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Driven Heisenberg Magnets: Nonequilibrium Criticality, Spatiotemporal Chaos and Control

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Abstract. – We drive a d-dimensional Heisenberg magnet using an anisotropic current. The continuum Langevin equation is analysed using a dynamical renormalization group and numerical simulations. We discover a rich steady-state phase diagram, including a critical point in a new nonequilibrium universality class, and a spatiotemporally chaotic phase. The latter may be 'controlled' in a robust manner to target spatially periodic steady states with helical order.

How does an imposed steady current of heat or particles alter the dynamics of the isotropic magnet? To answer this question, we extend the equations of motion for the classical $O(3)$ Heisenberg model [1] to include the effects of a uniform current in one spatial direction while retaining isotropy in the order parameter space. The resulting model is a natural generalization of the driven diffusive models of [2] to the case of a 3-component *axial*-vector order parameter and, as such, is an important step in the exploration of dynamic universality classes [3] far from equilibrium [4]. The form of the local molecular field in which spins precess in this driven state is strikingly different from that at equilibrium [1], and is responsible for all the remarkable phenomena we predict, including a novel nonequilibrium critical point and, in a certain parameter range, a type of turbulence.

Here are our results in brief: (i) Despite $O(3)$ invariance in the order-parameter space, the dynamics does not conserve magnetization; (ii) As a temperature-like parameter is lowered, the paramagnetic phase of the model approaches a nonequilibrium critical point in a new dynamic universality class; (iii) Below this critical point, in mean-field theory without stochastic forcing, paramagnetism, ferromagnetism and helical order are all linearly unstable; (iv) Numerical

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studies in space dimension $d = 1$ show spatiotemporal chaos in this last regime. This chaos, when 'controlled', is replaced by spatially periodic steady helical states. These predictions should be testable in experiments on isotropic magnets carrying a steady particle or heat current, as well as in simulations of a magnetized lattice-gas which we discuss at the end of this Letter.

To construct our equations of motion, recall that *at thermal equilibrium* at temperature T the probability of spin configurations $\{\vec{S}_i\}$ of a general nearest-neighbor Heisenberg chain with sites i is $\propto \exp(-H/T)$, with an energy function

$$H = -\sum_i J_i \vec{S}_i.\vec{S}_{i+1}, \tag{1}$$

where J_i is the exchange coupling between i and $i+1$. A spin $\vec{S}_i$ at i precesses as $\dot{\vec{S}}_i = \vec{S}_i \times \vec{h}_i$ where

$$\vec{h}_i = -\frac{\partial H}{\partial \vec{S}_i} = J_i \vec{S}_{i+1} + J_{i-1} \vec{S}_{i-1} \tag{2}$$

is the local molecular field. Replacing $J_i \to J(x)$ and $\vec{S}_i \to \vec{S}(x)$ in the continuum limit, yields [5] $\dot{\vec{S}}(x) = J(x)\vec{S} \times \partial_x^2 \vec{S} + (dJ/dx)\vec{S} \times \partial_x \vec{S} + ...$ For the physically reasonable case where J varies *periodically* about a mean value J_0, this reduces for long wavelengths to $\dot{\vec{S}}(x) = J_0 \vec{S} \times \partial_x^2 \vec{S}$, which is invariant under $x \to -x$, even if the H is not. The dynamics conserves $\sum_i \vec{S}_i$ since it commutes with H.

Now drive some background degrees of freedom in, say, the $\hat{\mathbf{x}}$ direction, *retaining isotropy in spin space*. These could be some mobile species – particles, vacancies, heat – or some nonconserved internal variables. Possible microscopic realizations are discussed towards the end of the paper. For now, note that the dynamics in this nonequilibrium state does not follow from an energy function, and must be constructed anew. If we average over these background variables, their effect should be simply to modify the equations for the $\vec{S}_i$ by allowing terms forbidden at *thermal* equilibrium. While many such terms are permitted, only two are *relevant*: (i) asymmetric exchange, i.e., $\vec{h}_i = J_+ \vec{S}_{i+1} + J_- \vec{S}_{i-1}$ yielding a precession rate $g\vec{S} \times \partial_x^2 \vec{S} + \lambda \vec{S} \times \partial_x \vec{S}$, with $\lambda \propto J_+ - J_-$ proportional to the driving rate; and (ii) nonconserving damping and noise. Both (i) and (ii) were ruled out [1] at thermal equilibrium only because the dynamics *had* to be generated by (1) and (2). Note that the λ term, while rotation-invariant in spin space, is not the divergence of a current [9]. The nonlinearity $\lambda \vec{S} \times \partial_x \vec{S}$ will thus *generate* [6,7,8] nonconserving noise and damping terms even if these are not put in at the outset.

