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Abstract: We formulate the Landau problem in the context of the noncommutative analog of a surface of constant negative curvature, that is AdS_2 surface, and obtain the spectrum and contrast the same with the Landau levels one finds in the case of the commutative AdS_2 space.

Keywords: Non-Commutative Geometry, Field Theories in Lower Dimensions, Space-Time Symmetries, Anyons.

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Contents

1. Introduction

Noncommutative spaces have been of current interest with various motivations, in particular they arise in the framework of M-theory and in interesting settings of string and branes [1, 2, 3] (the bibliography is so vast that we do not attempt comprehensive referencing).

A sector of the study of the physics in noncommutative spaces concerns the exploration of the consequences for the quantum mechanics of one particle [4, 6, 7].

The physics in the noncommutative spaces is closely related to the problem of a charged particle moving on a surface with constant magnetic field giving rise to Landau Levels (see for instance [5]). Hence it is interesting to study Landau Levels by comparing the settings of commutative and noncommutative spaces. This research has been carried out for the case of the plane [7, 8, 9], the sphere [7] and the torus [10].

In this paper we consider the Landau Levels problem in the case of a surface of negative constant curvature, that is AdS_2. The commutative case has been studied in various papers, [11, 12], and also been extended to cover the case of the higher genus Riemann surfaces, which can be realized by a tessellation of AdS_2 [13, 14]. We may note in passing that, as higher genus Riemann surfaces appear as building blocks of higher orders in string perturbation theory, this provides a further link between AdS spaces and string theory, besides the celebrated relation with conformal field theories.

We first of all recall in section 2 the results on the commutative AdS surface, by making an explicit derivation, using appropriate complex coordinates and giving the resulting eigenfunctions, eigenvalues and their (infinite) multiplicity. We may also recall that for higher genus Riemann surfaces one gets the same spectrum but with a finite multiplicity dictated by the Riemann-Roch theorem [14].

Then we give the algebraic formulation of the same problem, by expressing the hamiltonian in terms of the generators of SO$(2,1)$ and representing AdS_2 as an embedding of a surface in the flat $(2+1)$-dimensional Minkowski space.

In the case of the Riemann surfaces a quantization condition for the magnetic field naturally emerges. Indeed, the wave function can be regarded as a differential form on the surface (which is in fact the proper object of the Riemann-Roch theorem) and the requirement of definite monodromy for transport along noncontractible loops implies a quantization condition. One can also require some periodicity properties of the wave function in the noncompact case of AdS_2, like it is common to use peridodic boundary conditions in quantum mechanical problems on noncompact spaces, for instance on the plane. The results are actually the same, except one has to assume a quantized value for the magnetic field.

The algebric formulation of section 2 will allow us to properly define the analogous problem in the noncommutative setting, section 3.1. The commutation relations among the Minkowski space coordinates are taken to be the ones of the SO$(2,1)$, and the appropriate Casimir is fixed in order to define the embedding in this case, similarly to the construction for the noncommutative sphere done in ref. [7]. The resulting setting is described by two commuting SO$(2,1)$ algebrae. We have not attempted the construction of noncommutative higher genus Riemann surfaces.

We have first of all to define the hamiltonian for the noncommutative case: we assume it to be formally identical to the one defined on the commutative surface.

The next issue concerns how to define the constant magnetic field. Here we have studied two options.

In the first one, we fix the two Casimirs of the two commuting SO$(2,1)$ algebrae, similarly to what was done in ref. [7] for the sphere. With this option the hamiltonian for the noncommutative case may not be formally the same as for the commutative surface, and the commutative limit may require some care and adjustment of parameters appearing in the hamiltonian, see ref. [7]. Here, we see that this option can be in conflict with the requirement that a universal (commuting or noncommuting) form of the hamiltonian makes physical sense.

In the second option, we keep fixed one observable among a complete set of mutually commuting ones. This observable is formally identical to the magnetic field defined in the commutative case. In this case, retaining the same hamiltonian, formally identical to the one defined on the commutative surface, makes always physical sense, and the commutative limit is straightforward.

