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AN OPTIMALLY CONSTRAINED 4 NODE QUADRILATERAL THIN PLATE BENDING ELEMENT

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Abstract—Recent advances in the study of optimal integration of quadrilateral plate bending elements based on Mindlin's plate theory points to a four-noded quadrilateral plate bending element which can be based on thin-plate theory. This isoparametric element uses independent bi-linear interpolations for the transverse displacement W and face rotations θ_x and θ_y. An optimal set of Kirchhoff constraints are enforced at the centre of the element by a penalty function type formulation. The choice of the number of constraints and the effect of grid-orientation and grid-distortion with regard to this is studied in detail.

INTRODUCTION

The difficulties experienced in satisfying the C^1 continuity requirements of Kirchhoff plate bending elements led to attempts to derive simple elements based on the Mindlin plate theory These require only C° continuity for the independently interpolated shape functions for the transverse displacement W and face rotations θ_x and θ_y[1]. Alternatively, within the context of Kirchhoff theory, one can still work with such independent interpolations but must enforce the Kirchhoff conditions that relate the slopes to face rotations at a discrete number of points[2].

Unfortunately, Mindlin plate bending elements behave very erratically in extremely thin situations and many specially tailored re-constructions must be judiciously made to ensure that it does not lock. This has been the subject of a great deal of interest in recent years and many techniques known as reduced integration, selective integration, hybrid stress formulations, mixed Hellinger–Reissner and mixed modified Hellinger–Reissner formulations, independent shear strain interpolations have been devised to eliminate this phenomenon[1, 3–10].

A recent investigation into the mechanism of shear locking indicated that an optimal integration strategy can be found for a rectangular four-node bi-linear Mindlin plate bending element[11]. The analysis hinges upon the identification of true Kirchhoff constraints when the shear strain energy is integrated and the removal of other spurious constraints. The latter arises due to lack of consistency in field definitions and can be removed by an appropriate integration strategy[12].

Ideally, for a consistent definition of shear strain, one must have associated with each term of the polynomial expansion of the shear strain field, contributions from both W and θ_x and W and θ_y interpolations respectively. This can be achieved only if unequal order interpolations are used for W and the face rotations θ_x and θ_y. With equal order interpolations, as is normally the case, it would turn out that some of the terms in the shear strain field have contributions only from the interpolations for the face rotations θ_x and θ_y. In the penalty limit of extreme thinness, these terms act to severely constrain the behaviour of the face rotations θ_x and θ_y, thus 'locking' the solution.

An optimal integration strategy for the shear strain energy is one which would correctly retain all the true Kirchhoff constraints and remove all the spurious constraints when equal order interpolations are used. An under-integration may remove some of the valid Kirchhoff constraints and in this manner, introduce zero-energy mechanisms that can degrade the behaviour of the element in thick plate situations. Fortunately, for a rectangular 4 node element, an optimal integration strategy can be found if the local axes ξ, η are aligned with the global axes x, y. This strategy incorporates a 2×2 Gaussian integration of the bending energy and separate 1×2 and 2×1 integration rules for the shear energy contributions from the $(\theta_x - w_{,x})$ and $(\theta_y - w_{,y})$ terms respectively.

However, as the formulation of the shear strain energy and the subsequent identification of the Kirchhoff constraints in the thin plate theory is intrinsically dependent on the definition of a x and y orthogonal Cartesian system, the retention and enforcement of all true Kirchhoff constraints is correct only in a rectangular form of the element. With distortion to nonrectangular forms or with arbitrary orientation of the mesh with respect to an $x - y$ global system, it is impossible to devise a simple integration strategy that will correctly retain all the valid constraints. Interestingly, this deficiency is most severely felt only in very thin situations, where locking is brought in as a result, whereas for thick plate situations, the optimal integration (or even an exact integration of the shear strain energy) strategy does not restrict the use of a distorted mesh so severely.

