CERTAIN TRIGONOMETRIC SUMMATIONS

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THE trigonometric series considered below have been encountered in the course of certain investigations in room-acoustics and in view of the occurrence of similar series in diverse physical problems, we consider it of some interest to present here an elementary method of expressing them in terms of standard functions. Using these transformations, it is shown that some results due to Krishnan[1] and Goddard[2] are derivable.

We consider the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin (n+\theta) x}{(n+\theta)} \cdot \frac{\sin (n+\phi) x}{(n+\phi)}, \quad 0 < x < \pi \tag{1}$$

and the corresponding series in cosines

$$\sum_{n=1}^{\infty} \frac{\cos (n+\theta) x}{(n+\theta)} \cdot \frac{\cos (n+\phi) x}{(n+\phi)}, \quad 0 < x < \pi \tag{2}$$

which are denoted here by $A(\theta, \phi; x)$ and $B(\theta, \phi; x)$ respectively.

We use the following known results[3]

$$\text{(A)} \quad c(\theta, x) = \sum_{n=1}^{\infty} \frac{\cos (n+\theta) x}{(n+\theta)}$$

$$= \frac{1}{2} \int_{x}^{\pi} \frac{\cos (\theta - \frac{1}{2}) t}{\sin \frac{1}{2} t} dt + \frac{1}{2} \left[\psi\left(\frac{1+\theta}{2}\right) - \psi\left(\frac{\theta}{2}\right) \right] \cos \pi\theta - \frac{\cos \theta x}{\theta}$$

$$\text{(B)} \quad s(\theta, x) = \sum_{n=1}^{\infty} \frac{\sin (n+\theta) x}{(n+\theta)}$$

$$= \frac{1}{2} \int_{x}^{\pi} \frac{\sin (\theta - \frac{1}{2}) t}{\sin \frac{1}{2} t} dt + \frac{1}{2} \left[\psi\left(\frac{1+\theta}{2}\right) - \psi\left(\frac{\theta}{2}\right) \right] \sin \pi\theta - \frac{\sin \theta x}{\theta}$$

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where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \operatorname*{Lim}_{n \to \infty} \left[\log n - \frac{1}{z} - \frac{1}{1+z} - \frac{1}{2+z} - \ldots\ldots - \frac{1}{n+z} \right].$$

The series (1) can now be transformed thus

$$A(\theta, \phi; x) = \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{\cos(\theta - \phi)x - \cos[n + \tfrac{1}{2}(\theta + \phi)]\, 2x}{(n+\theta)(n+\phi)}$$

$$= \tfrac{1}{2} \frac{\cos(\theta - \phi)\, x}{(\theta - \phi)} \sum_{n=1}^{\infty} \left(\frac{1}{n+\phi} - \frac{1}{n+\theta} \right)$$

$$+ \tfrac{1}{2} \frac{1}{\theta - \phi} \sum_{n=1}^{\infty} \left(\frac{1}{n+\theta} - \frac{1}{n+\phi} \right) \cos[n + \tfrac{1}{2}(\theta + \phi)]\, 2x$$

Writing the terms $\dfrac{\cos[n + \frac{1}{2}(\theta + \phi)]\, 2x}{n+\theta}$ and $\dfrac{\cos[n + \frac{1}{2}(\theta + \phi)]\, 2x}{n+\phi}$ as $\dfrac{\cos[n + \theta - \frac{1}{2}(\theta - \phi)]\, 2x}{n+\theta}$ and $\dfrac{\cos[n + \phi + \frac{1}{2}(\theta - \phi)]2x}{n+\phi}$ respectively, $A(\theta, \phi; x)$ can be transformed as

$$A(\theta, \phi; x) = \tfrac{1}{2} \frac{\cos(\theta - \phi)\, x}{\theta - \phi} [\psi(1+\theta) - \psi(1+\phi)]$$

$$+ \tfrac{1}{2} \frac{\cos(\theta - \phi)\, x}{\theta - \phi} [c(\theta, 2x) - c(\phi, 2x)]$$

$$+ \tfrac{1}{2} \frac{\sin(\theta - \phi)\, x}{\theta - \phi} [s(\theta, 2x) + s(\phi, 2x)] \qquad (3)$$