By using the representation theory we obtain the spectrum in both options, section 3.2 and section 3.3 respectively. This is done in general for all possible representations of the algebra to yield the spectrum. Requiring, in addition, quantization of the eigenvalues in order to explore the possible noncommutative generalization of the features holding for Riemann surfaces, implies retaining only a quite small subset of the levels. Actually, the construction of the Landau Levels on the noncommutative version of the Riemann surfaces remains so far a completely unsolved problem, despite announcements which may have appeared in the title of some paper.

2. Landau levels in AdS_2

We consider a constant magnetic field on AdS_2, that is a magnetic field proportional to the curvature. We can describe AdS_2 by using complex coordinates $z, \bar{z}$ in the upper half plane $y > 0$ and taking the Poincare' metric $g_{z\bar{z}} = 1/y^2$:

$$ds^2 = \frac{dx^2 + dy^2}{y^2} . \tag{2.1}$$

The relevant covariant derivatives are

$$\nabla = \partial + \frac{B}{z - \bar{z}} \qquad \bar{\nabla} = \bar{\partial} + \frac{B}{z - \bar{z}} , \tag{2.2}$$

and

$$[\nabla, \bar{\nabla}] = \frac{B}{(2y^2)} . \tag{2.3}$$

We take the hamiltonian as

$$\begin{aligned} H &= -2g^{z\bar{z}}(\nabla\bar{\nabla} + \bar{\nabla}\nabla) - B^2 \\ &= -4g^{z\bar{z}}\nabla\bar{\nabla} + B(1 - B) \\ &= -4g^{z\bar{z}}\bar{\nabla}\nabla - B(1 + B) . \end{aligned} \tag{2.4}$$

Notice that, by taking into account the appropriate measure, the operators $-g^{z\bar{z}}\nabla\bar{\nabla}$ and $-g^{z\bar{z}}\bar{\nabla}\nabla$ are both semipositive definite, and therefore

$$H \geq |B|(1 - |B|) . \tag{2.5}$$

We take $B > 0$, since the case $B < 0$ is obtained by interchanging z and $\bar{z}$.

Consider the eigenvalue problem

$$H\Psi = E\Psi .$$

The lowest eigenstate, i.e. $E = B(1 - B)$, is obtained as a solution of $\bar{\nabla}\Psi_0 = 0$.

By defining $\tilde{\Psi}_0 = g_{z\bar{z}}^{B/2}\Psi_0$ we see that this means $\bar{\partial}\tilde{\Psi}_0 = 0$. This state is not unique. The different states can be labeled by the eigenvalues of an operator (that commutes with H), which we will, in the following, identify with a generator of SO(2,1):

$$J_3 = -\frac{i}{2}((1 + z^2)\partial + (1 + \bar{z}^2)\bar{\partial} + (z - \bar{z})B) . \tag{2.6}$$

The explicit form of the set of lowest level eigenstates is

$$\Psi_0^{(n)} = \frac{(z - \bar{z})^B}{(i + z)^{2B}} \left(\frac{-i + z}{i + z}\right)^n , \tag{2.7}$$

corresponding to the eigenvalues $J_3 = B + n$, with n any nonnegative integer.

We observe that under a holomorphic coordinate transformation $z \to z' = \frac{az+b}{cz+d}$ the wave function Ψ transforms like a differential form of the kind $T_{B/2}^{\bar{B}/2}$. That is, if z' is another local coordinate on the surface and the domain of z' intersect the domain of z, we first observe that $g_{z\bar{z}}dzd\bar{z} = g_{z'\bar{z}'}dz'd\bar{z}'$ and that

$$H' = UHU^{-1}\,,$$

with $U = (\frac{dz}{dz'} \cdot \frac{d\bar{z}'}{d\bar{z}})^{B/2}$. Therefore the wave function in the new coordinates is related to the wave function in the old ones by

$$\Psi' = U\Psi\,,$$

that is $\Psi(\frac{dz}{d\bar{z}})^{B/2}$ is invariant. It follows that $\tilde{\Psi} = (g_{z\bar{z}})^{B/2}\Psi$ transforms like a T_B form.

Also, the wave functions for the excited levels can be expressed as (we abbreviate, $g \equiv g_{z\bar{z}}$)

$$\Psi_l^{(n)} = g^{B/2-1}\partial g^{-(B/2-1)} \cdot g^{B/2-2}\partial g^{-(B/2-2)} \cdots g^{B/2-l}\partial g^{-(B/2-l)}\Phi\,, \tag{2.8}$$

where Φ is a $T^{\bar{B}/2}_{B/2-l}$ form. The eigenfunction equation requires $\tilde{\Phi} = (g_{z\bar{z}})^{B/2}\Phi$, which is a T_{B-l} form, to be holomorphic.