In contrast, the original bi-linear plate element of Hughes *et al.*[1] uses a uniform one point rule for the integration of all shear energy terms. In this way, only those Kirchhoff constraints that are not direction-dependent are retained. This is therefore an under-integration and the zero-energy mechanisms introduced thereby, restricts the use of the element beyond a moderately thick region. The element behaves very well in thin plate situations and suffers less from mesh distortion and mesh orientation than the optimally integrated element[11, 13].

A recent investigation into several three node triangular thin plate elements with three degrees of freedom per node (W, θ_x, θ_y) indicated that an element based on the discrete Kirchhoff theory model[2] behaves most efficiently[14]. It seems possible that an equivalent four-node quadrilateral with the same three degrees of freedom per node (W, θ_x, θ_y) can be constructed. This will use the same C_0 continuous bi-linear independent interpolations as used in the bi-linear Mindlin plate element[1, 11]. The shear strain energy is now replaced with equivalent Kirchhoff constraints derivable from the bi-linear interpolations and these are enforced in a penalty function form at the centroid of each element. This paper investigates such an element and describes its behaviour with regard to optimal number of constraints and the effects of mesh orientation and mesh distortions. It is seen that for most practical thin plate situations, a single constraint approach for each shear strain energy term is enough to produce an effective plate bending element.

FORMULATION

Following Ref. [1], the strain energy for an isotropic, linear elastic plate including shear deformation (after Mindlin plate theory) is

$$U(W,\theta_x,\theta_y)=\frac{1}{2}\frac{Et^3}{12(1-v^2)}\iint_A\Big[\theta_{x,x}^2+2v\theta_{x,x}\theta_{y,y}+\theta_{y,y}^2 + \frac{1-v}{2}(\theta_{x,y}+\theta_{y,x})^2\Big]\,dx\,dy + \frac{kGt}{2}\iint_A[(\theta_x-W_{,x})^2 + [\theta_y-W_{,y})^2]\,dx\,dy \qquad (1)$$

where x, y are the Cartesian co-ordinates, W is the transverse displacement, θ_x and θ_y are the face rotations, E is the Young's modulus and G the shear modulus, v is the Poisson's ratio, k is the shear correction factor, t is the plate thickness and A the area of the plate.

To derive a Kirchhoff plate theory equivalent based on a C_0 continuous formulation, we replace the shear parameter KGt by a penalty factor ϵ so that when ϵ takes extremely large limiting values, the Kirchhoff constraints $(\theta_x - W_{,x}) = 0$ and $(\theta_y - W_{,y}) = 0$ are satisfied in the element domain in a manner to be dictated by the type of interpolation function used for W, θ_x and θ_y.

Thus, in a penalty function formulation, we may interpret the problem as that of the minimisation of a function for bending strain energy augmented by a functional for the Kirchoff constraints enforced by a large penalty multiplier. The augmented functional will be

$$\bar{U}(W,\theta_x,\theta_y)=\frac{1}{2}\frac{Et^3}{12(1-v^2)}\iint_A[\theta_{x,x}^2+2v\theta_{x,x}\theta_{y,y}+\theta_{y,y}^2 + \frac{1-v}{2}(\theta_{x,y}+\theta_{y,x})^2]\,dx\,dy + \frac{1}{2}\epsilon\iint_A[(\theta_x-w_{,x})^2 + (\theta_y-w_{,y})^2]\,dx\,dy. \qquad (2)$$

The experience with the four-node bilinear Mindlin plate element[1, 11] shows that if the constraint terms are evaluated by an exact integration rule, then some of the constraints that emerge contain terms only from the interpolations for the face rotations. These spurious constraints impose severe restrictions on the behaviour of the face rotations and cause the "locking" of the solution in the penalty function limits[12].

