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The electroweak mixing angle in unified gauge theories

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Abstract. We study in detail the factors that influence the unification relations among the coupling parameters of strong and electroweak interactions. We find that the factor that decides the unification relations in a theory is the fermion content of the theory. The specific 'observed' group of strong and electroweak interactions used and the specific unification group in which these interactions are embedded are largely irrelevant. In particular, we find that the unification value of the electroweak mixing angle is the same for almost all models of interest. We also explicitly illustrate that the canonical value 3/8 of the mixing angle is a characteristic result of the currently popular sequential doublets scheme of fermions. Addition of extra fermion singlets reduces the mixing angle to 1/4. We propose this sequential triplets scheme of fermions as an interesting alternative to the current scheme.

Keywords. Grand unified models; neutral-currents; Weinberg-Salam model; left-right symmetric model; electroweak mixing angle.

1. Introduction

The SU(5) grand unification scheme (Georgi and Glashow 1974) is fast acquiring the status of an orthodoxy. The scheme incorporates strong interactions described by the unbroken colour group $SU(3)_C$, and the electromagnetic and weak (electroweak) interactions described by the minimal group $SU(2)_L \times U(1)$ (Weinberg 1967; Salam 1968). On unification under SU(5) the three coupling constants g_S, g_L and g_Y associated with the gauge groups $SU(3)_C$, $SU(2)_L$ and U(1), respectively, get related to each other. The weak mixing angle $\sin^2\theta$, defined to be e^2/g_L^2, and the ratio of the fine structure constant to the strong coupling constant, e^2/g_S^2, then get fixed at 3/8. This unification value of $\sin^2\theta$ and e^2/g_S^2 is expected to hold only at energies higher than the mass of the heaviest vector-bosons in the theory, which may be called the unification mass. Since in the SU(5) scheme baryon number is not conserved, and proton decay can occur in the first order of weak interactions, the unification mass has to be rather large ($\sim 10^{16}$ GeV) to make the proton sufficiently stable. Using the renormalisation group equation it has been shown that to obtain the correct value of $e^2/g_S^2 \approx 1/30$ at low energies (≈ 10 GeV), the unification mass, i.e., energy at which e^2/g_S^2 rises to 3/8, must indeed be of the order of 10^{16} GeV (Georgi *et al* 1974). The value of $\sin^2\theta$ at 10 GeV then turns out to be $\sim 0{\cdot}20$, in good agreement with the experimental number.

The above prediction for the mixing angle is the only experimentally accessible result of the SU(5) grand unification scheme. The model, however, is taken seriously enough to calculate quantities like baryon excess in the universe (e.g., Yoshimura 1978) and to speculate about the possible reasons for the near equality of the unification mass in SU(5) and the Planck mass (e.g., Zee 1979). Therefore, it should be of interest to see how far the only experimentally testable consequence of SU(5), i.e., the predicted value of the weak mixing angle $\sin^2\theta$, is unique to the SU(5) scheme. The question we ask is: Does this unification result depend on the specific choice of the unifying gauge group?

The question of the correct observed gauge group (i.e., the group relevant at the observed energies) for the strong and electroweak interactions is also by no means settled. Though the standard model for the strong and electroweak interactions based on the group $SU(3)_C \times SU(2)_L \times U(1)$ is quite plausible, alternative models which explain the experimental results equally well, can also be constructed. In particular the integrally charged quark model for strong interactions (Rajasekaran and Roy 1975; Pati and Salam 1976), and the left-right symmetric model for the electroweak interactions (Bajaj and Rajasekaran 1979a, and references cited therein) may be mentioned. Does the unification result for $\sin^2\theta$ depend critically on the choice of the observed group for strong and electroweak interactions?

In this paper (paper I) we try to answer these questions for the algebraic results obtained on unification. The renormalization effects will depend on the size of the unifying group and also on the observed group, but there is always sufficient freedom to adjust these effects to any desirable level. We shall, however, postpone the discussion of the renormalization effects to a subsequent paper (Bajaj and Rajasekaran 1979b, referred to as paper II in the text).

By deriving the unification relations among the coupling constants in a general manner (§ 2), we first show that these relations depend only on the set of fermions in the theory and on their classification under the observed group. These algebraic relations among the coupling constants are found to be the same for a large class of unification groups. Specifically, if the number of fermions in the theory is N_f, then the unification relations are the same for a maximal unifying group $SU(2N_f)$, and any simple or semi-simple unifying subgroup thereof.

