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BARLOW POINTS AND GAUSS POINTS AND THE ALIASING AND BEST FIT PARADIGMS

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Abstract—It is well known that finite elements show points where strains or stresses are of higher accuracy than elsewhere. Two competing models. the *aliasing* and the *best approximation* or *best fit* approaches are now available which attempt to explain how this emerges. Here. we compare the two and critically evaluate them.

INTRODUCTION

It has been known for some time that there are points in a finite element where strains/stresses are very accurate, sometimes exact [1]. These are called the Barlow points and it is also acknowledged that these may or may not coincide with the Gauss–Legendre points. Recently, Prathap [2, 3] offered a fresh interpretation on the variational basis for these points. An alternative interpretation was also put forward by MacNeal recently in terms of a concept called *aliasing* [4]. According to MacNeal [4], the term *aliasing* is borrowed from sample data theory where it is used to describe the misinterpretation of a time signal by a sampling device. An original sine wave is represented in the output of a sampling device by an altered sine wave of lower frequency—this is called the *alias* of the true signal. This concept can be extended to finite element discretization—the sample data points are now the values of the displacements at the nodes and the alias is the function which interpolates the displacements within the element from the nodal displacements. Barlow developed his theory of optimal points using an identical paradigm—the term *substitute function* is used instead of alias.

The aliasing concept can be used to derive the location of the optimal points, as Barlow did [1] and as MacNeal did recently [4]. Implicit in this approach is the assumption that the finite element method (FEM) seeks discretized displacement fields which are substitutes or aliases of the true displacement fields, by sensing the nodal displacements directly. Prathap's approach [2, 3] takes a different route—it recognizes that a proper variational basis of the displacement type FEM approach actually leads to the conclusion that the finite element method essentially seeks discretized strain/stress fields which are the substitutes/aliases of the true strain/stress fields and that it does this in a "best fit" or "best approximation" sense. In most cases, this coincides with a least squares error approximation, but a subtle distinction is implied here (see Refs [2, 3]). It is interesting now to see how these two alternative paradigms, the "displacement aliasing" approach and the "best approximation of strain" approach lead to subtle differences in interpreting the relationship between the Barlow points and the Gauss points.

Before we proceed to the main body of the argument it may be worthwhile to state what we mean by *a paradigm* here. The dictionary meaning of paradigm is *pattern* or *model* or *example.* Here, we use the word in the greatly enlarged sense in which the philosopher T. S. Kuhn introduced it in his classic study on scientific progress. In this sense, a *paradigm* is a "framework of suppositions as to what constitutes problems, theories and solutions." It can be a collection of metaphysical assumptions, heuristic models, commitments, values, hunches, which are all shared by a scientific community and which provides the conceptual framework within which they can recognize problems and solve them [6]. The *aliasing* and *best-fit* paradigms are therefore two competing scenarios which attempt to explain how the finite element method computes strains and stresses.

WHAT DOES THE FINITE ELEMENT METHOD DO—SAMPLE DISPLACEMENTS OR STRAINS/STRESSES?

It will be educational here to briefly review what is understood of the finite element discretization process. It is often believed that the finite element method, i.e. the discretization process it implies, seeks approximations to the displacement fields and that the strains/stresses are computed by differentiating these fields. Thus, Barlow's argument [1] implicitly accepts that elements are "capable of representing the nodal displacements in the field to a good degree of accuracy." MacNeal [4] assumes that each finite element samples the displacements at the nodes and internally, within the element, the displacement field is interpolated using the basis functions. The strain fields are computed from these, using a process that involves differentiation. The course of the argument runs further — that as a result, displacements are more accurately computed than the strain and stress field. This follows from the generally accepted axiom that derivatives of functions are less accurate than the original functions. It is also argued that strains/stresses are usually most inaccurate at the nodes and that they are of greater accuracy near the element centres — this, it is thought, is a consequence of the mean value theorem for derivatives.