In conclusion the wave functions of the operator $J \circ J = j(1-j)$ are in correspondence with T_j holomorphic forms.

On a Riemann surface the Riemann-Roch theorem tells us that the dimensionality of the holomorphic T_j forms is $(2j-1)(h-1)$ where h is the genus of the surface and $j = B, B-1, \ldots$ Therefore, on the Riemann surface this quantity must be an integer. We expect this quantization to generalize to the possible noncommutative version of the Riemann surface.

Further, requiring periodicity for transport along noncontractible loops fixes j to be integer (it can also be half-integer if we only require periodicity up to $\pm$ signs, giving rise to the "spin structures" of the half-forms, which are familiar from perturbative String theory). Periodic boundary conditions are very natural for eigenvalue problems and we can choose to require it even for the AdS_2 surface (which can be considered as a limiting case of a very large Riemann surface). Therefore we are led to consider the case of j to be integer (and hence we will restrict B to integer values) and futher investigate the noncommutative version of this requirement.

In the language of group theory, considering j integer means considering representations of the group SO(2, 1), rather than just of its algebra, see ref. [15] for an illuminating discussion. Thus this requirement is immediatly transferred to the noncommutative case, where it implies requiring the quantum numbers labeling the representations to be integers (or half-integers, if we allow for $\pm$ signs). We will contrast the relevance of the more general (non integer) representations of the algebra and the richer set of spectrum it gives with the restricted set applicable for Riemann surfaces etc. in our analysis.

This discrete part of the spectrum, which we will call Landau Levels, comprises the eigenvalues

$$E_j = j(1-j)\,, \tag{2.9}$$

with $j = B - l$ up to the maximal $l = B - 1$, each having a degeneracy corresponding to the eigenvalues $J_3 = (B-l) + n$, with n any nonnegative integer.

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The corresponding wavefunctions are

$$\begin{aligned}\Psi_l^{(n)} &= (\partial - (B/2-1)\partial lng_{z\bar z})\cdot(\partial - (B/2-2)\partial lng_{z\bar z})\times \\ &\cdots\times(\partial-(B/2-l)\partial lng_{z\bar z})(i+z)^{2l}\Psi_0^{(n)}\,. \end{aligned} \tag{2.10}$$

Besides the above discrete levels, there is a continuum spectrum with nonnegative values for E.

The above results can be cast in an algebraic form, by making use of the invariance group of AdS_2, that is SO$(2,1)$. The AdS_2 manifold is conveniently described by embedding it in flat Minkowski manifold with coordinates x_1, x_2, x_3 with the constraint:

$$x\circ x = x_1^2 + x_2^2 - x_3^2 = -1\,, \tag{2.11}$$

where we have defined $V\circ W \equiv V_1W_1 + V_2W_2 - V_3W_3$ for two vectors V and W.

The SO$(2,1)$ generators J_1, J_2, J_3 satisfy the commutation relations:

$$[J_1,J_2] = -iJ_3 \qquad [J_2,J_3] = iJ_1 \qquad [J_3,J_1] = iJ_2\,, \tag{2.12}$$

$$[J_1,x_2] = -ix_3 \qquad [J_2,x_3] = ix_1 \qquad [J_3,x_1] = ix_2\,. \tag{2.13}$$

We are considering here the standard commuting operators for x, therefore

$$[x_l, x_n] = 0\,. \tag{2.14}$$

The relation with the previous formalism in terms of operators in complex coordinates are:

$$x_1 = i\frac{z+\bar z}{z-\bar z} \qquad x_2 = i\frac{1-z\bar z}{z-\bar z} \qquad x_3 = i\frac{1+z\bar z}{z-\bar z}\,, \tag{2.15}$$

and

$$\begin{aligned} J_1 &= i(z\partial + \bar z\bar\partial) \\ J_{3,2} &= -\frac{i}{2}\left((1\pm z^2)\partial + (1\pm\bar z^2)\bar\partial \pm B(z-\bar z)\right). \end{aligned} \tag{2.16}$$

We can verify that $x\circ J = -B$ and that $H = J\circ J$, therefore eq. (2.5) tells us that $J\circ J + B(B-1)\geq 0$.