The dependence of these relations on the observed group is analysed next (§ 3). We find that for any electroweak group $SU(2)_L \times U(1) \times G'$ the weak mixing angle defined as $\sin^2\theta = e^2/g_L^2$ has the same unification value as for the minimal group $SU(2)_L \times U(1)$, if the classification of fermions under $SU(2)_L$ remains same in all cases. This result is of special relevance for the left-right symmetric models based on the group $SU(2)_L \times SU(2)_R \times U(1)$, for which we have earlier shown that the neutral-current couplings have the same functional dependence on $\sin^2\theta$ as in the Weinberg-Salam (W-S) model (Bajaj and Rajasekaran 1979a). Therefore, we prove in detail the equivalence of $\sin^2\theta$ for $SU(2)_L \times U(1)$ and $SU(2)_L \times SU(2)_R \times U(1)$ and show that the generalisation of this result to $SU(2)_L \times U(1) \times G'$ is straightforward. We also show that the unification value of $\sin^2\theta$ (appropriately defined) does not depend on whether the colour group $SU(3)_C$ is broken (integrally charged quark model) or not (fractionally charged quark model).

We illustrate these results in two examples. We first take the standard scheme according to which fermions come as sequential doublets of $SU(2)_L$ (§ 4). We show that in this case the SU(5) result, $e^2/g_S^2 = \sin^2\theta = 3/8$, is obtained without any reference to the details of the unification group or of the observed group. As our second example (§ 5), we take the case where fermions appear as sequential triplets of $SU(3)_L$ forming a doublet and a singlet of $SU(2)_L$, in each case. In constructing this example, our motivation was to look for a model that gives a low unification value of $\sin^2 \theta$, so that large renormalisation corrections need not be invoked. We find that for this simple example the value of $\sin^2 \theta$ is 1/4, rather near the experimental number. Once again we do not need to specify either the unification group or the observed group.

In the last section (§ 6), we have summarised the results of this paper, and have discussed their significance.

2. Algebraic relations in the unification limit

In this section, we give a general analysis of the algebraic relations among the coupling constants of the strong, weak and electromagnetic interactions that follow when these interactions are embedded in a unified gauge group.

Let $G_0 = G_A \times G_B \times G_C \ldots$ (where G_A, G_B, $G_C \ldots$ are simple) be the observed group of strong, weak and electromagnetic interactions with corresponding coupling constants g_A, g_B, $g_C \ldots$. If G_0 is embedded in a unifying group G with a single coupling constant g_G, then the various observed coupling constants, g_A, $g_B \ldots$, get related to g_G. There are many choices possible for the unifying group G. But the trick that facilitates a general analysis is to consider a maximal unifying group $G_{\max} = SU(2N_f)$, where N_f is the total number of fermions in the theory and $2N_f\,(=N)$ is the total number of left-handed and right-handed components (Fritzsch and Minkowski 1975). The algebraic relations among the coupling constants g_A, $g_B \ldots$, of the factor groups G_A, $G_B \ldots$, are derived for $G_{\max}$, but are automatically valid for any unifying group G which is a subgroup of $G_{\max}$.

The left-handed and right-handed fermions together form an N-dimensional multiplet ψ of $G_{\max}$. The interaction of the fermions with the gauge-bosons $W_{G\alpha}$ ($\alpha = 1, 2 \ldots$) of the unifying group $G = G_{\max}$ can be written as*:

$$L_G = i\, g_G\, (\bar{\psi}\, \gamma\, T_{G\alpha} \psi)\, W_{G\alpha}, \tag{1}$$

where $T_{G\alpha}$ are the generators of G in the N-dimensional fermion representation. We normalise $T_{G\alpha}$ such that $\mathrm{Tr}\,(T_{G\alpha})^2$ is independent of α, and

$$\mathrm{Tr}\,(T_{G\alpha}\, T_{G\beta}) = \delta_{\alpha\beta}\, \mathrm{Tr}\, T_G^2. \tag{2}$$

The observed interaction Lagrangian on the other hand is

$$L_0 = i \sum_{A,B,\ldots} g_A\, (\bar{\psi}\, \gamma\, T_{Ai} \psi)\, W_{Ai}, \tag{3}$$

where T_{Ai} are $N \times N$ matrices representing the operation of the generators of the group G_A on the fermion multiplet ψ and W_{Ai} are the corresponding gauge fields.

*The space-time four-vector index μ has been suppressed wherever convenient.