However, several years of experimenting with finite element design [2] have taught this writer that in actual fact, the Ritz approximation process and the displacement type **FEM,** which can be interpreted as a piecewise Ritz procedure, does exactly the opposite — it is the strain fields which are computed, almost independently, as it were, within each element. This can be derived in a formal way — many attempts have been made to give expression to this idea (e.g. Barlow [1] and Moan [7]), hut the present writer feels that the most intellectually satisfying proof can be arrived **at** by starting with a mixed principle known as the Hu–Washizu theorem **[S].** This proof has been taken up in greater detail in Ref. **[3]** and will be reviewed only briefly in the next section. Having said that the Ritz type procedures determine strains, it follows that the displacement fields are then constructed from this in an integral sense — the stiffness equation actually reflecting this integration process and the continuity of fields across element boundaries and suppression of the field values at domain edges being reflected by the imposition of boundary conditions. It must therefore be argued that displacements are more accurate than strains because integrals of smooth functions are generally more accurate than the original data. We have thus turned the whole argument on its head.

The "best-fit" rule

In Ref. **[3]** it was shown that the "best-fit'' manner, in which finite elements compute strains, can be shown to follow from an interpretation using the Hu–Washizu theorem. **To** see how we progress from the continuum domain to the discretized domain, it was found that it is best to develop the theory from the generalized Hu–Washizu theorem [8] rather than the minimum potential theorem. We proceed thus:

Let the continuum linear elastic problem have an exact solution described by the displacement field u, strain field ϵ and stress field σ (we project that the strain field ϵ is derived from the displacement field through the strain–displacement gradient operators of the theory of elasticity and that the stress field is derived from the strain field through the constitutive laws). Let us now replace the continuum domain by a discretized domain and describe the computed state to be defined by the quantities $\bar{u}$, $\bar{\epsilon}$ and $\bar{\sigma}$, where again, we take that the strain fields and Stress fields are computed from the strain–displacement and constitutive relationships. It **is** clear that $\bar{\epsilon}$ is an approximation of the true strain field ϵ. What the Hu–Washizu theorem does, following the interpretation given by de Veubeke [9], is to introduce a "dislocation potential" to augment the usual total potential. This dislocation potential is based on a third independent stress field $\bar{\bar{\sigma}}$ which can be considered to be the Lagrange multiplier, removing the lack of compatibility appearing between the true strain field ϵ and the discretized strain field $\bar{\epsilon}$. Note that $\bar{\bar{\sigma}}$ is now an approximation of 6. The three-field Hu–Washizu theorem can be stated as

$$\delta\pi = 0, \tag{1}$$

where

$$\pi = \int \{\bar{\sigma}^{\mathrm{T}}\bar{\epsilon}/2 + \bar{\bar{\sigma}}^{\mathrm{T}}(\epsilon - \bar{\epsilon}) + P\}\,\mathrm{d}V$$

and ***P*** is.the potential energy of the prescribed loads.

In the simpler minimum total potential principle. which is the basis for the derivation of the displacement type finite element formulation in most textbooks, only one field (i.e. the displacement field u), is Subject to variation. However, in this more general three field approach, all three fields are subject to variation and lead to three sets of equations which can be grouped and classified as follows:

Variation on	*Nature*	*Equation*	
u	Equilibrium	$\nabla\bar{\bar{\sigma}}$ + terms from $\boldsymbol{P} = 0$	(3a)
$\bar{\bar{\sigma}}$	Orthogonality (compatibility)	$\int \delta\bar{\bar{\sigma}}^{\mathrm{T}}(\bar{\epsilon} - \epsilon)\,\mathrm{d}V = 0$	(3b)
$\bar{\epsilon}$	Orthogonality (equilibrium)	$\int \delta\bar{\epsilon}^{\mathrm{T}}(\bar{\bar{\sigma}} - \bar{\sigma})\,\mathrm{d}V = 0.$	(3c)

Table 1. Barlow and Gauss points for one-dimensional case, Scenario A

p	Node locations	u	$\bar{u}$	ϵ	$\bar{\epsilon}$	Gauss points	Barlow points "Best-fit"	Barlow points Aliasing
1	± 1	ξ^2	ξ	5	1	0	0	0
2	$0, \pm 1$	ξ^3	ξ^2		5	$\pm 1/\sqrt{3}$	$\pm 1/\sqrt{3}$	$\pm 1/\sqrt{3}$
3	$\pm 1/3, \pm 1$	ξ^4	ξ^3	ξ^3	ξ^2	$0, \pm\sqrt{3/5}$	$0, \pm\sqrt{3/5}$	$0, \pm\sqrt{5/3}$

$1, \xi, \ldots, \xi^4$ indicate polynomial orders from constant to quartic.