For a general dimension $d \equiv d_\perp + 1$, with anisotropic driving along one direction ($\parallel$) only, the above arguments yield, to leading orders in a gradient expansion, the generalized Langevin equation

$$\begin{aligned}
\frac{\partial \vec{S}}{\partial t} &= \left(r_\parallel \partial_\parallel^2 + r_\perp \nabla_\perp^2 \right) \vec{S} - v\vec{S} - \frac{u}{6}(\vec{S} \cdot \vec{S})\vec{S} - \lambda \vec{S} \times \partial_\parallel \vec{S} \\
&\quad + g_\parallel \vec{S} \times \partial_\parallel^2 \vec{S} + g_\perp \vec{S} \times \nabla_\perp^2 \vec{S} + \vec{\eta},
\end{aligned} \tag{3}$$

where we have allowed for spatial anisotropy in the coefficient of the usual spin precession term. The Gaussian, zero-mean nonconserving noise $\vec{\eta}$ satisfies $\langle \eta_\alpha(\mathbf{x}, t)\eta_\beta(\mathbf{x}', t') \rangle = 2B\, \delta_{\alpha\beta}\, \delta^d(\mathbf{x} - \mathbf{x}')\delta(t - t')$.

In the *equilibrium, isotropic* limit, $\lambda = u = v \equiv 0$, $r_\parallel = r_\perp \equiv r$, the noise strength vanishes at zero wavenumber, and (3) has a critical point where the renormalized $r \to 0$. In the *driven*

state, since the dynamics and noise are nonconserving, the critical point is $v = 0$, which in general takes place on a curve in the temperature/driving-force plane. As the drive is taken to zero there should be a crossover from nonequilibrium to equilibrium critical behavior. Our primary interest is in the behavior at a given nonzero driving rate, for which it suffices to vary the temperature-like parameter v in (3), keeping the rest fixed (with $r_\parallel$, $r_\perp$, $u > 0$). For $v > 0$ (the paramagnetic phase) all correlations clearly decay on finite lengthscales $\sim 1/\sqrt{v}$ and time-scales $\sim 1/v$, and nonlinearities are irrelevant. Let us focus first on the nature of correlations on the critical surface $v = 0$. Here, we expect an anisotropic scaling form for the correlation function $C(x, t) \equiv \langle \vec{S}(\mathbf{x} + \mathbf{x}', t + t') \cdot \vec{S}(\mathbf{x}', t') \rangle$: $C(\mathbf{x}, t) = x_\parallel^{2\chi} F(t/x_\parallel^z, x_\perp/x_\parallel^\zeta)$, where F is a scaling function. In the *linear* approximation to (3) the roughening, growth, and anisotropy exponents are respectively $\chi = 1 - d/2$, $z = 2$, and $\zeta = 1$, and F is analytic in its arguments.

We now include the effect of the nonlinear terms in (3) via a standard implementation of the dynamical renormalization group (DRG) [10] based on a perturbation expansion in λ and u. Rescaling $x_\parallel = bx'_\parallel$, $\mathbf{x}_\perp = b^\zeta \mathbf{x}'_\perp$, $t = b^z t'$ and $\vec{S} = b^\chi \vec{S}'$, where $b > 1$ is an arbitrary parameter, the coefficients in Eq. (3) transform as $r'_\parallel = b^{z-2} r_\parallel$, $r'_\perp = b^{z-2\zeta} r_\perp$, $B' = b^{z-2\chi-\zeta(d-1)-1} B$, $u' = b^{4-d} u$, $\lambda' = b^{\chi+z-1} \lambda$, $g'_\parallel = b^{\chi+z-2} g_\parallel$ and $g'_\perp = b^{\chi+z-2\zeta} g_\perp$. λ and u are thus *relevant* for dimension $d < 4$, and $g_\parallel$ and $g_\perp$ are *irrelevant* for d near 4. In units where the ultraviolet cutoff is 1, λ and u enter the perturbation theory in the dimensionless combinations $\tau \equiv (1/2\pi^3) \lambda^2 B / \sqrt{r_\parallel^3 r_\perp^3}$ and $\kappa \equiv (1/2\pi^3) u B / \sqrt{r_\parallel r_\perp^3}$, *At the critical point* $v = 0$, setting the irrelevant $g_\parallel$ and $g_\perp$ to zero, for $d = 4 - \epsilon$, we find [7, 8] to $O(\epsilon)$ the differential recursion relations