It is well known [16] that the unitary representations of SO$(2,1)$ algebra are of two kinds: the discrete ones $D_j^{\pm}$, in which $J\circ J = j(1-j)$ and $J_3 = \pm(j, j+1,\ldots,j+n,\ldots)$ with $j\geq 1/2$, (but restricted to positive integer or half integer, if we look for representation of the group instead) and n nonnegative integer, and the continuum ones C_j in which $J\circ J$ is real positive.

Therefore we find that the Landau Levels, we have obtained correspond to D_j^+ with $j\leq B$.

Note that if the surface is more in general described by the constraint $x\circ x = x_1^2 + x_2^2 - x_3^2 = -r^2$ (previously $r=1$), then the metric is $ds^2 = r^2\frac{dx^2+dy^2}{y^2}$ and the hamiltonian (2.4) is now $H = r^2 J\circ J$ while $x\circ J = -rB$. Thus the discrete spectrum is $E_j = r^2 j(1-j)$, with the same $j = B-l$ as for $r=1$.

It is convenient to redefine the hamiltonian to be

$$H_0 = J \circ J \,, \tag{2.17}$$

for generic r, keeping the same definition $x \circ J = -rB$, so that the discrete spectrum is always $E_l = -(B-l)(B-l-1)$. We will keep the hamiltonian of eq. (2.17) in the generalization to the noncommutative surface described in the next section.

3. Landau levels in the noncommutative AdS_2

3.1 Definition of the problem

In order to define the noncommutative AdS_2, we introduce a set of noncommuting coordinates R_j with the SO$(2,1)$ algebra as relevant noncommuting rules:

$$[R_1, R_2] = -iR_3 \qquad [R_2, R_3] = iR_1 \qquad [R_3, R_1] = iR_2 \,, \tag{3.1}$$

$$[J_1, R_2] = -iR_3 \qquad [J_2, R_3] = iR_1 \qquad [J_3, R_1] = iR_2 \,, \tag{3.2}$$

where the J_i are the SO$(2,1)$ generators satisfying the algebra eq. (2.12).

Now, instead of requiring $x_1^2+x_2^2-x_3^2 = -r^2$ which describes AdS_2 in the commutative case, we require a fixed negative value for the Casimir $R \circ R \equiv R_1^2 + R_2^2 - R_3^2$. We know from the SO$(2,1)$ representation theory [16] that such a negative Casimir is of the form $R \circ R = r(1-r)$. Of the two discrete representations ($D_r^{\pm}$ distinguished by positive or negative R_3) we choose D_r^+, and then $R_3 = r, r+1, \ldots$ and so on.

We still maintain the hamiltonian to be:

$$H_0 = J \circ J \equiv J_1^2 + J_2^2 - J_3^2 \,, \tag{3.3}$$

as it is formally in the commutative case.

We note that the system is described by two mutually commuting SO$(2,1)$ algebrae, $K_i = J_i - R_i$ and R_i.

The relevant decompositions are (see ref. [17] for a useful summary on combining SO$(2,1)$ representations):

$$D_k^+ \otimes D_r^+ = \sum_{m=0} D_j^+ \,, \qquad j = k + r + m \,, \quad m \text{ integer} \,, \tag{3.4}$$

and

$$D_k^- \otimes D_r^+ = \left(\sum_{j}^{|r-k|} D_j^{\pm}, \; j = |r-k| \pmod 1 + n, \; n \text{ integer} \right) \oplus \int C_j \tag{3.5}$$

with D_j^+ for $r > k$ and D_j^- for $k > r$.

The other relevant formula is:

$$C_k \otimes D_r^+ = \left(\sum_{j=r+n} D_j^+ \,, \quad n = 0, 1, \ldots \right) \oplus \int C_j \,. \tag{3.6}$$

In the commutative case we fixed $B = -(x \circ J)/r$ and studied the spectrum with this additional constraint. Now we must analogously decide what to fix to represent the constant magnetic field.