Figure 1 showr a four-node isoparametric quadrilateral element. Since an isoparametric formulation is the basis of this bi-linear element, the interpolation functions used for the displacement and rotations are the same functions which map the element from the (ξ, η) plane to the (x, y) plane. We may write the interpolation functions as

$$W = a_0 + a_1\xi + a_2\eta + a_3\xi\eta \qquad (3a)$$

$$\theta_x = b_0 + b_1\xi + b_2\eta + b_3\xi\eta \qquad (3b)$$

$$\delta_y = c_0 + c_1\xi + c_2\eta + c_3\xi\eta \qquad (3c)$$

where the constants a_0 to C_3 are related to the nodal variables w_1 to w_4, θx_1 to θx_4 and θy_1 to θy_4.

Consider first, the constraint associated with the shear strain energy from the $(\theta_x - W_{,x})^2$ term. We may write this in terms of the bi-linear interpolation functions as

$$(\theta_x - W_{,x}) = \left(b_0 - a_1\frac{\partial\xi}{\partial x} - a_2\frac{\partial\eta}{\partial x}\right) + \left(b_1 - a_3\frac{\partial\eta}{\partial x}\right)\xi + \left(b_2 - a_3\frac{\partial\xi}{\partial x}\right)\eta + b_3\xi\eta. \qquad (4)$$

If this term had been evaluated by an exact integration (equivalent to a 2×2 Gaussian integration scheme) over the region of the element, we can expect four constraints in the penalty limit $\epsilon\to\infty$, for an arbitrary orientation of the element. These constraints will be related to constraints of the form

$$b_0 - a_1\frac{\partial\xi}{\partial x} - a_2\frac{\partial\eta}{\partial x} = 0 \qquad (5a)$$

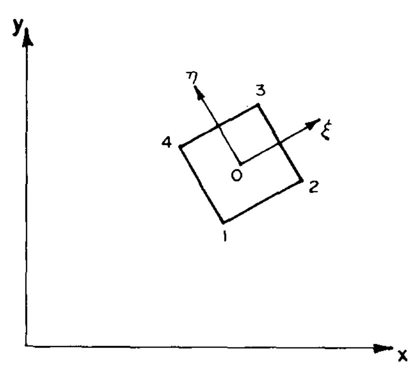

Fig. 1. Orientation of four-node isoparametric quadrilateral element.

$$b_1 - a_3 \frac{\partial \eta}{\partial x} = 0 \tag{5b}$$

$$b_2 - a_3 \frac{\partial \xi}{\partial x} = 0 \tag{5c}$$

$$b_3 = 0. \tag{5d}$$

If the element were rectangular and aligned with the (ξ, η) system to coincide with the (x, y) system, we have $\partial\eta/\partial x = 0$ so that the constraints represented by eqns (5b) and (5d) become

$$b_1 = 0$$

$$b_3 = 0.$$

These two constraints are clearly the spurious constraints of the sort recognised for the bi-linear Mindlin plate[11] and which were removed there by a 1×2 Gaussian integration of the shear strain energy associated with the $(\theta_x - W_{,x})^2$ term. An identical rule is now required for this penalty plate bending element. Note that this optimal rule exists only for a rectangular plan-form correctly aligned with the global system. For any other alignment or distortion from the rectangular plan-form, such a simple elimination of the spurious constraints by an optimal integration rule may not be possible. Instead, we explore the possibility of satisfying some of these constraints at the centroid of the element in a penalty function form after suitably weighting them with terms that account for the changes in grid alignment. We obtain some hints for these weights in the following analysis.