In general, T_{Ai} form reducible representations of the algebra G_A. Normalisation of T^{Ai} also is such that Tr $(T_{Ai})^2$ is independent of i and

$$\mathrm{Tr}\,(T_{Ai}\,T_{Aj}) = \delta_{ij}\,\mathrm{Tr}T_A^2. \tag{4}$$

Since the observed interactions are obtained from L_G by the spontaneous breaking of group G to the level of G_0, it is clear that L_0 is contained in L_G. The fields W_{Ai} are in general linear combinations of $W_{G\alpha}$. However, since the original gauge fields $W_{G\alpha}$ were arbitrary with respect to linear orthogonal transformations we can, without loss of generality, identify each of the 'physical' gauge fields W_{Ai} with some $W_{G\alpha}$. So L_0 is recognised to be a part of L_G, and by comparing (1) and (3) we get

$$g_G\,T_{G\alpha} = g_A\,T_{Ai},\; g_G\,T_{G\beta} = g_B\,T_{Bj},\; \text{etc.,}$$

and hence, using (2) and (4),

$$g_G^2\,\mathrm{Tr}\,T_G^2 = g_A^2\,\mathrm{Tr}\,T_A^2 = g_B^2\,\mathrm{Tr}\,T_B^2 = \ldots \tag{5}$$

The above equation (5) is the basic algebraic relationship connecting each of the coupling constants g_A, $g_B, \ldots$ of the observed interactions among themselves, and to the coupling constant g_G of the unifying group. Since the details of the symmetry breaking or of the symmetry groups in the intermediate stages do not play any role in the above derivation of equation (5), it is clear that this result will hold for any unifying group (which will be a subgroup of $G_{\max}$) occurring in the intermediate stages. Therefore, the relationship among the coupling constants g_A, $g_B \ldots$ does not depend on the specific choice of the unifying group, nor on the manner in which it is broken to the observed group G_0. The only relevant factors in determining the algebraic relations among the observed coupling constants g_A, $g_B \ldots$ are

(i) the set of fermions in the theory, and

(ii) the classification of fermions under various components G_A, $G_B \ldots$ of G_0, which fix Tr T_A^2, Tr T_B^2, ...

3. Comparison of standard model with the alternatives

In this section we show that the unification value of the weak mixing angle remains the same in a large class of models. In particular, we compare the standard model, wherein the strong interactions are described by the unbroken colour SU(3) and the electroweak interactions by the minimal $\mathrm{SU(2)}_L \times \mathrm{U(1)}$ with the left-right symmetric model in which the electroweak group is $\mathrm{SU(2)}_L \times \mathrm{SU(2)}_R \times \mathrm{U(1)}$ instead of $\mathrm{SU(2)}_L \times \mathrm{U(1)}$.

We have shown earlier that the neutral-current sector of the left-right symmetric models mimics the neutral-current sector of the W-S model making the two models phenomenologically indistinguishable (Bajaj and Rajasekaran 1979a). Here, using the general results of § 1, we show that even the predicted unification value of the mixing angle is the same for both the models.

The standard model is based on the observed group G_0=SU(3)$_C$ × SU(2)$_L$ × U(1) with coupling constants g_S, g_L, and g_Y, respectively. The interaction between the fermions and the gauge fields in this model is given by

$$L_0\,(W\text{–}S) = i\,[g_S \sum_{i=1}^{8} (\bar{\psi}\,\gamma_\mu\,T_{Ci}\,\psi)\,V_i^\mu + g_L \sum_{i=1}^{3} (\bar{\psi}\,\gamma_\mu\,T_{Li}\,\psi)\,W_i^\mu$$

$$+\, g_Y (\bar{\psi}\,\gamma_\mu\,T_Y\,\psi)\,B^\mu], \tag{6}$$

where T_{Ci}, T_{Li} and T_Y are $N \times N$ matrices representing the operation of the usual SU(3), SU(2) and U(1) generators $(\lambda_i/2)$, $(\tau_i/2)$ and Y, respectively on the fermion multiplet ψ: and V_i, W_i, B are the corresponding gauge fields in appropriate representation. The explicit forms of the matrices will depend on the choice of the fermion multiplet ψ and will be exhibited in §§ 4 and 5.

The left-right symmetric model is described by the group $G_0' =$ SU(3)$_C$ × SU(2)$_L$ × SU(2)$_R$ × U(1) with corresponding coupling constants g_S', g_L', g_R' and g_Y'. The interaction of the fermions with the gauge fields is given by

$$L_0\,(L\text{–}R) = i\left[\,g_S' \sum_{i=1}^{8} (\bar{\psi}\,\gamma_\mu\,T_{Ci}'\,\psi)\,V_i^\mu + g_L' \sum_{i=1}^{3} (\bar{\psi}\,\gamma_\mu\,T_{Li}'\,\psi)\,W_{Li}^\mu\right.$$

$$\left.+\, g_R' \sum_{i=1}^{3} (\bar{\psi}\,\gamma_\mu\,T_{Ri}'\,\psi)\,W_{Ri}^\mu + g_Y' (\bar{\psi}\,\gamma_\mu\,T_Y'\,\psi)\,B^\mu\right], \tag{7}$$

where, as above, T''s are $N \times N$ matrices representing the operation of the generators of the corresponding factor groups of G_0' on ψ; and V_i, W_{Li}, W_{Ri} and B are the appropriate gauge fields.