$$\frac{\partial r_\parallel}{\partial l} = r_\parallel(z - 2 + \frac{\pi}{4}\tau),$$

$$\frac{\partial r_\perp}{\partial l} = r_\perp(z - 2\zeta + \frac{5\pi}{48}\tau),$$

$$\frac{\partial B}{\partial l} = B[z - 2\chi - \zeta(d-1) - 1 + \frac{\pi}{32}\tau],$$

$$\frac{\partial \tau}{\partial \ell} = \tau(\epsilon\zeta - \frac{35}{64}\pi\tau),$$

$$\frac{\partial \kappa}{\partial \ell} = \kappa(\zeta\epsilon - \frac{11}{24}\pi\zeta\kappa - \frac{\pi}{2}\tau) + \frac{27}{16}\pi\zeta\tau^2. \tag{4}$$

Since we are working at $v = 0$, we seek a fixed point that is *stable* with respect to perturbations in the *remaining* directions in the parameter space. For $\epsilon = 4 - d > 0$ we find the nontrivial stable fixed point $\tau^* = 64\epsilon/(35\pi)$, $\kappa^* = 36[1 + \sqrt{1409}]\epsilon/385\pi$. The critical exponents for $d < 4$, to lowest order in ϵ, are $z = 2 - 16\epsilon/35$, $\zeta = 1 - 2\epsilon/15$ (anisotropic scaling) and [since u plays no role at $O(\epsilon)$] $\chi = 1 - d/2$. These exponents clearly place this critical point in a new universality class. A more detailed analysis, including the approach to the critical point, will appear elsewhere [8].

We now investigate the low-temperature $v < 0$ phase, *in the absence of noise*. It is convenient to work with dimensionless variables, obtained by rescaling $x_\perp$, $x_\parallel$, t and $\vec{S}$ in Eq. (3); this leaves λ as the only parameter in the equation of motion. There are two static, spatially homogeneous steady states — a 'paramagnetic steady state' $\langle S_\alpha \rangle = 0$, and a 'ferromagnetic steady state' $\langle S_1 \rangle = \langle S_2 \rangle = 0$ and $\langle S_3 \rangle = 1$. It is straightforward to see from (3) that both these stationary solutions are linearly unstable [7, 8]. We next look for static, spatially inhomogeneous steady states, a natural candidate being the helical state. Defining
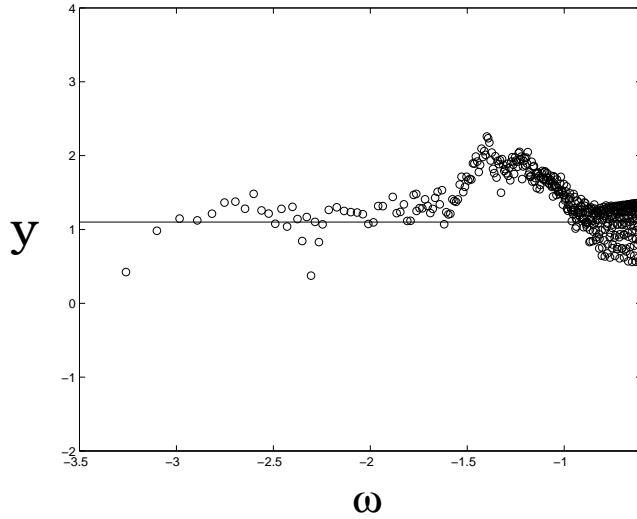
Fig. 1 – Log-log plot of $y = \sqrt{\omega^2 |M_3(\omega)|^2}$ versus ω showing the $1/\omega^2$ dependence of the power spectrum over approximately 1.5 decades.