Equation (3a) shows that the variation on the displacement field u requires that the independent stress field $\bar{\sigma}$ must satisfy the equilibrium equations (∇ signifies the operators that describe the equilibrium condition). Equation (3c) is a variational condition to restore the equilibrium imbalance between $\bar{\sigma}$ and $\bar{\bar{\sigma}}$. In the displacement type formulation, we choose $\bar{\bar{\sigma}} = \bar{\sigma}$. This satisfies the orthogonality condition, seen in eqn (3c), identically. This leaves us with the orthogonality condition in eqn (3b). We can now argue that this tries to restore the compatibility imbalance between the exact strain field ϵ and the discretized strain field $\bar{\epsilon}$. In the displacement type formulation this can be stated as

$$\int \delta\bar{\sigma}^{\mathrm{T}}(\bar{\epsilon} - \epsilon)\,\mathrm{d}V = 0. \tag{4}$$

Thus we see very clearly that the strains computed by the finite element procedure are a variationally correct (in a sense, a least squares correct) "best approximation" of the true state of strain.

Reference [3] shows how to determine from this "best-fit" orthogonality condition, why Gauss point sampling gives strains of a higher accuracy than at any other point within an element domain, using the same beam element example that was used originally by Barlow [1]. In this note, we shall confine attention to the aliasing and best fit paradigms to see how they operate to determine the optimal points. We shall designate the optimal points determined by the aliasing algorithm as ξ_a, the Barlow points (aliasing), and the points determined by the best-fit algorithm as ξ_b, the Barlow points (best-fit). Note that ξ_a are the points established by Barlow [1] and MacNeal [4], while ξ_b will correspond to the points given in Refs [2, 3]. Note that the natural coordinates system ξ is being used here for convenience.

A one-dimensional problem

We shall take up the simplest problem, a bar under axial loading. This is the one-dimensional problem that corresponds to the problem used by MacNeal [4] to determine the Barlow points (aliasing) and compare them with the Gauss points corresponding to the order of polynomial chosen to do the discretization.

As in MacNeal [4], we shall assume that the bar is replaced by a single element of varying polynomial order for its basis function (i.e. having varying number of equally spaced nodes). Thus, from Table 1, we see that $p = 1, 2, 3$ corresponds to the basis functions of linear, quadratic and cubic order, implying that the corresponding elements have 2, 3, 4 nodes, respectively. These elements are therefore capable of representing a constant, linear and quadratic state of strain/stress, where strain is taken to be the first derivative of the displacement field. We shall adopt the following notation: the true displacement, strain and stress field will be designated by u, ϵ and σ. The discretized displacement, strain and stress field will be designated by $\bar{u}$, $\bar{\epsilon}$ and $\bar{\sigma}$. The *aliased* displacement, strain and stress field will be designated by u^a, ϵ^a and σ^a. Nodal displacements will be represented by u_i.

We shall examine three scenarios. In the simplest, Scenario A, the true displacement field u is exactly one polynomial order higher than what the finite element is capable of representing—it is in this case, and only in this case, that the Barlow points can be determined exactly in terms of the Gauss points. In Scenario B, we consider the case where the true field u is two orders higher than the discretized field $\bar{u}$. The definition of an identifiable optimal point becomes difficult. In both Scenarios A and B, we assume that the rigidity of the beam is a constant, i.e. $\sigma = D\epsilon$. In Scenario C, we take up a case where the rigidity can vary, i.e. $\sigma = D(\xi)\epsilon$. We shall see that once again, it becomes difficult to identify the optimal points by any simple rule.