1) We may follow the philosophy of Nair and Polychronakos ref. [7] and fix the value of the two Casimirs. Since $R \circ R$ is fixed by definition, this amounts to parameterize the magnetic field by the choice of $K \circ K$.

Note that the approach of ref. [7] is such that it allows a redefinition of the hamiltonian by an overall constant which can be positive or negative depending on the range of the parameters, in particular of the magnetic field.

Our approach is to keep always the definition of the hamiltonian as in eq. (3.3).

2) Alternatively we we may stick to the choice similar to the one in the commutative case, and use the commuting set of observables $J \circ J,\ K \circ K,\ R \circ R,\ J_3$ to keep fixed

$$K \circ K - J \circ J = R \circ R - 2R \circ J\,.$$

That is, like in the commutative case, we define and keep fixed $B \equiv -(x \circ J)/r$. This, we consider, is more appropriate to our definition of the problem, in keeping with the hamiltonian of eq. (3.3).

In this case the limit $r \to \infty$ (at fixed B) is expected to reproduce the physics of the Landau Levels on the commutative AdS_2 surface: in fact by defining $x_i \equiv R_i/r$ we get approximate commuting coordinates.

Let us explore the resulting spectrum for both the choices.

Following the discussion of the previous Section, we may require the eigenvalue j (and also r for consistency) to be integer or half-integer. This will be probably relevant for the setting of the Landau Levels problem on a noncommutative Riemann surface, which is still to come. Since the derivation is essentially the same, we consider j to be real as the general case. Requiring integer or half-integer j would simply imply to discard the levels in which j is not so; the allowed levels would be much sparser then in the real j case, also depending on a fine tuning of the values of r and B.

3.2 Case 1

Let us start with $K \circ K > 0$. Since $J_i = K_i + R_i$, the resulting spectrum of the hamiltonian is obtained from eq. (3.6). We get a continuum nonnegative part of the spectrum C_j and an unbounded discrete spectrum D_j^+ with $J \circ J = j(1-j)$, $j = r, r+1, \ldots$ up to infinity; therefore the hamiltonian (3.3) is unbounded from below and above.

If $K \circ K < 0$, we have to consider two cases, corresponding to the representations D_k^- and D_k^+.

For D_k^-, the relevant decomposition is eq. (3.5). We get a nonnegative continuum spectrum C_j as well as a finite discrete set of Landau Levels, $D_j^{\pm}$, with $J \circ J = -j(j-1)$; $j = |r-k|, |r-k|-1, \ldots, |r-k| \pmod 1$ and hence bounded from below.

For D_k^+ the relevant decomposition is eq. (3.4), giving thus again an unbounded negative discrete spectrum, and therefore the hamiltonian (3.3) is unbounded from below.

3.3 Case 2

Here we do not choose a particular value for $K \circ K$ and therefore the spectrum is composed of various parts, which must be consistent with the magnetic constraint

$$K \circ K - J \circ J = N \equiv -r(r-1) + 2rB\,. \tag{3.7}$$

We take N to be positive or negative. There are still several possibilities in the parameter space.

a) Since $N = -r(r-1) + 2rB$ and we would like to discuss in particular the case r large with B fixed (because in this limit we recover the commutative AdS_2), we begin by assuming $N < 0$. Let us define $M = -N > 0$: the magnetic constraint reads

$$J \circ J - K \circ K = M \equiv r(r-1) - 2rB\,. \tag{3.8}$$

In particular, when both $J \circ J$ and $K \circ K$ are in the discrete part of the spectrum, the constraint reads

$$\left(k - \frac{1}{2}\right)^2 - \left(j - \frac{1}{2}\right)^2 = M\,. \tag{3.9}$$

We analyze different ranges for M.

ai) $\sqrt{M} \geq r$: this means $B < -1/2$.

We find that D_k^+ cannot occur because (3.4) is incompatible with the constraint (3.8) which would require $(j - 1/2)^2 - (k - 1/2)^2 = -M$ which is impossible since $j > k$.

The case D_k^- is possible.

First of all, from (3.5) we can have the continuum C_j, with j real positive and $j^2 = M - k(k-1)$; since k is an arbitrary positive parameter, we get the part of the continuum spectrum for $j \leq M$.