$\rho \equiv \sqrt{S_1^2 + S_2^2}$ and $\phi \equiv \tan^{-1}(S_2/S_1)$, (3) for $g_\parallel = g_\perp = 0$ becomes

$$\frac{\partial \rho}{\partial t} = \nabla^2 \rho - \rho(\nabla\phi)^2 + \rho - (\rho^2 + S_3^2)\rho - \lambda\rho S_3 \partial_\parallel \phi\,,$$

$$\frac{\partial \phi}{\partial t} = \nabla^2 \phi + \frac{2}{\rho}(\nabla\rho)\cdot(\nabla\phi) + \frac{\lambda}{\rho}(S_3 \partial_\parallel \rho - \rho \partial_\parallel S_3)\,,$$

$$\frac{\partial S_3}{\partial t} = \nabla^2 S_3 + S_3 - (\rho^2 + S_3^2)S_3 + \lambda\rho^2 \partial_\parallel \phi\,. \tag{5}$$

A regular helix $\rho = a$, $\phi = px_\parallel$ and $S_3 = b$ (a, b and p are arbitrary constants) is a steady state solution if $2b^2 = 1 - a^2(1+\lambda^2) \pm \sqrt{(a^2(\lambda^2 + 1) - 1)^2 - 4a^4}$, and $2p = -\lambda b \pm \sqrt{\lambda^2 b^2 - 4(R^2 - 1)}$, where $R = \sqrt{a^2 + b^2}$ is the magnitude of each spin. The only free parameter a is bounded by $a < (3+\lambda^2)^{-1/2}$ from the requirement that b be real. Unfortunately even this steady state shows a linear instability, triggered by the growth of S_3 [7].

Having failed to find any stable static steady states analytically, we solve (3) numerically for $d = 1$ without noise, for a range of generic initial conditions. To avoid numerical instabilities we adopt an operator splitting method [11] — we solve the dissipative part using the standard Euler method and the drive part [8] by rotating each spin by an azimuthal angle $|\mathbf{h}(x,t)| \triangle t$ about its computed local magnetic field $\mathbf{h}$. With our choice of $\triangle x = 1$ and $\triangle t = 0.0001$ on a system of size $N = 200$ with periodic boundary conditions, we find that we avoid numerical instabilities and finite size effects.

We find that the time series of the magnetization and energy density $E = N^{-1} \int dx (\nabla\vec{S})^2$ never settle to a constant value; the motion could therefore be either temporally (quasi)periodic or chaotic. Figure.1 shows that the power spectrum of $M_3 \equiv \int_x S_3$ goes as $1/\omega^2$. The power spectrum of E also shows a similar behavior. This suggests that the dynamics is temporally chaotic [12].

Space-time plots of local quantities, such as the signed local pitch, $\mathrm{sgn}(p) \equiv \mathrm{sgn}(\partial_x \phi)$ (fig.2), strongly suggest the presence of spatiotemporal chaos [12]. These results are preliminary, and
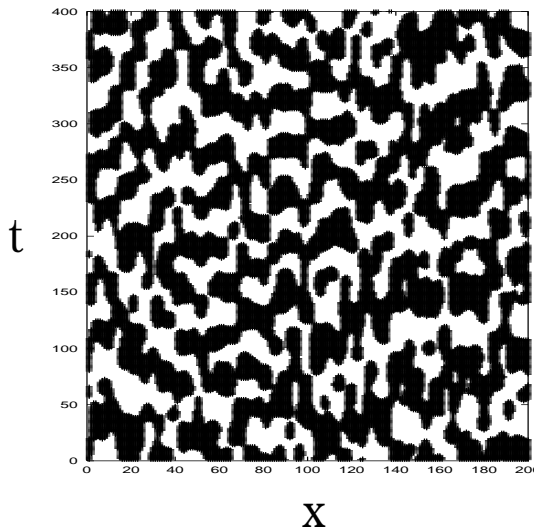
Fig. 2 – Space-time plot of the signed local pitch, $\mathrm{sgn}(p) \equiv \mathrm{sgn}(\partial_x \phi)$ (black $(+1)$, white (-1)), revealing spatiotemporal chaos.

only for $d = 1$. We shall characterize this behavior in greater detail elsewhere [8], including studies of the dependence of the number of positive Lyapounov exponents on system size and the behavior for $d > 1$.