About the fermion content of the two models we assume that the set of fermions is the same for both the standard and the left-right symmetric model. This ensures that the maximal unifying group $G_{\max}$ is the same in both cases. We further assume that the classification of fermions under the common factors SU(3)$_C$ and SU(2)$_L$ of the observed groups is also the same in the two models. Though not essential for the proof of our main results, we also assume that the fermions transform symmetrically under SU(2)$_L$ × SU(2)$_R$. Thus if the left-handed fermions are assigned to the representation $\underline{n}$ of SU(2)$_L$ and the right-handed fermions are singlets of SU(2)$_L$, in the standard model, then in the left-right symmetric model, the left-handed and right-handed fermions transform as $(\underline{n}, \underline{1})$ and $(\underline{1}, \underline{n})$ respectively of SU(2)$_L$ × SU(2)$_R$.

These simple assumptions about the fermion content of the two models force a number of correspondences between the unification results for the two models. These results follow simply from the basic equation (5). First, consider the standard model. In this case, the coupling constant ratios are given by

$$g_S^2/g_L^2 = \mathrm{Tr}\;T_L^2/\mathrm{Tr}\;T_C^2;\;\; g_Y^2/g_L^2 = \mathrm{Tr}\;T_L^2/\mathrm{Tr}\;T_Y^2. \tag{8}$$

The traces of T_L^2 and T_C^2 can be evaluated merely by using the third components T_{L3} and T_{C3}. Also, it is more convenient to rewrite these formulae (8) in terms of the electric charge matrix Q, the electric coupling constant e, and the weak-electromagnetic mixing parameter $\sin^2\theta$, defined by

$$Q = T_{L3} + T_Y, \quad 1/e^2 = 1/g_L^2 + 1/g_Y^2, \quad \sin^2\theta = e^2/g_L. \tag{9}$$

Using equations (8) and (9), we get

$$\sin^2\theta = \mathrm{Tr}\ T_{L3}^2/\mathrm{Tr}\ Q^2, \quad e^2/g_S^2 = \mathrm{Tr}\ T_{C3}^2\ /\mathrm{Tr}\ Q^2. \tag{10}$$

In the left-right symmetric model, the basic unification relations obtained on using equation (5) are:

$$\begin{aligned} g_L'^2/g_R'^2 &= \mathrm{Tr}\ T_{R3}'^2/\mathrm{Tr}\ T_{L3}'^2; \\ g_S'^2/g_L'^2 &= \mathrm{Tr}\ T_{L3}'^2/\mathrm{Tr}\ T_{C3}'^2; \\ g_Y'^2/g_L'^2 &= \mathrm{Tr}\ T_{L3}'^2/\mathrm{Tr}\ T_Y'^2. \end{aligned} \tag{11}$$

Again we define

$$\begin{aligned} Q' &= T_{L3}' + T_{R3}' + T_Y'; \\ 1/e^2 &= 1/g_L'^2 + 1/g_R'^2 + 1/g_Y'^2; \\ \sin^2\theta' &= e^2/g_L'^2; \end{aligned} \tag{12}$$

and rewrite the latter two of equations (11) in the form

$$\sin^2\theta' = \mathrm{Tr}\ T_{L3}'^2/\mathrm{Tr}\ Q'^2; \quad e^2/g_S'^2 = \mathrm{Tr}\ T_{C3}'^2\ /\mathrm{Tr}\ Q'^2. \tag{13}$$

It is important to realise that the primed matrices of the left-right symmetric model as well as the unprimed matrices of the standard model act on the same N-dimensional representation ψ of $G_{\max}$. Further, since the classification of particles under $SU(3)_C$ and $SU(2)_L$ is identical in the two models, we have

$$T_{C3}' = T_{C3}, \tag{14}$$

and

$$T_{L3}' = T_{L3}. \tag{15}$$

Since the set of fermions is the same, the charge matrices are also the same. Therefore

$$Q' = Q. \tag{16}$$

However, note that because of the difference between the definitions of Q (equation (9)) and Q' (equation (12)), $T_Y' \neq T_Y$. But T_Y' and T_Y can always be eliminated in favour of Q' and Q, respectively.

