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TECHNICAL NOTE

A VARIATIONAL BASIS FOR BARLOW POINTS

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Abstract—This paper review the developments from recent studies on the derivation of consistent strain fields in modelling constrained media elasticity and on deriving consistent stress resultants and thermal stress fields. The review will consider the Hu–Washizu principle as the variational basis for the existence of the so-called Barlow points where stresses (or strains) have the highest accuracy.

INTRODUCTION

An interesting development that has emerged from some recent studies on the derivation of consistent strain fields in modelling of constrained media elasticity (to eliminate locking) [I] and on deriving consistent stress resultants [2] and thermal stress fields *[3]* was that the minimum total potential energy principle was inadequate to describe the capabilities of the displacement based finite element formulation. We had to turn to a general variational theorem called the Hu–Washizu principle {4} which is based on three independent fields, and to the Hellinger–Reissner two-field principle [5] which can be derived as a restricted **case** of the former, to explain many significant features of the displacement-based procedure as used in these studies **[I–31.** Their usefulness arises from the fact that they now allow simultaneous approximations on displacements, strains and stresses. This provides the basis for the power and versatility of these mixed rules in several key applications in finite element modelling. The Gauss–Legendre integration rule and the Legendre polynomials are seen to occupy a central position in such an interpretation. It is seen not only as a means of optimal numerical integration of element arrays, but also as a means of establishing points of accurate strain-stress recovery

In this paper these developments will be reviewed, as related to the use of the Hu–Washizu principle as the variational basis for the existence of the so-called Barlow points, i.e. points where stresses (or strains) have the highest accuracy. These are of a very fundamental nature and their significance is not well understood or appreciated by the finite element community.

THE MINIMUM TOTAL POTENTIAL THEOREM

The displacement-based finite element method has until quite recently, looked only to the minimum total potential energy principle (a single field principle in that the displacement field is the only unknown) to find a variational basis. However, this has been found inadequate to explain many interesting features of the method as well as to resolve many challenges. These are issues such as the displacement method Producing strains which are 'best-approximations' of the actual continuum strains. the inability of the conventional approach to avoid locking [1], the presence of stress oscillations in problems involving varying stiffness [2] and in thermal stress computation [3], etc. The more general multi-field mixed principles such as the Hu–Washizu principle [4] and the Hellinger–Reissner principle [5] (where strains and or stresses are also ...

been described as the basis of the formulation of the hybrid and mixed elements. In this paper we introduce one more aspect that supports the viewpoint that the complete theoretical basis for the displacement-based elements must rest on the Hu–Washizu theorem. This then allows many of the features and challenges faced by the method to be explained or resolved **[I–31.**

ACCURACY OF NUMERICAL INTEGRATION

To start with, we shall introduce a short statement about the use of numerical integration in a displacement-type finite element formulation. The construction of element matrices and arrays in the finite element method requires the integration of functions which are considered smooth and integrable over the element domain. Often, these functions are not easily integrable in an exact analytical way and it has been the practice to use a numerical integration (quadrature) formula to accomplish this. An important advantage of the numerical integration route is that general programs can be written for various classes of problems using universally applicable shape function routines.

Many rules of quadrature are available, e.g. the Newton–Cotes rules, of which the trapezoidal, and Simpson's rules are the most well known, the Gauss–Legendre rules, etc. **All** these are based on replacing the exact integral with a computation using sampling points at which the integrand is computed and multiplying these with appropriate weighting **terms** to determine the integral.

It is known that the Gauss–Legendre quadrature formula is the most efficient rule from the point of view of computational cost for a given accuracy for one-dimensional problems [6}. This actually follows from the fact that the theory to determine the optimal sampling points and weights leads to finding. the zeros of the corresponding Legendre polynomials. **These** points are used as sampling points for the Gauss–Legendre rule. Later, we shall see that the same fact has other useful implications for the use of the Gaussian rule for finite element applications. In multi-dimensional problems, the Gaussian rule is not always the most efficient, but despite this these rules are almost always favoured.

THE 'BEST-APPROXIMATION' RULE AND OPTIMAL POINTS FOR STRAIN RECOVERY

The displacement-type finite element procedure in unconstrained media ...

strain-fields, yields strain predictions which are variationally accurate smoothed approximations of the true strain fields within each element domain. We can show that this emerges from the basic variational or weighted residual nature of the displacement-type discretization process. To see how we progress from the continuum domain lo the discretized domain, we will find presently that it is best to develop the theory from the generalized Hu–Washizu mixed theorem [4] rather than the minimum potential theorem as is done in most textbooks on the finite element method.

