1q_2P^0_{32}P^t_{21} + q_1^3P^t_{21}P^t_{32}P^t_{21} + q_1^2q_2P^t_{21}P^t_{32}P^0_{21} + q_1^2q_2P^0_{21}P^t_{32}P^t_{21} \\
&\quad + q_1q_2^2P^0_{21}P^t_{32}P^0_{21} + 2q_1q_2^2P^t_{21}P^0_{32}P^0_{21} + q_2(q_1^2+q_2^2)P^0_{21}P^0_{32}P^0_{21}
\end{aligned} \tag{A28}$$

where we have used the identities

$$P^t_{21}P^0_{32}P^0_{21} = P^0_{21}P^0_{32}P^t_{21}, \tag{A29}$$

$$P^0_{21}P^0_{32}P^0_{21} = P^t_{21}P^0_{32}P^t_{21}. \tag{A30}$$

These identities can be proved by decomposing the total exchange operators into orbital and spin exchanges:

$$P^t_{ij} = P^0_{ij}P^s_{ij} \tag{A31}$$

and then using the commutativity of P^0 and P^s:

$$[P^0_{ij}, P^s_{kl}] = 0. \text{ (for any } ij \text{ and } kl). \tag{A32}$$

A few important features of the formula (A28) may be noted.

(i) Among the 36 possible states obtained by permuting the space and spin indices $(p, s, q;\ \lambda, \mu, \nu)$ of the 3-particle state, there exist nonvanishing inner products only for the 19 permutations explicitly written down in (A28). All other permutations lead to vanishing inner products, since they lead to pure spin exchanges.

(ii) The operators on the right of (A28) have been written in the standard form arising from the products in (13); but there are other equivalent forms for some of the terms. For instance, note

$$P^t_{21}P^t_{32}P^t_{21} = P^t_{32}P^t_{21}P^t_{32}. \tag{A33}$$

(iii) The matrix element is a monomial of the form $q_1^J q_2^K$ except when identities [15] of the type (A29) or (A30) connect terms occurring in the standard form to other terms also occurring in the standard form. In these exceptional cases, either the monomial $q_1^J q_2^K$ acquires a numerical coefficient larger than unity or is replaced by a polynomial of more than a single term. Examples are the last two terms on the right of (A28): $2q_1q_2^2$ and $q_2(q_1^2+q_2^2)$.

In the n-particle sector, the expansion in eq. (13) contains

$$\prod_{i=1}^{n}(2^i-1)$$

terms which is smaller than the total number of permutations of space and spin indices which is $(n!)^2$. This actual number of distinct nonvanishing matrix elements is further reduced because of the identities of the type (A29) or (A30). For 3 particles, there were only two; for 4 particles, there are more than 60 identities.

Appendix 3. Positivity of the metric in the three and four particle sectors

The three-particle sector with distinct space indices k, m, p and distinct spin indices α, β, γ consists of 36 states. We consider the smaller problem in which the space indices are distinct, but the spin index ranges over only two values (denoted by σ and $\bar{\sigma}$) so that the total number of states is only 18.

As already pointed out in § 4, the region of positivity is larger if the range of indices is smaller. So, for the two-parameter algebra with the space index ranging over 1 to ∞, the largest region of positivity is likely to occur when the range of the spin index is two. (If the range of the spin index is one, the two-parameter algebra is reduced to a one-parameter algebra).

We introduce a convenient notation. The spin indices σ and $\bar{\sigma}$ will not be explicitly written but will be indicated respectively by the absence or presence of the bar above the spatial index. Thus.

$$|k\,m\,\bar{p}\rangle = c_k^\dagger c_m^\dagger c_{\bar{p}}^\dagger|0\rangle \equiv c_{k\sigma}^\dagger c_{m\sigma}^\dagger c_{p\bar{\sigma}}^\dagger|0\rangle \tag{A34}$$

$$\langle k\,m\,\bar{p}| = \langle 0|\,c_{\bar{p}}c_m c_k \equiv \langle 0|\,c_{p\bar{\sigma}}c_{m\sigma}c_{k\sigma}. \tag{A35}$$

The 18×18 matrix of inner products in this three-particle sector is given below:

$$M_3 = \begin{array}{c|ccc} & |km\bar{p}\rangle & |k\bar{m}p\rangle & |\bar{k}mp\rangle \\ \hline \langle km\bar{p}| & A & D & E \\ \langle k\bar{m}p| & D^T & B & F \\ \langle \bar{k}mp| & E^T & F^T & C \end{array} \tag{A36}$$

where A, B, C, D, E and F are 6×6 matrices given below and the superscript T denotes transpose:

$A=$	$\lvert 12\bar{3}\rangle$	$\lvert 13\bar{2}\rangle$	$\lvert 31\bar{2}\rangle$	$\lvert 32\bar{1}\rangle$	$\lvert 23\bar{1}\rangle$	$\lvert 21\bar{3}\rangle$
$\langle 12\bar{3}\rvert$	1	q_2	$(q_1+q_2)q_2$	x	$(q_1+q_2)q_2$	q_1+q_2
$\langle 13\bar{2}\rvert$		1	q_1+q_2	$(q_1+q_2)q_2$	x	$(q_1+q_2)q_2$
$\langle 31\bar{2}\rvert$			1	q_2	$(q_1+q_2)q_2$	x
$\langle 32\bar{1}\rvert$				1	q_1+q_2	$(q_1+q_2)q_2$
$\langle 23\bar{1}\rvert$					1	q_2
$\langle 21\bar{3}\rvert$						1

$B=$	$\|1\bar{2}3\rangle$	$\|1\bar{3}2\rangle$	$\|3\bar{1}2\rangle$	$\|3\bar{2}1\rangle$	$\|2\bar{3}1\rangle$	$\|2\bar{1}3\rangle$
$\langle 1\bar{2}3\|$	1	q_2	q_2^2	y	q_2^2	q_2
$\langle 1\bar{3}2\|$		1	q_2	q_2^2	y	q_2^2
$\langle 3\bar{1}2\|$			1	q_2	q_2^2	y
$\langle 3\bar{2}1\|$				1	q_2	q_2^2
$\langle 2\bar{3}1\|$					1	q_2
$\langle 2\bar{1}3\|$						1

$C=$	$\|\bar{1}23\rangle$	$\|\bar{1}32\rangle$	$\|\bar{3}12\rangle$	$\|\bar{3}21\rangle$	$\|\bar{2}31\rangle$	$\|\bar{2}13\rangle$
$\langle \bar{1}23\|$	1	q_1+q_2	$(q_1+q_2)q_2$	z	$(q_1+q_2)q_2$	q_2
$\langle \bar{1}32\|$		1	q_2	$(q_1+q_2)q_2$	z	$(q_1+q_2)q_2$
$\langle \bar{3}12\|$			1	q_1+q_2	$(q_1+q_2)q_2$	z
$\langle \bar{3}21\|$				1	q_2	$(q_1+q_2)q_2$
$\langle \bar{2}31\|$					1	q_1+q_2
$\langle \bar{2}13\|$						1

A, B and C are symmetric matrices and we have defined

$$x=q_2(q_1^2+q_2^2+q_1q_2)$$
$$y=q_1^3+q_1^2q_2+q_2^3$$
$$z=(q_1+q_2)^2q_2$$

$D=$	$\|1\bar{2}3\rangle$	$\|1\bar{3}2\rangle$	$\|3\bar{1}2\rangle$	$\|3\bar{2}1\rangle$	$\|2\bar{3}1\rangle$	$\|2\bar{1}3\rangle$
$\langle 12\bar{3}\|$	0	q_1	q_1q_2	u	v	0
$\langle 13\bar{2}\|$	q_1	0	0	v	u	q_1q_2
$\langle 31\bar{2}\|$	v	0	0	q_1	q_1q_2	u
$\langle 32\bar{1}\|$	u	q_1q_2	q_1	0	0	v
$\langle 23\bar{1}\|$	q_1q_2	u	v	0	0	q_1
$\langle 21\bar{3}\|$	0	v	u	q_1q_2	q_1	0

$E=$	$\|\bar{1}23\rangle$	$\|\bar{1}32\rangle$	$\|\bar{3}12\rangle$	$\|\bar{3}21\rangle$	$\|\bar{2}31\rangle$	$\|\bar{2}13\rangle$
$\langle 12\bar{3}\|$	0	0	q_1^2	w	0	0
$\langle 13\bar{2}\|$	0	0	0	0	w	q_1^2
$\langle 31\bar{2}\|$	0	0	0	0	q_1^2	w
$\langle 32\bar{1}\|$	w	q_1^2	0	0	0	0
$\langle 23\bar{1}\|$	q_1^2	w	0	0	0	0
$\langle 21\bar{3}\|$	0	0	w	q_1^2	0	0