If $k > r$ we can also have a discrete part of the spectrum: in this case from (3.5) we have D_j^-. The constraint (3.9) is solved by:

$$j_l = \frac{1}{2}\left(\left|\frac{M}{n_l} - n_l\right| + 1\right) \tag{3.10}$$

$$k_l = \frac{1}{2}\left(\frac{M}{n_l} + n_l + 1\right). \tag{3.11}$$

We can always choose to restrict $n_l \leq \sqrt{M}$. Since from (3.5) we have $j_l = k_l - r - l$ we get $n_l = r + l$, with l a nonnegative integer; this is consistent with $\sqrt{M} \geq r$. Since the minimum (maximum) possible j_l is obtained for the maximum (minimum) possible n_l, we get the following discrete part of the spectrum $J \circ J = -j_l(j_l - 1)$:

$$\frac{1}{2} \leq j_l = \frac{1}{2}\left(\frac{M}{r+l} - r - l + 1\right) \leq \frac{1}{2}\left(\frac{M}{r} - r + 1\right). \tag{3.12}$$

with l integer restricted by $r + l \leq \sqrt{M}$.

If $r > k$ there is no discrete spectrum for $\sqrt{M} \geq r$.

In fact we would have from (3.5) $j_l = r - k_l - l$; by using (3.10), (3.11) this gives $n_l = \frac{M}{r-1-l} \leq M$.

Now $n_l(\min) = \frac{M}{r-1}$ corresponding to $j_l(\max) = (r - \frac{M}{r-1})/2$, whereas $j_l(\min) \geq j_l(n_l = \sqrt{M}) = 1/2$.

Therefore, this is possible iff $1/2 \leq (r - \frac{M}{r-1})/2$ which implies $r \geq \sqrt{M} + 1$.

Finally, the case C_k is also possible. It gives the part of the continuum spectrum for $j \geq M$, since $j^2 = M + k^2$ with k arbitrary. We do not get a discrete part from it because, from (3.8), this would require $-j(j-1) = M + k^2$, which is not possible.

Summarizing: for $\sqrt{M} > r$ we get the entire continuum spectrum ($0 \leq j^2 \leq \infty$) and the discrete part of the spectrum reported in eq. (3.12).

aii) $r - 1 \leq \sqrt{M} \leq r$: this means $-1/2 \leq B \leq 1/2 - 1/(2r)$.

From the analysis of the case ai) we conclude that in this case we have only the continuum spectrum.

aiii) $\sqrt{M} \leq r - 1$: this means $1/2 - 1/(2r) \leq B$ (and also $B < r - 1/2$ in order to have $M > 0$).

In this case from the analysis of the case ai) we conclude that we have the continuum spectrum, and the following discrete part of the spectrum $J \circ J = -j_l(j_l - 1)$, coming from the the representation D_k^+ and $r > k$:

$$\frac{1}{2} \leq j_l = \frac{1}{2}\left(r - l - \frac{M}{r-1-l}\right) \leq \frac{1}{2}\left(r - \frac{M}{r-1}\right). \tag{3.13}$$

with l integer restricted by $\frac{M}{r-1-l} \leq \sqrt{M}$.

Let us now consider the above results in the commutative limit $r \to \infty$ at fixed B, looking at the lowest part of the spectrum.

By taking the limit of eqs. (3.12) and (3.13), and keeping the integer l fixed (note that $\frac{M}{r-1-l} \sim r - 2B + l$ and that $\frac{M}{r+l} \sim r - 1 - 2B - l$), we find for $|B| > \frac{1}{2}$ the approximate discrete spectrum $J \circ J = -j_l(j_l - 1)$ with:

$$j_l \sim |B| - l \tag{3.14}$$

which is indeed the same result as for the commutative AdS_2. In the figures 1 and 2 we compare the discrete spectrum for the commutative and noncommutative case.

For $|B| < 1/2$ we only get the continuum spectrum.

b) Let us turn now for completeness to study the case

$$N \equiv -r(r-1) + 2rB > 0$$

which implies B large in the commutative limit $r \to \infty$.