Constraint (5d) is eliminated altogether as it is a spurious constraint. Constraints from eqns (5b) and (5c) may be true or spurious depending on the alignment of the grid. In an element correctly aligned so that (ξ, η) corresponds to (x, y), a spurious constraint emerges from eqn (5b), namely

$$b_1 = 0.$$

A rotation by a right angle so that (ξ, η) will coincide with $(-x, y)$ will uncover a spurious constraint from eqn (5c), namely

$$b_2 = 0$$

whereas eqn (5b) would now be the true constraint, with terms from both interpolation functions. This suggests that a weighting of the constraint represented by eqn (5b) by a term $\partial\eta/\partial x$ or $\partial\xi/\partial y$ and a weighting of the constraint represented by eqn (5c) by a term $\partial\xi/\partial x$ or $\partial\eta/\partial y$. An examination of the curvature compatibility concept will give a correct clue to this. This concept had been successfully used by MacNeal[4] to derive the 1×2 and 2×1 integration schemes for his 4 node quadrilateral shell element. The curvature compatibility requirement in a thin-plate case for this term can be written as

$$(\theta_x - W_{,x})_{,y} = \left(b_1 - a_3 \frac{\partial \eta}{\partial x}\right)\frac{\partial \eta}{\partial y} + \left(b_2 - a_3 \frac{\partial \xi}{\partial x}\right)\frac{\partial \eta}{\partial y} + b_3 \eta \frac{\partial \xi}{\partial y} + b_3 \xi \frac{\partial \eta}{\partial y} = 0.$$

Clearly, at the origin ($\xi = 0$, $\eta = 0$), the valid curvature compatibility constraints that apply on the element are

$$\left(b_1 - a_3 \frac{\partial \eta}{\partial x}\right)\frac{\partial \xi}{\partial y} = 0 \tag{6a}$$

$$\left(b_2 - a_3 \frac{\partial \xi}{\partial x}\right)\frac{\partial \eta}{\partial y} = 0. \tag{6b}$$

Recognising that these are weighted forms of eqns (5b) and (5c) above, we recommend that the following three constraints evaluated at the origin ($\xi = 0$, $\eta = 0$) of each element form the most rational set of Kirchhoff type constraints for the $(\theta_x - W_{,x})^2$ terms. They are

$$\left(b_0 - a_1 \frac{\partial \xi}{\partial x} - a_2 \frac{\partial \eta}{\partial x}\right) = 0 \tag{7a}$$

$$\left(b_1 - a_3 \frac{\partial \eta}{\partial x}\right)\frac{\partial \xi}{\partial y} = 0 \tag{7b}$$

$$\left(b_2 - a_3 \frac{\partial \xi}{\partial x}\right)\frac{\partial \eta}{\partial y} = 0. \tag{7c}$$

In a similar fashion, a rational set of Kirchhoff type constraints for terms arising from the $(\theta_y - W_{,y})^2$ terms are

$$\left(c_0 - a_1 \frac{\partial \xi}{\partial y} - a_2 \frac{\partial \eta}{\partial y}\right) = 0 \tag{8a}$$

$$\left(c_1 - a_3 \frac{\partial \eta}{\partial y}\right)\frac{\partial \xi}{\partial x} = 0 \tag{8b}$$

$$\left(c_2 - a_3 \frac{\partial \xi}{\partial y}\right)\frac{\partial \eta}{\partial x} = 0. \tag{8c}$$

Thus, in the penalty function formulation adopted here, the squares of each of these six terms are evaluated at each element centroid and are added and multiplied by a large penalty parameter ϵ. The use of all six constraints eqns (7a)–(7c) and (8a)–(8c) is thus equivalent to the optimal integration rule is 1×2 and 2×1 for shear strain energy) used for the Mindlin plate element[11]. In order to have an equivalent of the 1 point rule for shear energy of the earlier Hughes element[1], we also include a provision for incorporating only the constraints represented by eqns (7a) and (8a). These two sets will be called the 6-constraint element or the 2-constraint element, for any general orientation of the element. The bending energy is evaluated in both cases by a 2×2 Gaussian integration rule.

NUMERICAL EXPERIMENTS

This element is a DKT type version of earlier elements[1, 3, 11] whose behaviour in terms of convergence and accuracy and in terms of stress predictions have been very well documented in those references. Therefore, no attempt will be made to repeat them here. The emphasis in the present study will be to examine the errors of the second kind that can emerge under certain conditions in this formulation. The appearance of these errors will be considered in the context of the number of the constraints enforced and in terms of the effect of grid distortion and grid orientation.