As a consequence of (14), (15) and (16), we get, on comparing (10) and (13),

$$\sin^2\theta' = \sin^2\theta, \tag{17}$$

and

$$e^2/g_S'^2 = e^2/g_S^2. \tag{18}$$

Hence the algebraic consequences of unification, namely the values of the weak mixing angle and the ratio of the strong coupling to the fine structure constant are proved to be the same for the left-right symmetric as well as the standard model.* Notice that for this result we had to assume only that the classification of fermions under $SU(3)_C$ and $SU(2)_L$ is the same for both models. The additional assumption of left-right symmetric classification implies

$$T'_{L3} = T'_{R3}, \tag{19}$$

which coupled with the first of equations (11) gives

$$g'^2_L = g'^2_R. \tag{20}$$

Hence, once the fermions are classified in a left-right symmetric manner, there is no freedom in a grand unified model to choose the gauge coupling constants g'_L and g'_R differently.

The unification values of the mixing angle $\sin^2\theta$, and of the ratio e^2/g_S^2 are in fact the same for a much wider class of models. For any model based on an electroweak group $SU(2)_L \times U(1) \times G'$, one can define a mixing angle θ such that

$$\sin^2\theta = e^2/g_L^2. \tag{21}$$

The unification equation (5), now implies

$$g_G^2 \operatorname{Tr} T_G^2 = g_L^2 \operatorname{Tr} T_L^2 = g_S^2 \operatorname{Tr} T_C^2 = \ldots \tag{22}$$

Extension of the argument which led to equation (5) to the level where electromagnetic U(1) is the observed group gives

$$g_G^2 \operatorname{Tr} T_G^2 = e^2 \operatorname{Tr} Q^2. \tag{23}$$

Comparing (22) and (23), one gets

$$\sin^2\theta = \operatorname{Tr} T_L^2/\operatorname{Tr} Q^2 \quad \text{and} \quad e^2/g_S^2 = \operatorname{Tr} T_C^2/\operatorname{Tr} Q^2. \tag{24}$$

So it is clear that unification values of $\sin^2\theta$ and e^2/g_S^2 are independent of G'. These values will be the same for all models based on the group $SU(3)_C \times SU(2)_L \times U(1) \times G'$, provided these models involve the same set of fermions and the classification of fermions under the factors $SU(3)_C$ and $SU(2)_L$ is the same for all models. Our results for the left-right symmetric model are clearly a special case of this general result. However, unlike in the case of left-right symmetric models, the equality of

*A similar result has been obtained by Chanowitz *et al* (1977) using SO(10) as the unification group. Elias (1977) also studied the unification of the left-right symmetric model, but because of the choice of a phenomenologically inappropriate definition of the mixing angle the simple result (17) seems to have been missed.

$\sin^2\theta$ may not imply the equality of neutral-current couplings, in general. These equalities are of interest only in cases where at least some neutral-current sector has been shown to have the same dependence on $\sin^2\theta$ ($=e^2/g_L^2$, equation (2.1)) as in the standard model. This obviously covers the infinite class of models in which the ν-induced sectors can be identified with the standard model (Georgi and Weinberg 1978).

Finally we may remark that the unification value of the mixing angle will be the same in both the fractionally charged and the integrally charged quark models. In the latter case, at the final stage of symmetry breaking, $SU(3)_C$ also breaks such that the charge operator becomes

$$Q = Q_{\text{colour}} + Q_{\text{flavour}}, \tag{25}$$

where
$$Q_{\text{colour}} = T_{C3} + (1/\sqrt{3})\, T_{C8}, \tag{26}$$

and Q_{flavour} is the same as the charge operator in the fractionally charged models. In this case $\sin^2\theta$ defined as e^2/g_L^2 (equation (21)) will of course be different from the standard model. However, the mixing angle that appears in the neutral-current in the integrally-charged quark model is usually defined as (see, e.g., Rajasekaran and Roy 1975)

$$\sin^2\theta = (g_L^2 + g_Y^2)/g_L^2, \tag{27}$$

where g_L and g_Y refer to the coupling constants defined in the standard model case (equation (6)). On unification, then,

$$\sin^2\theta = \operatorname{Tr} T_L^2/(\operatorname{Tr} T_L^2 + \operatorname{Tr} T_Y^2) = \operatorname{Tr} T_L^2/\operatorname{Tr} Q_{\text{flavour}}^2. \tag{28}$$

A similar expression can obviously be obtained for the case when the electroweak model is left-right symmetric. Now, since Q_{flavour} is the same as the charge in the fractionally-charged quark model, unification value of $\sin^2\theta$ in the integrally charged model (equation (28)) is the same as in the standard model (equation (10)).