The magnetic constraint now reads:

$$K \circ K - J \circ J = N > 0\,. \tag{3.15}$$

The analysis parallels the one done for the case a). When both $J \circ J = -j(j-1)$ and $K \circ K = -k(k-1)$ are in the discrete spectrum this constraint reads:

$$\left(j - \frac{1}{2}\right)^2 - \left(k - \frac{1}{2}\right)^2 = N\,, \tag{3.16}$$

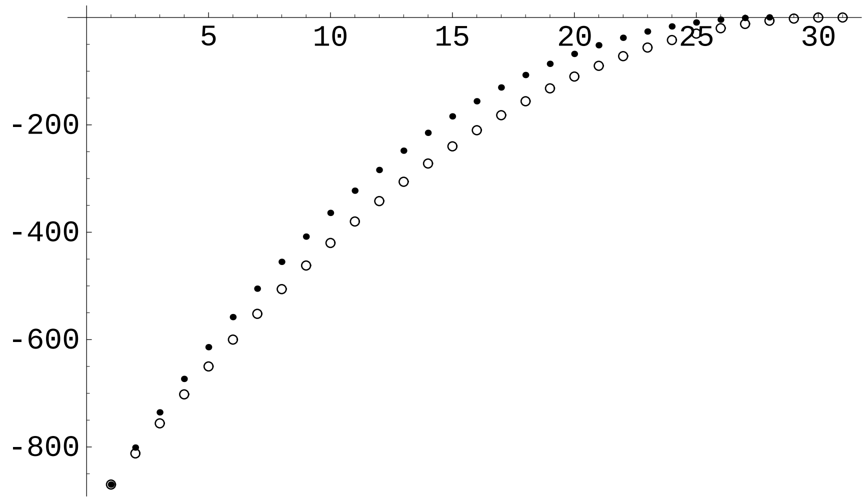

Figure 1: Plot of the discrete levels $H_0 = -j_l(j_l - 1)$ as a function of l for the case $r = 150$, $B = -30$ (full points), see eq. 3.12, compared with the levels of the commutative case (open circles).

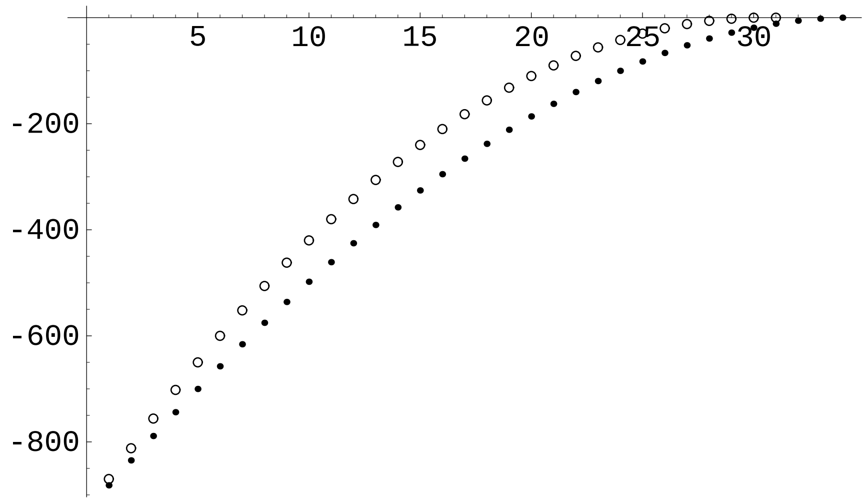

Figure 2: Same as figure 1 but for the case $r = 150$, $B = 30$, see eq. 3.13.

which can be solved by writing:

$$j_l = \frac{1}{2}\left(\frac{N}{n_l} + n_l + 1\right) \tag{3.17}$$

$$k_l = \frac{1}{2}\left(\frac{N}{n_l} - n_l + 1\right). \tag{3.18}$$

With no loss of generality we have assumed $n_l \leq N$.

Also here we consider the three ranges of N.

bi) $\sqrt{N} > r$. This means $B > r - 1/2$.

Now we can have D_k^+ since the decomposition (3.4) is allowed, implying $n_l = r + l \leq \sqrt{N}$, with l nonnegative integer.

The maximum $j_l(\max) = (N/r + r + 1)/2$ is obtained for $n_l(\min) = r$, whereas the minimum $j_l(min) \geq \sqrt{N} + 1/2$ corresponds to the maximum possible n_l.