A recent investigation into the behaviour of finite elements for continuum problems which require description by more than one independent displacement field has shown that in certain limiting situations, the solutions are severely constrained[15]. This constraining effect is found to propagate in a $(l/t)^2$ fashion for beam/arch/plate/shell elements, where l is an element dimension (length of arch/beam elements or side of plate/shell elements) and t is the element thickness[15, 16]. In the penalty limit of extreme thinness in which l/t is taken to approach an infinitely large value, the constraining effect is so great that the results become meaningless and the solution is said to have "lock". This phenomenon has been well known for some time in various contexts as "shear locking", "parasitic shear" and more recently, "in-plane locking" or "membrane locking" when applied to a curved beam/arch/shell problems[17, 18]. These errors of the second kind exist only where exact integration of the element matrices are made and in many cases, can be removed by an optimal reduced integration strategy of the critical energy terms membrane energy for thin curved beams[12, 17, 18], shear energy for Mindlin type plates and shells[1–11]. In a DKT type formulation, these critical energy terms are replaced by equivalent constraints, appropriately enforced in a penalty function manner. It is therefore possible that in the present element which is based on independent displacement functions, there could be "locking" in a similar fashion as the Kirchhoff constraints are being enforced in the penalty limit, unless all constraints are correctly enforced. This may not be the case where the elements are distorted from a rectangular form or misaligned with respect to the original global orthogonal coordinate system. This is the subject of numerical experimentation in this section.

A quarter of a simply supported square plate of length $L = 10.0$, $E = 0.1092 \times 10^7$, $G = 0.42 \times 10^6$ and $\nu = 0.3$ is idealised by a 2×2 mesh (see Fig. 2). The transverse deflection W at the centre of the plate under the action of a uniform distributed load $p = 1$ is studied. Figure 3 shows the parameters that govern the grid layout and the grid orientation. The distance a is a measure of the distortion brought into the grid. When $a = 2.5$, an idealisation by square elements is obtained. When $a = 3.0$, an idealisation by four quadrilaterals is obtained. To test the effectiveness of the weighting scheme, it is necessary to study different orientations of the element local coordinate system to the plate global coordinate system. This is done by rotating the numbering scheme as shown in Fig. 3 so that three types of orientations are derived. For each example, a two-constraint element and a six-

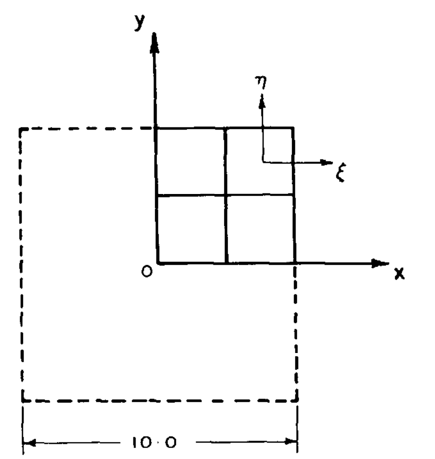

Fig. 2. Simply-supported plate under uniformly distributed load. Quarter-plate is idealised by 2×2 mesh.