$F=$	$\|\bar{1}23\rangle$	$\|\bar{1}32\rangle$	$\|\bar{3}12\rangle$	$\|\bar{3}21\rangle$	$\|\bar{2}31\rangle$	$\|\bar{2}13\rangle$
$\langle 1\bar{2}3\|$	0	0	q_1q_2	q_2v	v	q_1
$\langle 1\bar{3}2\|$	0	0	q_1	v	q_2v	q_1q_2
$\langle 3\bar{1}2\|$	v	q_1	0	0	q_1q_2	q_2v
$\langle 3\bar{2}1\|$	q_2v	q_1q_2	0	0	q_1	v
$\langle 2\bar{3}1\|$	q_1q_2	q_2v	v	q_1	0	0
$\langle 2\bar{1}3\|$	q_1	v	q_2v	q_1q_2	0	0

where

$$u = q_1 q_2(q_1 + 2q_2)$$
$$v = q_1(q_1 + q_2)$$
$$w = q_1^2(q_1 + q_2).$$

By a series of straightforward but tedious manipulations, we have reduced the determinant of the above 18 × 18 matrix to the following product of factors:

$$\det M_3 = (\det Z_3)(\det X^{(+)})(\det X^{(-)})(\det Y)^2 \tag{A37}$$

where

$$\det Z_3 = \{1 - (q_1 + q_2)^6\}\{1 - (q_1 + q_2)^2\}^6. \tag{A38}$$

Actually, Z_3 is the 6 × 6 matrix of inner products in the three-particle sector for the q-mutator algebra:

$$c_k c_m^\dagger - (q_1 + q_2) c_m^\dagger c_k = \delta_{km}. \tag{A39}$$

Further, $X^{(+)}$ and $X^{(-)}$ are 2 × 2 matrices while Y is a 4 × 4 matrix. Their matrix elements are given below:

$$X_{11}^{(\pm)} = (1 - q^2 + 3q_2^2) \mp (q - q^3 - 3q_2 - 2q_2^3 - q q_2^2 + 3q^2 q_2),$$
$$X_{22}^{(\pm)} = (1 - q^2 - q_2^2 + 4q q_2) \mp (-q + q^3 - q_2 + q q_2^2 - 3q^2 q_2^2),$$
$$X_{12}^{(\pm)} = (2q q_2 - 2q_2^2) \mp (-q + q^3 + q_2 + 2q q_2^2 - 3q^2 q_2),$$
$$X_{21}^{(\pm)} = \mp (q q_2^2 - q_2^3),$$

where

$$q = q_1 + q_2,$$
$$Y_{11} = 1 + q_1^3 + (1 - 2q_2) q_1 q_2 - [1 + (1 - q_2) q_2] q_2,$$
$$Y_{12} = -[1 - q_1 - 2q_2^2] q_1 - q_1^3 + (1 - q_2^2) q_2,$$
$$Y_{13} = -[1 - (1 - q_1 - q_2)(q_1 - q_2)] q_1,$$
$$Y_{14} = -[(2 - q_1) q_1 + q_2^2] q_1,$$
$$Y_{21} = -(1 + q_1) q_1,$$
$$Y_{22} = 1 + [q_1 + (1 + 2q_2) q_2] q_1 - q_1^3 + (1 - q_2 - q_2^2) q_2,$$
$$Y_{23} = -(1 - 2q_1) q_1,$$
$$Y_{24} = [1 - (1 - q_1) q_1 - (1 + q_2) q_2] q_1,$$
$$Y_{31} = -(1 + q_1 + q_2^2) q_1,$$
$$Y_{32} = -(1 - 2q_1 - q_2^2) q_1,$$
$$Y_{33} = 1 + (1 - q_1) q_1^2 - (1 - 2q_2) q_1 q_2 - (1 + q_2 - q_2^2) q_2,$$
$$Y_{34} = [1 - (1 - q_1) q_1] q_1 - 2q_1 q_2^2 + (1 - q_2^2) q_2,$$
$$Y_{41} = -2(1 + q_1) q_1,$$
$$Y_{42} = (1 + q_1 + q_1^2) q_1,$$
$$Y_{43} = (1 + q_1) q_1,$$
$$Y_{44} = 1 + q_1^3 - (1 + 2q_2) q_1 q_2 + (1 - q_2 - q_2^2) q_2.$$

The boundary of the positivity of M_3 is determined by the zeros of $\det M_3$ which lie closest to the origin $q_1 = q_2 = 0$ in figure 1. The first factor $\det Z_3$ on the right of (A37) has real zeros only for $q_1 + q_2 = \pm 1$, which correspond to the straight lines BC and FG in figure 1; $\det X^{(+)}$ has real zeros along the lines BG and GF while the real zeros of $\det X^{(-)}$ occur along the lines FC and BC. It is the last factor $\det Y$ which is found to have zeros not only along the boundaries of the square BCFG but also along the curved lines lying inside BCFG shown in figure 1. Thus, these curved lines become the boundary of the region of positivity for the three-particle sector.

We have also done the calculations for the three-particle sector with spin index taking three values and for the four-particle sector with spin index taking two values which lead to 36×36 and 144×144 matrices respectively. Our numerical results show that the boundary of the region of positivity shrinks further.

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[15] We must distinguish these identities (A29) or (A30) from identities such as (A33) which connect the terms occurring in the standard form to other equivalent terms not occurring in the standard form.