We thus get a discrete part of the spectrum $J \circ J = -j_l(j_l - 1)$:

$$\sqrt{N} + \frac{1}{2} \leq j_l = \frac{1}{2}\left(\frac{N}{r+l} + r + l + 1\right) \leq \frac{1}{2}\left(\frac{N}{r} + r + 1\right). \tag{3.19}$$

with l nonnegative integer restricted by $r + l \leq \sqrt{N}$.

Instead D_K^- is not allowed: from the decomposition (3.5) we find that all the possibilities are ruled out, since D_j^- would imply $j \leq k - r$ contradicting the constraint (3.15) which gives $j > k$, and the same constraint (3.15) would require for C_j that $j^2 = -k(k-1) - N$ which is nonsense.

As for the last possibility D_j^+, impying $r > k$, the analysis is slightly longer: from the parametrization (3.17) and (3.18) we get $n_l = \frac{N}{r-1-l} \leq \sqrt{N}$ (with nonnegative integer l) which in turn implies $\sqrt{N} \leq r - 1$, which is outside the range of N considered here.

We can also have C_k. The relevant decomposition is eq. (3.6) from which we get the continuum spectrum C_j, that is $J \circ J = j^2$, with any value for j since k is arbitrary and $j^2 = k^2 - N$.

Moreover, from eq. (3.6) we also get another part of the discrete spectrum D_j^+, that is $J \circ J = -j_l(j_l - 1)$, with $j_l = r + l$, with l nonnegative integer.

The magnetic constraint (3.15) gives now:

$$\left(j_l - \frac{1}{2}\right)^2 = N + \frac{1}{4} - k^2 \Rightarrow l = \frac{1}{2} + \sqrt{N + \frac{1}{4} - k^2} - r\,.$$

Since k is arbitrary, this gives a possible range $0 \leq l \leq 1/2 + \sqrt{N + 1/4} - r$.

Summarizing, we find for $\sqrt{N} > r$ the continuum spectrum $J \circ J = j^2$ for any j, and two parts of the discrete spectrum $J \circ J = -j_l(j_l - 1)$, namely the part described in eq. (3.19) and another part in which

$$r \leq j_l = r + l \leq \frac{1}{2} + \sqrt{N + \frac{1}{4}}\,. \tag{3.20}$$

Since l is zero or integer, the two parts do not overlap.

bii) $r - 1 < \sqrt{N} \leq r$. This means $r - 3/2 + 1/2r < B \leq r - 1/2$.

From the analysis of the case bi) we conclude that in this case we have only the continuum spectrum.

biii) $\sqrt{N} \leq r - 1$. This means $r/2 - 1/2 < B \leq r - 3/2 + 1/2r$.

From the analysis of the case bi) we see that D_k^+ is not allowed, whereas D_k^- is allowed for $r > k$, giving the discrete spectrum D_j^+, while the continuum C_j is forbidden by the constraint (3.15).

The eigenvalues for j_l are of the form of eq. (3.17) with $n_l = \frac{N}{r-1-l}$ with the nonnegative integer l bounded by $n_l \leq \sqrt{N}$.

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The maximum j_l corresponds to the minimum n_l, that is for $l = 0$, and the minimum j_l is obtained from the maximum $n_l \leq \sqrt{N}$ implying $\sqrt{N} + 1/2 \leq j_l$.

We thus get the discrete spectrum $J \circ J = -j_l(j_l - 1)$ with

$$\sqrt{N} + \frac{1}{2} \leq j_l = \frac{1}{2}\left(\frac{N}{r-1-l} + r - l\right) \leq \frac{1}{2}\left(\frac{N}{r-1} + r\right), \qquad (3.21)$$

with the nonnegative integer l bounded by $l \leq r - 1 - \sqrt{N}$.

Finally we can have C_k. From the analysis of the case bi) we see that we do not get here discrete spectrum, but only the continuum part C_j, with $j^2 = k^2 - N$ which is always possible for any j.

Summarizing, we find that for $\sqrt{N} \leq r - 1$ the continuum spectrum $J \circ J = j^2$ for any j, and the discrete spectrum $J \circ J = -j_l(j_l - 1)$ described in eq. (3.21).

Note. After this paper appeared on the net, we received a paper (hep-th/0201070) by B.Morariu and A.P.Polychronakos, which implied a contrast with our results. The present revised version clarifies some points raised by hep-th/0201070.

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