Let the continuum linear elastic problem have an exact solution described by the displacement field u, strain field ϵ and stress field σ (we project that the strain field ϵ is derived from the displacement field through the strain–displacement gradient operators of the theory of elasticity and that the stress field is derived from the strain field through the constitutive laws). Let us now replace the continuum domain by a discretization domain and describe the computed state to be defined by the quantities $\bar{u}$, $\bar{\epsilon}$ and $\bar{\sigma}$, where again, we take that the strain fields and stress fields are computed from the strain–displacement and constitutive relationships. It is clear that $\bar{\epsilon}$ is an approximation of the true strain field ϵ. What the Hu–Washizu theorem does, following the interpretation given by de Veubeke [7], is lo introduce a 'dislocation potential' to augment the usual total potential. This dislocation potential is based on a third independent stress field $\hat{\sigma}$ which can be considered to be the Lagrange multiplier removing the lack of compatibility appearing between the true strain field ϵ and the discretized strain field $\bar{\epsilon}$. Note that $\hat{\sigma}$ is now an approximation of $\bar{\sigma}$. The three-field Hu–Washizu theorem can be stated as

$$\delta\pi = 0, \tag{1}$$

where

$$\pi = \int \{\bar{\sigma}^T\bar{\epsilon}/2 + \hat{\sigma}^T(\epsilon - \bar{\epsilon}) + P\}\,\mathrm{d}V \tag{2}$$

and P is the potential energy of the prescribed loads.

In the simpler minimum total potential principle, which is the basis for the derivation of the displacement-type finite element formulation in most textbooks, only one field (i.e. the displacement field u), is subject to variation. However, in this more general three-field approach, all three fields are subject lo variation and leads to three sets of equations which can be grouped and classified as follows:

Variation on	*Nature*	*Equation*	
u	equilibrium	$\nabla\hat{\sigma}$ + terms from $P = 0$	(3a)
$\hat{\sigma}$	orthogonality (compatibility)	$\int \delta\hat{\sigma}^T(\bar{\epsilon} - \epsilon)\,\mathrm{d}V = 0$	(3b)
$\bar{\epsilon}$	orthogonality (equilibrium)	$\int \delta\bar{\epsilon}^T(\hat{\sigma} - \bar{\sigma})\,\mathrm{d}V = 0.$	(3c)

Equation (3a) is easy to rationalize—the variation on the displacement field u requires that the independent stress field $\hat{\sigma}$ must satisfy the equilibrium equations (∇ signifies the operators that describe the equilibrium condition). We can interpret eqn (3c) as a variational condition to restore the equilibrium imbalance between $\hat{\sigma}$ and $\bar{\sigma}$. In the displacement-type formulation, we choose $\hat{\sigma} = \bar{\sigma}$. This satisfies the orthogonality condition seen in eqn (3c) identically. This leaves us with the orthogonality condition in eqn (3b). We can now argue that this tries to restore the compatibility strain field $\bar{\epsilon}$. In the displacement-type formulation this can be stated as

$$\int \delta\bar{\sigma}^T(\bar{\epsilon} - \epsilon)\,\mathrm{d}V = 0 \tag{4}$$

Thus, we see very clearly that the strains computed by the finite element procedure are a variationally correct (in a sense, a least-squares correct) 'best approximation' of the true state of strain.

Our task is now to point out from this why Gauss point sampling gives strains of a higher accuracy than at any other point within an element domain. This fact was first observed by Barlow [8] and these optimal points are often called Barlow points. Note that Barlow, and subsequent researchers, describe these as points of optimal stress recovery. However, the proof so far described here reveals that these points are firstly points of optimal strain recovery. Stresses are computed in a secondary way from the Strains and it is necessary lo establish that points of optimal stress coincide with points of optimal strain. This is done quite simply by introducing the constitutive relationship $\sigma = D\epsilon$ and $\bar{\sigma} = D\bar{\epsilon}$ in eqn (4) to give

$$\int \delta\bar{\epsilon}^T(\bar{\sigma} - \sigma)\,\mathrm{d}V = 0. \tag{5}$$

It is worthwhile noting here that Moan [9] did not talk in terms of strains or stresses but argued that the principal derivatives in the energy functional are 'best-approximated' in a least-squares sense. It was also argued that this approximation of the principle derivatives takes place almost independently within each element. Since strains are based on principal derivatives, we can argue that it is more meaningful to recognize that the finite element method performs the smoothing or 'best-approximation' operation on the strains directly. This has been seen in the proof so far from eqns (4) or (5).