4. Sequential doublets scheme

In this and the next section, we illustrate the general results already derived for two specific fermion schemes. As our first example, we consider the sequential doublets scheme which is the favoured scheme for fermions at present. The quarks and leptons are assumed to occur in a sequence of $SU(2)_L$ doublets

$$\begin{pmatrix}\{u\}\\ \{d\}\end{pmatrix},\quad \begin{pmatrix}\{c\}\\ \{s\}\end{pmatrix},\quad \begin{pmatrix}\{t\}\\ \{b\}\end{pmatrix}\ldots,$$

$$\begin{pmatrix}\nu_e\\ e\end{pmatrix},\quad \begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix},\quad \begin{pmatrix}\nu_\tau\\ \tau\end{pmatrix}\ldots$$

Each quark comes in three varieties of colour, the curly bracket, $\{\ \}$, denoting the colour triplet. Let N_q be the number of quark flavours, so that $3N_q$ is the total number of coloured quarks and let N_l be the number of lepton flavours. The total number of fermions is

$$N_f = 3N_q + N_l\,.$$

The maximal unification group is $G_{\text{max}} = SU(2N_f)$ where $2N_f$ is the total number of left-handed and right-handed fermion components. The fermions belong to the $2N_f$ -dimensional representation of this group:

$$\psi = (\psi_L, \psi_R),$$

where $$\psi_L = (\underbrace{\{u_L\},\ \{d_L\},\ \{c_L\},\ \{s_L\},\ \dots}_{\longleftarrow\ 3N_q\ \longrightarrow}\ ;\ \underbrace{\nu_{eL},\ e_L,\ \nu_{\mu L},\ \mu_L,\ \dots}_{\longleftarrow\ N_l\ \longrightarrow}),$$

and ψ_R is given by a similar multiplet. For typographical convenience, the column vectors have been written as row vectors.

The electric charge operator in the fermionic representation is a $2N_f \times 2N_f$ matrix:

$$Q = \begin{pmatrix} Q_L & 0 \\ 0 & Q_R \end{pmatrix},$$

where Q_L and Q_R are $N_f \times N_f$ matrices:

$$Q_L = Q_R = \begin{array}{c} \left[\begin{array}{c|c} \begin{matrix} \frac{2}{3}\{1\} & & & & \\ & -\frac{1}{3}\{1\} & & & \\ & & \frac{2}{3}\{1\} & & \\ & & & -\frac{1}{3}\{1\} & \\ & & & & \ddots \end{matrix} & \\ \hline & \begin{matrix} 0 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & \ddots \end{matrix} \end{array}\right] \\ \begin{array}{cc} \longleftarrow 3N_q \longrightarrow & \longleftarrow N_l \longrightarrow \end{array} \end{array}$$

The curly matrix $\{1\}$ stands for the unit matrix in colour space:

$$\{1\} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

We now find

$$\mathrm{Tr}\ Q = 2\ \mathrm{Tr}\ Q_L = N_q - N_l,$$

$$\mathrm{Tr}\ Q^2 = 2\ \mathrm{Tr}\ Q_L^2 = \frac{5}{3} N_q + N_l.$$

Since Q is a generator of SU($2N_f$), we require

$$\mathrm{Tr}\ Q = 0,$$

which implies

$$N_q = N_l = N_f/4.$$

So, we have

$$\mathrm{Tr}\ Q^2 = \frac{2}{3} N_f.$$

The matrix corresponding to the third component of colour SU(3) is

$$\mathrm{T}_{C3} = \begin{pmatrix} T_{C3}\ (L) & 0 \\ 0 & T_{C3}\ (R) \end{pmatrix},$$

where

$$T_{C3}(L) = T_{C3}(R) = \left(\begin{array}{ccc|ccc} \{\lambda_3\} & & & & & \\ & \{\lambda_3\} & & & & \\ & & \ddots & & & \\ \hline & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \ \ddots \end{array}\right),$$
$$\longleftarrow 3N_q \longrightarrow \longleftarrow N_l \longrightarrow$$

and $\{\lambda_3\} = \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$; $\mathrm{Tr}\ \{\lambda_3\}^2 = \frac{1}{2}$.

Hence $\mathrm{Tr}\ (T_{C3})^2 = 2\ \mathrm{Tr}\ (\mathrm{T}_{C3}\ (L))^2 = 2N_q\ \mathrm{Tr}\ \{\lambda_3\}^2 = N_q = N_f/4.$