The helix solutions of Eq. (5) for $v < 0$ are an infinite family of unstable spatially periodic steady states (parametrised by a) of the type discussed in [13]. Can chaos in our model be *controlled* so as to *stabilize* and *target* [13] these helical states? The control of spatiotemporal chaos in PDEs [13, 14] is not nearly as well-developed as that in finite dimensional dynamical systems [13]. Accordingly, it is significant that we are able to stabilize, target, and hence control spatiotemporal chaos in our model, as we now show.

For instance, in order to stabilize a specific helical configuration (with fixed a, b and p), we could in principle wait till the dynamics (presumably ergodic) eventually leads to this configuration, after which we apply small perturbations to prevent S_3 from deviating from the value b (recall that the instability of the helical state was led by S_3). This prescription successfully *stabilizes* the prescribed helix.

In order to *target* this prescribed helix, we add to (5) terms which would arise from a uniaxial spin anisotropy energy $V_3 = r_3(S_3^2 - b^2)^2$ or $V_3 = r_3(S_3 - b)^2$. We find that a sufficiently large and positive r_3 forces S_3 to take the value b exponentially fast starting from arbitrary initial configurations. The subsequent evolution, given by Eq. (5) on setting $S_3 = b$, can be recast as purely relaxational dynamics,

$$\frac{\partial \rho}{\partial t} = -\frac{\delta F}{\delta \rho} \quad , \quad \frac{\partial \phi}{\partial t} = -\frac{1}{\rho^2}\frac{\delta F}{\delta \phi} , \tag{6}$$

where the 'free-energy functional' F has the form of a *chiral XY model*,

$$\begin{aligned} F \ = \ & \frac{1}{2}\int_{\mathbf{x}}\left[(\nabla\rho)^2 + \rho^2(\nabla\phi)^2 - (\rho^2 + b^2) + \frac{1}{2}(\rho^2 + b^2)^2 \right. \\ & \left. +\lambda b\rho^2 \partial_\| \phi \right] . \end{aligned} \tag{7}$$

Using the chain-rule, it is easy to see that $dF/dt = \int_{\mathbf{x}} \left[-(\delta F/\delta S_1)^2 - (\delta F/\delta S_2)^2 \right] < 0$, Hence F is a Lyapunov functional for the dynamics. Completing the squares, we see that $\partial_{\parallel}\phi$ appears in F in the combination $(1/2)\rho^2(\partial_{\parallel}\phi + \lambda b/2)^2$, which is minimized by the helix $\phi = -(1/2)\lambda b x_{\parallel}$. Starting from any initial configuration, the system plummets towards this unique helical minimum of F.

Let us now see whether our control is robust against noise. We modify (6) by including the noise $\vec{\eta}$ in Eq. (3), and ask for the statistics of small fluctuations with Fourier components $\tilde{\rho}_{\mathbf{k}}(t)$ and $\tilde{\phi}_{\mathbf{k}}(t)$ about the controlled helical state, where $2\pi/L < k < \Lambda$ for a system of linear extent L. It is clear from (6) that the means $\langle\tilde{\rho}_{\mathbf{k}}\rangle$ and $\langle\tilde{\phi}_{\mathbf{k}}\rangle$ decay exponentially to zero: the relaxation time for $\tilde{\rho}_{\mathbf{k}}$ is finite at small k, whereas that for $\tilde{\phi}_{\mathbf{k}}$ goes as k^{-2}. To calculate the variances, note that the dynamics is governed in the mean by the Lyapounov functional (7), and that the noise is spatiotemporally white. It follows [15] that the steady-state configuration probability $P[\rho,\phi] \propto e^{-cF}$ where c is an effective inverse temperature. $F \simeq \int [\text{const}(\tilde{\rho})^2 + \text{const}(\nabla\tilde{\phi})^2]$ for small fluctuations about the helical minimum, i.e., P is approximately gaussian, so that $\langle|\tilde{\phi}_{\mathbf{k}}|^2\rangle \sim k^{-2}$ and $\langle|\tilde{\rho}_{\mathbf{k}}|^2\rangle \sim \text{const.}$ for small k. Thus the variance $\langle\tilde{\rho}^2\rangle = \int_k \langle|\tilde{\rho}_{\mathbf{k}}|^2\rangle$ is L-independent for $L \to \infty$ in any dimension d, whereas $\langle\tilde{\phi}^2\rangle = \int_k \langle|\tilde{\phi}_{\mathbf{k}}|^2\rangle$ diverges as L and $\ln L$ respectively for $d = 1$ and 2, and is finite for $d > 2$. Thus *occasional excursions from the controlled state as a result of the noise do not lead to an instability of the targeted state for $d > 2$*; the behavior for $d \leq 2$ is no worse than for a thermal equilibrium XY model.