Herrmann [10] interpreted the finite element procedure as a stress error minimization procedure. In this, the best approximate solution from a family of trial solutions ($\bar{\sigma}$) is obtained by the minimization of a measure of error of the approximate stress field $\bar{\sigma}$. This will lead to an expression of the form

$$\int \delta\bar{\sigma}^T(\bar{\sigma} - \sigma)\,\mathrm{d}V = 0 \tag{6}$$

We can now compare eqn (6) with eqns (4) and (5). It would appear that the mixed principles derive the 'best approximation' theorem in a more physically meaningful way.

The beam element (after Barlow [8])

We shall now re-interpret the beam element example by Barlow [8] to show how optimal strain (and hence stress) points can be derived from eqns (4) or (5) using the orthogonal property of the Legendre polynomials. Consider a simple beam element based on cubic polynomials.

It is easy to show from this that the discretized strain field is a linear function

$$\bar{\epsilon} = \bar{w}_{,xx} = a_0 + a_1\xi.$$

Following Barlow, let us assume that this element has been used to model a beam region in which the strain distribution is actually quadratic (i.e. corresponding to a quartic displacement field for w). We therefore have a continuum strain which can be written as

$$\epsilon = w_{,xx} = (b_0 + b_2/3) + b_1\xi - b_2/3(1 - 3\xi^2).$$

Note that the true bending strain field has been exp[illegible] usual polynomial form. There is a very useful idea behind this step. When the continuum strain field ϵ and the strain field $\bar{\epsilon}$ are introduced into eqn (4). it is seen that the finite element procedure determines the coefficients a_i from the coefficients b_i to satisfy this orthogonality condition. This can be done by directly equating the terms of the coefficients of the respective Legendre polynomials. Thus

$$a_0 = b_0 + b_2/3 \quad \text{and} \quad a_1 = b_1. \tag{9}$$

This is possible because of the orthogonal nature of the Legendre polynomials, so that the coefficient associated with the quadratic polynomial in the true Strain field does not 'do work' on the coefficients of the discretized strain field, The important corollary from this is that in this specific example points at which the second-order Legendre function vanishes (in this case, $\xi = \pm 1/\sqrt{3}$) are points at which the discretized strain field matches the true strain field. Thus, in such a case, although the discretized strain field has only a 'linear' accuracy over the element domain, there are two points at which it yields Strains and stresses of a 'quadratic' accuracy.

Note that the argument so far is exactly valid (i.e. the optimal points are also the pints of exact strain recovery) only if $\bar{\epsilon}$ is exactly one order lower than ϵ. In a general case. the true continuum strain field ϵ may be several orders higher than the discretized strain field $\bar{\epsilon}$. In this case, the Gauss points corresponding to the finite element formulation (i.e. in a linear strain element, the points correspoinding to a two-point rule, etc.) are points where the strains are recovered to one order higher than that represented by its discretization strain field. In many cases, as in the beam element [8], we can say that at these points the strains are sampled at the same accuracy as the displacements at the nodes.

We find that the development of this argument from the Hu-Washizu theorem and the orthogonality condition derived hence [i.e. eqns (4) or (5)] to be more satisfying than those given by Barlow [8], Moan [9], Herrmann [10], or in the many textbooks that deal with this topic so far, e.g. Zienkiewicz and Taylor [11].

CONCLUDING REMARKS

In this paper we have introduced another example which demonstrates that a wmplete degree of confidence in the theoretical soundness of the displacement-type finite element formulation wuld be found only in the Hu–Washizu interpretation of the procedure.

Moan [9] had noted the importance of being able to know points of minimum error in the strains and stresses. The fact that the sampling points of the Gauss–Legendre quadrature rule are also the points at which strains and stresses are optimally sampled, and/or are consistently sampled in the case of constrained media elasticity [1], recommends the use of this rule in all finite element applications.

We have also seen in earlier work applications where the Hu-Washizu interpretation is essential to provide a completely rational basis—these are modelling of the constrained media elasticity [1]. stress resultant computation, where the modulus matrix varies over the element domain [2], and thermal stress recovery in problems where the temperature field varies over the element domain [3].

One important feature we have observed so far is that an interpolation field for the stresses (or stress resultants as the case may be) which is of higher order than the strain fields on which it must 'do work in the energy or virtual work principle is actually self-defeating because the higher order terms cannot be 'send. This is precisely the basis for de Veubeke's famous *limitation principle* [7], that 'it is useless lo look for a better solution by injecting additional degrees of freedom in the stresses'. We see clearly from our present studies that one cannot get stresses which are of higher order than are reflected in the strain expressions.

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