The series (2) can be written as

$$B(\theta, \phi; x) = \sum_{n=1}^{\infty} \frac{\cos(\theta - \phi)\, x - \sin(n+\theta)\, x \sin(n+\phi)\, x}{(n+\theta)(n+\phi)}$$

$$= \cos(\theta - \phi)\, x \sum_{n=1}^{\infty} \frac{1}{(n+\theta)(n+\phi)} - A(\theta, \phi; x)$$

so that we have finally

$$B(\theta, \phi; x) = \tfrac{1}{2} \frac{\cos(\theta - \phi)\, x}{\theta - \phi} [\psi(1+\theta) - \psi(1+\phi)]$$

$$- \tfrac{1}{2} \frac{\cos(\theta - \phi)\, x}{\theta - \phi} [c(\theta, 2x) - c(\phi, 2x)]$$

$$- \tfrac{1}{2} \frac{\sin(\theta - \phi)\, x}{\theta - \phi} [(s(\theta, 2x) + s(\phi, 2x)] \qquad (4)$$

When $\theta = \phi$, $A(\theta, \theta; x)$ and $B(\theta, \theta; x)$ are easily obtained from (3) and (4) by passing to the limit. Then

$$A(\theta, \theta; x) = \tfrac{1}{2} \frac{d\psi(1+\theta)}{d\theta} + \tfrac{1}{2} \frac{dc(\theta, 2x)}{d\theta} + xs(\theta, 2x) \tag{3}$$

$$B(\theta, \theta; x) = \tfrac{1}{2} \frac{d\psi(1+\theta)}{d\theta} - \tfrac{1}{2} \frac{dc(\theta, 2x)}{d\theta} - xs(\theta, 2x) \tag{4}$$

We now derive certain trigonometric summations:

I. The transformation (3) can be used to obtain the elegent summation*

$$\sum_{n=1}^{\infty} \left[\frac{\sin(n-\theta)x}{n-\theta} - \frac{\sin(n+\theta)x}{n+\theta}\right]\left[\frac{\sin(n-\phi)x}{n-\phi} - \frac{\sin(n+\phi)x}{n+\phi}\right]$$

$$= \pi\left[\frac{\sin(\theta-\phi)x}{\theta-\phi} - \frac{\sin(\theta-\phi)x}{\theta-\phi}\right] \tag{5}$$

II. Adding $A(\theta, \theta; x)$ and $A(-\theta, -\theta; x)$ we obtain

$$\sum_{n=1}^{\infty} \frac{\sin^2(nx+\alpha)}{(nx+\alpha)^2} = \frac{\pi}{x}, \tag{6}$$

where $\alpha = \theta x$, a result used by Krishnan in Light Scattering (*loc. cit.*).

III. Taking $\phi = -\theta$ in (3) we get

$$\sum_{n=1}^{\infty} \frac{\sin(n+\theta)x \sin(n-\theta)x}{n^2-\theta^2} = \frac{\sin^2\theta x}{2\theta^2} + \frac{\pi}{4}\frac{\sin 2\theta x}{\theta}.$$

In particular, when $\theta = \alpha m$ and $x = \pi/\alpha$, where α is real and > 1, and m is a positive integer, the above relation becomes

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\pi/\alpha)}{n^2-\alpha^2 m^2} = 0 \tag{8}$$

a result used by Goddard (*loc. cit.*),

IV. By employing the transformation (4) we can obtain the following summation analogous to (5)

$$\sum_{n=1}^{\infty} \left[\frac{\cos(n-\theta)x}{n-\theta} - \frac{\cos(n+\theta)x}{n+\theta}\right]\left[\frac{\cos(n-\phi)x}{n-\phi} - \frac{\cos(n+\phi)x}{n+\phi}\right]$$

$$= \tfrac{1}{2}\left(\frac{1}{\theta} - \pi\cot\pi\theta\right)\left[\frac{\cos(\theta-\phi)x}{\theta-\phi} - \frac{\cos(\theta+\phi)x}{\theta+\phi}\right]$$

$$- \tfrac{1}{2}\left(\frac{1}{\phi} - \pi\cot\pi\phi\right)\left[\frac{\cos(\theta-\phi)x}{\theta-\phi} + \frac{\cos(\theta+\phi)x}{\theta+\phi}\right]$$

$$- \left[\frac{\sin(\theta-\phi)x}{\theta-\phi} - \frac{\sin(\theta+\phi)x}{\theta+\phi}\right] \tag{9}$$

REFERENCES

1. Krishnan, K. S. .. *Jour. Ind. Math. Soc.*, 1948, **12**, 79.
2. Goddard, L. S. .. *Proc. Camb. Phil. Soc.*, 1945, **41**, 148.
3. Hardy, G. H. .. *An Introduction to the Theory of Infinite Series*, 392
   See Bromwich, T. J. I'A (McMillan, 1926).

*It was necessary to sum this particular series in our problem referred to.