Having arrived at the continuum equations Eq. (3) based only on symmetry arguments and conservation laws, we now suggest ways in which the driving nonlinearity in (3) may be realised. (a) Consider the isotropic magnet on a lattice whose unit cell lacks $\hat{\mathbf{x}} \to -\hat{\mathbf{x}}$ symmetry. Now subject the spins to a spatiotemporally random, isotropic, nonconserving noise source. The lack of invariance under x-inversion together with this nonequilibrium noise should give rise, via an interplay of nonequilibrium noise and the electrons participating in the exchange interaction, to the $\vec{S} \times \partial_x \vec{S}$ term. We admit, however, that we do not have a microscopic derivation of this. (b) Consider the asymmetric exclusion process (ASEP) [2] on a 1-dimensional lattice, each site i of which can be either vacant ($n_i(t) = 0$) or occupied by one particle ($n_i(t) = 1$) at time t. Each particle has *an attached Heisenberg spin* and may hop to the nearest neighbor at the right (left), if vacant, with probability p (q). The spin $\vec{S}_i(t)$ at an occupied site i is the spin of the occupying particle. The local field at site i is $\vec{h}_i(t) = J_{i-1,i}\vec{S}_{i-1}(t) + J_{i+1,i}\vec{S}_{i+1}(t)$. The required asymmetric exchange is achieved by making the Js dynamical. Recall that the exchange coupling between, say, $i-1$ and i is operative at time t only if $n_{i-1}(t)n_i(t) = 1$. Imagine that the local exchange coupling emanating from a given site depended on some internal degree of freedom with a short relaxation time, as follows: $J_{i,i\pm1}(t) = J_1$ if $n_i(t-1) = 1$; if not, $J_{i,i\pm1}(t) = J_2$. Assume for simplicity that the particles can hop only to the right. Then a configuration 111 at sites $i-1$, i, and $i+1$ at time t was either already present at time $t-1$ or arose from 011 (by a right hop from $i-2$). Thus, for 111 configurations, $J_{i-1,i}$ will be a weighted average of J_1 and J_2, while $J_{i+1,i}$ will be J_1. In the continuum limit, and averaging over the particle dynamics, we will get the driving term in (3), with $\lambda \propto p - q$. Such averaging is justified if the hopping species has a faster dynamics than the spins (for example, a smaller dynamical exponent). Alternatively, allowing evaporation-deposition in the ASEP renders the "particles" fast without altering qualitatively the derivation above for the effective asymmetric exchange. Of course, particle non-conservation induces spin non-conservation trivially in this case. The work of [5] on moving space curves suggests another promising approach to finding realizations of our model.

In conclusion, we have studied the interplay of dissipation, precession, and spatially anisotropic driving on the dynamics of a classical Heisenberg magnet in d space dimensions.

We have found a nonequilibrium critical point which we have shown, in an expansion in $\epsilon = 4 - d$, to be in a new dynamical universality class. We have presented evidence of spatiotemporal chaos in the mean-field dynamics of the model, at least in $d = 1$, and have shown how this chaos can be controlled to yield helical order. Our work reinforces the idea that spatiotemporal chaos is a generic feature of driven, dissipative, spatially extended systems with nonlinear reactive terms. Further properties of this remarkable model, including the Lyapounov spectrum of the chaotic state, the possibility of complex ordered states or spatiotemporal chaos for $d > 1$, possible experimental realizations, and the crossover between equilibrium and driven behavior, will be discussed elsewhere [8].

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