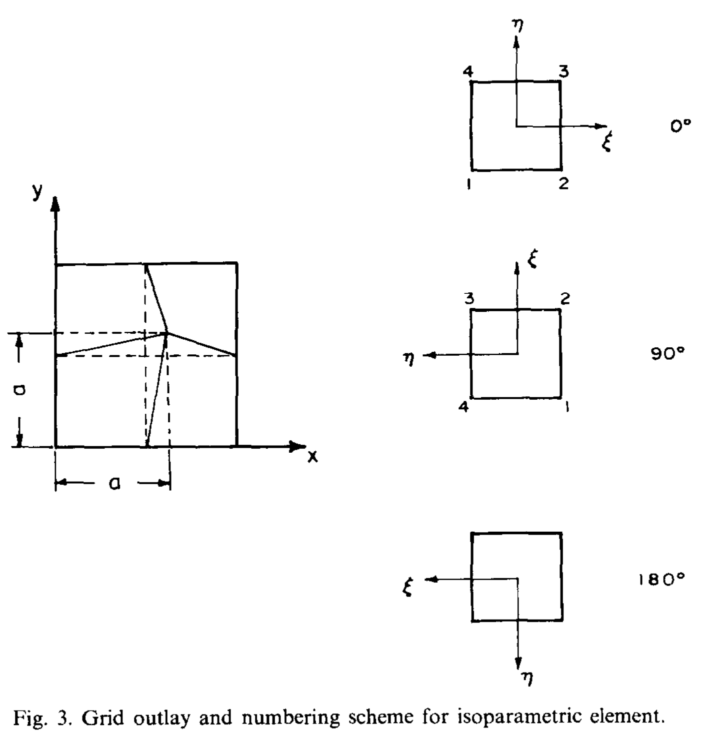

Fig. 3. Grid outlay and numbering scheme for isoparametric element.

constraint element form the basis of the finite element modelling.

Figure 4 shows the results for a plate of thickness $t = 0.1$, as the grid distortion parameter a is altered for the 2-constraint and 6-constraint elements. It is seen that for all these cases of axes orientations, the results are identical showing that the weighting scheme, which is based on the terms in the Jacobian defining the transformation, is an optimal one. Had the weighting scheme not been adopted, then even for an undistorted mesh (i.e. $a = 2.5$), some of the additional constraints in the 6-constraint case would have locked under certain alignments of the element to global axes systems. This was seen in the optimally integrated element[12]. The 2-constraint element has errors that change slowly with the distortion parameter whereas the 6-constraint element has errors that deteriorate rapidly as the same parameter is altered. It is necessary now to investigate whether these errors are of the first kind (errors due to discretisation that vanish rapidly as mesh size is reduced) or errors of the second kind (errors due to incorrect or "spurious" constraining that is magnified rapidly as a structural parameter such as l/t changes). It would appear that the 6-constraint element has errors of the second kind, which are presumably absent in the 2-constraint element. This can be studied by comparing the errors as the plate thickness is reduced. In order to enforce the penalty parameter in an $(l/t)^2$ fashion, the penalty multiplier ϵ is magnified in this fashion.

It has been shown in an earlier exercise[16] that errors of the second kind are best studied through the use of an additional stiffening parameter defined in the following form,

$$e = \frac{W_{\text{theory}}}{W_{\text{FEM}}} - 1.$$

It was seen in Ref. [16] that in a variety of problems, these errors can easily be recognised in a logarithmic plot of e against the relevant structural parameter that magnifies the error, usually the l/t ratio. For this example, using a 2×2 mesh throughout, we expect that if errors of the second kind are present, they would propagate in an $(l/t)^2$ fashion, and therefore, in an $(L/t)^2$ fashion.

Figure 5 shows the results of the same plate example as the thickness is varied from $t = 0.1$ to $t = 0.0001$. It is seen that for the undistorted mesh ($a = 2.5$) there are no errors of the second kind for both the 2-constraint and the 6-constraint element. For the distorted mesh ($a = 2.9$), there are no errors of the second kind with the 2-constraint element. However, with the 6-constraint element, there are errors of the second kind that vary exactly in a $(L/t)^2$ fashion. Figure 6 shows how the locking is initiated as the mesh is distorted. It is clear that even for a small distortion, severe constraints are brought in that can completely destroy the solution at high penalty parameters for a 6-constraint element. It appears that a weighting scheme that can remove errors of the second kind from a 6-constraint element