The matrix corresponding to the third component of $SU(2)_L$ is

$$T_{L3} = \begin{pmatrix} T_{L3}(L) & 0 \\ 0 & T_{L3}(R) \end{pmatrix}, \text{ where } T_{L3}(R) = 0 \text{ and}$$

$$T_{L3}(L) = \left(\begin{array}{ccc|ccc} (\tau_3) & & & & & \\ & (\tau_3) & & & & \\ & & \ddots & & & \\ \hline & & & \tau_3 & & \\ & & & & \tau_3 & \\ & & & & & \ddots \end{array}\right),$$
$$\longleftarrow 3N_q \longrightarrow \longleftarrow N_l \longrightarrow$$

$$(\tau_3) = \tfrac{1}{2}\begin{pmatrix} \{1\} & 0 \\ 0 & -\{1\} \end{pmatrix}; \quad \{1\} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}; \quad \tau_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So, the trace is

$$\mathrm{Tr}\,(T_{L3})^2 = \tfrac{1}{4}\,(3N_q + N_l) = N_f/4.$$

Thus, we have

$$\sin^2\theta = \mathrm{Tr}\,(T_{L3})^2/\mathrm{Tr}\;Q^2 = \frac{3}{8},$$

$$e^2/g_S^2 = \mathrm{Tr}\,(T_{C3})^2/\mathrm{Tr}\;Q^2 = \frac{3}{8}.$$

This SU(5) result is thus seen to depend only on the fermion content. As we already know, these results are common to both the standard and left-right symmetric models.

Large renormalisation corrections have to be invoked in order to bring down the value of $\sin^2\theta$ from the above unification value 0·375 to the recent empirically determined value $0{\cdot}23 \pm 0{\cdot}01$ (see the review, Musset 1979). These renormalisation corrections are analysed in detail in paper II. It suffices here to point out that the computation of the renormalisation corrections involves doubtful extrapolations over energy regions of many orders of magnitude. So, it seems reasonable to ask as to what schemes for the fermions would give values of $\sin^2\theta$ near the empirically determined value, even without the renormalisation corrections. That provides the motivation for our second example given in the next section.

5. Sequential triplets scheme

To obtain a different unification value of $\sin^2\theta$, we have to change the fermion content of the theory. Here we consider the case where corresponding to every doublet of the last section, there is also an $SU(2)_L$ singlet so that the fermions appear as sequential triplets:

$$\begin{pmatrix}\{u\}\\\{d\}\\\{x\}\end{pmatrix}, \quad \begin{pmatrix}\{c\}\\\{s\}\\\{y\}\end{pmatrix}, \quad \begin{pmatrix}\{t\}\\\{b\}\\\{z\}\end{pmatrix}\ldots,$$

$$\begin{pmatrix}\nu_e\\e\\E\end{pmatrix}, \quad \begin{pmatrix}\nu_\mu\\\mu\\M\end{pmatrix}, \quad \begin{pmatrix}\nu_\tau\\\tau\\T\end{pmatrix}\ldots.$$

The left-handed triplet $(u, d, x)_L$ decomposes into the usual doublet $(u, d)_L$ and a new singlet x_L under $SU(2)_L$, and other triplets decompose similarly. Curly brackets denote, as in § 4, colour triplets. Such schemes having extra leptons and quarks have been considered in literature in various other contexts (e.g., Khare *et al* 1979; Pandit 1976; Gupta and Mani 1974; Schechter and Ueda 1973) and it is possible to arrange the model in a way such that the interactions of the usual quarks and leptons remain largely unaltered.

We shall again use N_q and N_l to denote the number of quark flavours and lepton flavours, respectively, so that the total number of fermions is given by

$$N_f = 3N_q + N_l.$$

The maximal unification group is $SU(2N_f)$ and the $2N_f$ dimensional fermion representation is

$$\psi = (\psi_L, \psi_R),$$

$$\psi_L = (\underbrace{\{u_L\}, \{d_L\}, \{x_L\}, \ldots}_{3N_q};\ \underbrace{\nu_{eL}, e_L, E_L, \ldots}_{N_l}),$$

and similarly for ψ_R.

The charge matrix is

$$Q = \begin{pmatrix} Q_L & 0 \\ 0 & Q_R \end{pmatrix}$$

where $$Q_L = Q_R = \begin{pmatrix} \frac{2}{3}\{1\} & & & & & & & \\ & -\frac{1}{3}\{1\} & & & & & & \\ & & -\frac{1}{3}\{1\} & & & & & \\ & & & \ddots & & & & \\ & & & & 0 & & & \\ & & & & & -1 & & \\ & & & & & & +1 & \\ & & & & & & & \ddots \end{pmatrix}$$
$$\longleftarrow 3N_q \longrightarrow \longleftarrow N_l \longrightarrow$$

We have chosen the charge of the singlet quarks $x, y, z\ldots$ to be $-\frac{1}{3}$ and the singlet leptons $E, M, T\ldots$ to be $+1$. This follows if Q is a generator of an SU(3) group of which the triplets form irreducible representations. We shall assume this. Now, the trace of Q is zero separately for quarks and leptons and so N_q need not equal N_l. Nevertheless, we shall choose $N_q = N_l$ by invoking lepton-quark symmetry. So,

$$\mathrm{Tr}\, Q^2 = 2\, \mathrm{Tr}\, Q_L^2 = \tfrac{4}{3}\,(N_q + N_l) = \tfrac{2}{3}\, N_f.$$