Thus, for Scenarios A and B, the orthogonality condition becomes simply

$$\int \delta\bar{\epsilon}^{\mathrm{T}}(\bar{\epsilon} - \epsilon)\,\mathrm{d}V = 0, \tag{5}$$

whereas for Scenario C, it becomes

$$\int \delta\bar{\epsilon}^{\mathrm{T}} D(\xi)(\bar{\epsilon} - \epsilon)\,\mathrm{d}V = 0. \tag{6}$$

Note that we can consider eqn (5) as a special case of the orthogonality condition in eqn (6) with the weight function $D(\xi) = \mathrm{I}$. It is well known that this

Table 2. The Legendre polynomials P,

Order of polynomial	Polynomial P
0	1
	$(3 - 30\xi^2 + 35\xi^4)$

case corresponds to one in which a straightforward application of Legendre polynomials can he made. This point was observed very early by Moan [7]. In this case, one can determine the points where $\bar{\epsilon} = \epsilon$ as those corresponding to points which are the zeros **of** the Legendre polynomials. See Table 2 for a list **of** unnormalized Legendre polynomials. We shall show below that in eqn (5), the points of minimum error are the sampling points of the Gauss-Legendre integration rule only if $\bar{\epsilon}$ is exactly one polynomial order lower than ϵ. And in eqn (6). the optimal points no longer depend on the nature of the Legendre polynomials, making it difficult to identify the optimal points.

Scenario A

This **is** the example worked out in detail in MacNeal [4] (see Table 7.1 in Ref. [4] and Table 1 here). We shall consider FEM solutions using a linear (two-noded), a quadratic (three-noded) and a cubic (four-noded) element. The true displacement field **is** taken to be one order higher than the discretized field in each case.

Linear ***element*** *($p = 1$)*

$$u = \text{quadratic} = b_0 + b_1\xi + b_2\xi^2$$

$$\epsilon = \text{linear} = u_{,\xi} = b_1 + 2b_2\xi = \sum_{s=0}^{p=1} \epsilon_s P_s(\xi)$$

Note that we have written ϵ in terms of the Legendre polynomials for future convenience. Note also that we have simplified the algebra by assuming that strains can be written as derivatives in the natural co-ordinate system. It **is** now necessary to work out how the algebra differs for the *aliasing* and *best-fit* approaches.

Aliasing: at $\xi_i = \pm 1$, $u_i^a = u_i$; then points where $\epsilon^a = \epsilon$ are given by $\xi_a = 0$. (The algebra is very elementary and is left to the reader to work out.) Thus. the Barlow point (aliasing) are $\xi_a = 0$, for this case.

Best-fit: $\bar{u}$ = linear, is undetermined at first. Let $\bar{\epsilon} = c_0$, as the element is capable of representing only a constant strain. Equation (5) will now give $\bar{\epsilon} = c_0 = b_1$. Thus, the optimal point is $\xi_b = 0$, the point where the Legendre polynomial $P_1(\xi) = \xi$ vanishes. Therefore, the Barlow point (best-fit) for this example **is** $\xi_b = 0$.

Quadratic element ($p = 2$).

$$u = \text{cubic} = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3$$

$$\epsilon = \text{quadratic} = u_{,\xi}$$

$$= (b_1 + b_3) + 2b_2\xi - b_3(1 - 3\xi^2)$$

$$= \sum_{s=0}^{p=1} \epsilon_s P_s(\xi)$$

Aliasing: at $\xi_i = 0, \pm 1$, $u_i^a = u_i$: then points where $\epsilon^a = \epsilon$ are given by $\xi_a = \pm 1/\sqrt{3}$. (Again, the algebra is left to the reader to work out.) Thus. the Barlow points (aliasing) are $\xi_a = \pm 1/\sqrt{3}$, for this case.

Best-fit: $\bar{u}$ = quadratic. Let $\bar{\epsilon} = c_0 + c_1\xi$, as this element is capable of representing a linear strain. Equation **(5)** will now give $\bar{\epsilon} = (b_1 + b_3) + 2b_2\xi$. Thus, the optimal points are $\xi_b = \pm 1/\sqrt{3}$, the points where the Legendre polynomial $P_2(\xi) = (1 - 3\xi^2)$ vanishes. Therefore, the Barlow points (best-fit) for this example are $\xi_b = \pm 1/\sqrt{3}$.

Note that in these two examples, i.e. for the linear and quadratic elements, the Barlow points from both schemes coincide with the Gauss points (the points where the corresponding Legendre polynomials vanish). In our next example we will find that this will not be so.