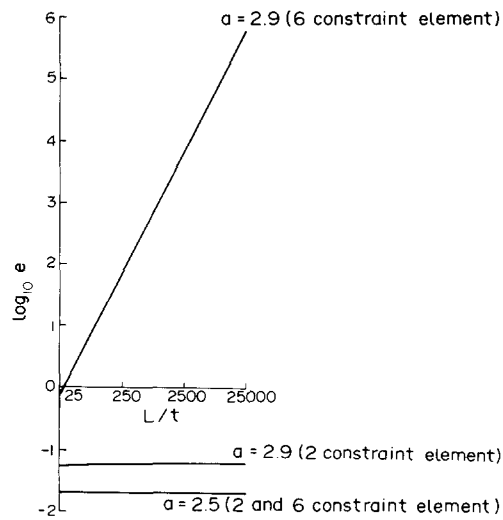

Fig. 5. Variation of additional stiffness parameter with (L/t) ratio for s.s. plate under u.d.

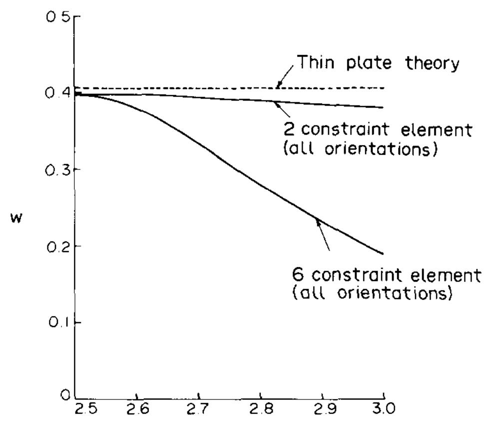

Fig. 4. Central deflection W vs distortion parameter a for 2-constraint and 6-constraint elements under 0°, 90°, 180° orientations.

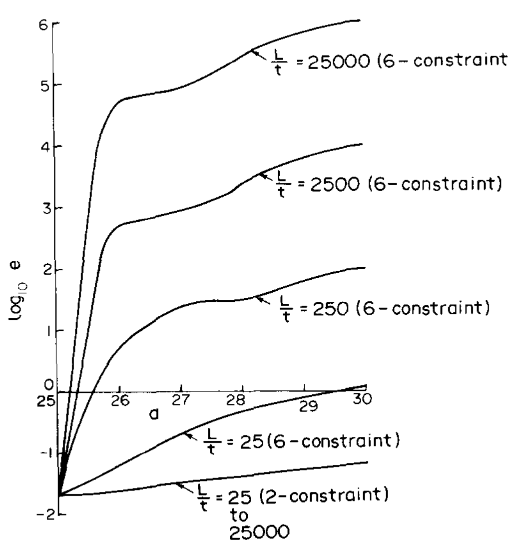

Fig. 6. Variation of additional stiffness parameter with mesh distortion parameter a for s.s. plate under *u.d.*

in an arbitrarily distorted quadrilateral may not be possible. However, with a 2-constraint element, no such difficulties arise. Thus, for very thin plates, where Kirchhoff type theory is valid, this 2-constraint element is identical to the 1 point Mindlin plate element of Hughes *et al.*[1]. However, if thicker plates are to be analysed, the optimally integrated element[12], which is free of zero energy mechanisms is needed.

CONCLUSIONS

As isoparametric four node quadrilateral element based on Kirchhoff thin plate theory and using independent interpolation functions for the transverse displacement W and face rotations θ_x and θ_y has been derived. The independent displacements are coupled through Kirchhoff constraints enforced in a penalty function approach to give a DKT-equivalent element. Numerical experiments confirm that a 6-constraint element, while optimal from a theoretical point of view, introduces undesirable constraints when the elements are distorted from a rectangular form. The optimal element would therefore have only 2 constraints relating to the $\theta_x = W_{,x}$ and $\theta_y = W_{,y}$ terms and such an element would uniformly reproduce thin plate behaviour under reasonable mesh distortions and orientations and for all values of penalty parameters.

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