The matrix T_{C3} is of the same form as in the doublets scheme and so

$$\mathrm{Tr}\, (T_{C3})^2 = N_f/4.$$

On the other hand, T_{L3} is different. We have

$$T_{L3} = \begin{pmatrix} T_{L3}\,(L) & 0 \\ 0 & T_{L3}\,(R) \end{pmatrix};$$

where $\quad T_{L3}\,(R) = 0,$

and

$$T_{L3}(L) = \left(\begin{array}{ccccc|ccccc} (\tau_3) & & & & & & & & & \\ & \{0\} & & & & & & & & \\ & & (\tau_3) & & & & & & & \\ & & & \{0\} & & & & & & \\ & & & & \ddots & & & & & \\ \hline & & & & & \tau_3 & & & & \\ & & & & & & 0 & & & \\ & & & & & & & \tau_3 & & \\ & & & & & & & & 0 & \\ & & & & & & & & & \ddots \\ \multicolumn{5}{c|}{\longleftarrow 3N_q \longrightarrow} & \multicolumn{5}{c}{\longleftarrow N_l \longrightarrow} \end{array}\right),$$

(τ_3) and τ are as defined before and

$$\{0\} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}.$$

Hence $\quad \mathrm{Tr}\, (T_{L3})^2 = \tfrac{1}{6}\,(3N_q + N_l) = N_f/6.$

The coupling constant ratios are

$$\sin^2 \theta = \mathrm{Tr}\, (T_{L3})^2 / \mathrm{Tr}\, Q^2 = 1/4,$$

$$\frac{e^2}{g_S^2} = \mathrm{Tr}\, (T_{C3})^2 / \mathrm{Tr}\, Q^2 = 3/8.$$

These are the results for both the standard and left-right symmetric models, if in addition to fermion doublets there exist singlets under $SU(2)_L$ such that the fermions form $SU(3)_L$ triplets.

As promised, $\sin^2\theta$ in the sequential triplets scheme is already near the empirically determined value, $0{\cdot}23 \pm 0{\cdot}01$ (Musset 1979). So, this scheme of the fermions is the appropriate one for those unified models which do not involve large renormalisation corrections. The possibility of such unified models is also studied in paper II.

6. Conclusions and comments

We have studied in detail the factors that influence the unification relations among the coupling-parameters of strong and electroweak interactions. We may summarise the results as follows:

(i) The fermion content of a theory almost completely fixes the unification relations. In particular, these relations do not depend on the specific choice of the unifying group. One can derive the unification relations using a maximal unifying group $G_{\max} = SU(2N_f)$, where N_f is the number of fermions in the theory. The relations thus derived will be valid for all unifying groups that are subgroups of $G_{\max}$.

(ii) The physically relevant unification results do not depend on the specific choice of the 'observed' group of strong and electroweak interactions either. In particular, the unification value of the weak mixing angle is the same for all electroweak groups which contain $SU(2)_L \times U(1)$ as a subgroup, and in which the set of fermions and the classification of fermions under $SU(2)_L$ is the same. Also, the value of the mixing angle is the same for both the integrally-charged and the fractionally-charged quark models.

(iii) The conclusion of para 2 is important because, as we have already shown (Bajaj and Rajasekaran 1979a and 1980) the left-right symmetric models are in good agreement with neutral-current data with more or less the same value of $\sin^2\theta$. In addition, it is known that a large class of models based on $SU(2)_L \times U(1) \times G'$ can be arranged to give the same neutral-current couplings in the ν-induced sectors (Georgi and Weinberg 1976). In these models again the empirically determined $\sin^2\theta$ will have the same value as in the standard model. Here we show that the 'theoretical' value of $\sin^2\theta$, determined by embedding the strong and electroweak interactions in a grand unified group, is also the same for all these models.

(iv) All this leads to the conclusion that as far as empirically relevant results of unification are concerned there is nothing sacrosanct about the minimal unifying group SU(5), or about the standard model of strong and electroweak interactions.

(v) To illustrate these conclusions we explicitly show that the canonical value of 3/8 for $\sin^2\theta$ can be obtained by only specifying that the fermions form sequential doublets under $SU(2)_L$. No reference to the unifying group or to the 'observed' group need be made. Introduction of additional fermion singlets reduces $\sin^2\theta$ to 1/4, which is rather near the empirically determined value of $\sin^2\theta$. This sequential triplet scheme for fermions may, therefore, be of interest in unification theories that avoid large renormalisation corrections (e.g., see Fritzsch and Minkowski 1975).

The above conclusions and comments are valid only for the algebraic results of unification. The question of renormalisation corrections to these results is taken up in paper II.

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