Cubic element ($p = 3$)

$$u = \text{quadratic} = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + b_4\xi^4$$

$$\epsilon = \text{cubic} = u_{,\xi}$$

$$= (b_1 + b_3) + (2b_2 + 12b_4/5)\xi - b_3(1 - 3\xi^2)$$

$$-4b_4/5(3\xi - 5\xi^3)$$

$$= \sum_{s=0}^{p=3} \epsilon_s P_s(\xi).$$

Aliasing: at $\xi_i = \pm 1/3, \pm 1$, $u_i^a = u_i$: then points where $\epsilon^a = \epsilon$ are given by $\xi_a = 0, \pm\sqrt{5/3}$. Thus, the Barlow points (aliasing) are $\xi_a = 0, \pm\sqrt{5/3}$, for this case. Note that the points where the Legendre polynomial $P_3(\xi) = (35 - 5\xi^3)$ vanishes are $\xi_G = 0, \sqrt{3/5}$.

Best-fit: $\bar{u}$ = cubic. Let $\bar{\epsilon} = c_0 + c_1\xi + c_2(1 - 3\xi^2)$, as this element **is** capable of representing a quadratic strain. Equation *(5)* will now give $\bar{\epsilon} = (b_1 + b_3) +$ $(2b_2 + 12b_4/5)\xi - b_3(1 - 3\xi^2)$. Thus. the Barlow points (best-fit) for this example are $\xi_b = 0, \sqrt{3/5}$, the points where the Legendre polynomial $P_3(\xi) =$ $(35 - 5\xi^3)$ vanishes.

Therefore, we have an example where the aliasing paradigm does not give the correct picture about the way the finite element process computes strains However, the best-fit paradigm shows that, as long as the discretized strain **is** one order lower than the true strain, the corresponding Gauss points are the optimal points. Table I summarizes the results obtained so far.

The experience ***of*** this writer and many of his colleagues is that the best-fit model is the one that corresponds to reality—that if one were to actually solve a problem where the true strain varies cubically using a four-noded element which offers a discretized strain which **is** of quadratic order, the points of optimal strain actually coincide with the Gauss points.

Scenario B

So far, we have examined simple scenarios where the true strain is exactly one polynomial order higher than the discretized strain with which it is replaced. If $P_p(\xi)$, denoting the Legendre polynomial of order p, describes the order by which the true strain exceeds the discretized strain, the simple rule is that the optimal points are obtained as $P_p(\xi_b) = 0$. These are therefore the set of p Gauss points at which the Legendre polynomial of order p vanishes. Consider now a case where the true strain is two orders higher than the discretized strain—e.g. a quadratic element ($p = 2$) modelling a region where the strain and stress field vary cubically. Thus, we have

$$\epsilon = (b_1 + b_3) + (2b_2 + 12b_4/5)\xi - b_3(1 - 3\xi^2)$$

$$- 4b_4/5(3\xi - 5\xi^3).$$

$$\bar{\epsilon} = c_0 + c_1\xi$$

Equation (5) allows us to determine the coefficients c_i in terms of b_i; it turns out that

$$\bar{\epsilon} = (b_1 + b_3) + (2b_2 + 12b_4/5)\xi;$$

a representation made very easy by the fact that the Legendre polynomials are orthogonal and that therefore $\bar{\epsilon}$ can be obtained from ϵ by simple inspection. It is not, however, a simple matter to determine whether the optimal points coincide with other well known points like the Gauss points. In this example, we have to seek the zeros of

$$b_3(1 - 3\xi^2) + 4b_4/5(3\xi - 5\xi^3)$$

Since b_3 and b_4 are arbitrary, depending on the problem, it is not possible to seek universally valid points where this would vanish, unlike in the case of Scenario **A** earlier. Therefore, in such cases, it is not worthwhile to seek points of optimal accuracy. It is sufficient to acknowledge that the finite element procedure yields strains $\bar{\epsilon}$, which are the most reasonable one can obtain in the circumstances.

Scenario C

So far, we have confined attention to problems where σ is related to ϵ through a simple constant rigidity term. Consider an exercise where (the one dimensional bar again) the rigidity varies because the cross-sectional area varies or because the modulus of elasticity varies or both, i.e. $\sigma = D(\xi)\epsilon$. The orthogonality condition that governs this case is given by eqn (6). Thus, it may not be possible to determine universally valid Barlow points *a priori* if $D(\xi)$ varies considerably.

CONCLUSIONS

In this paper, we have critically evaluated two competing paradigms for the basis of the optimal points in finite element stress predictions. It would seem that the best-fit or best-approximation model is the one that has a more rational basis.

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