Kaun Banega Crorepati – A Million Dollars for a Mathematician

2. Poincaré Conjecture

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I will now embark on explaining as best as I can to the non-mathematician what the Poincaré conjecture is all about.

Poincaré Conjecture

The Poincaré conjecture is a problem in topology, an area which, as I mentioned earlier, is essentially the creation of Poincaré. The topologist studies geometric objects looking for properties that remain unchanged when the object is moved, stretched, contracted, bent – when it is subjected to a very wide class of 'transformations' called 'topological transformations'.

Before I explain what a topological transformation is, I must first say what a *geometric object* is for a topologist. Familiar objects such as triangles, polyhedra, circles, cubes and spheres which figure in Euclidean geometry are of course among geometric objects for topologists; one goes much farther: any subset, any aggregate of points of any shape or size in 3-dimensional space is a geometric object. But topology does not stop even there; it takes in the study of subsets of Euclidean spaces of all possible dimensions. The n-dimensionsal (Euclidean) space $\mathbb{R}^n$ is the collection of all possible n-tuples $(x_1, ..., x_n)$ of real numbers: just as a point in our familiar 3-dimensional space is a triple (x_1, x_2, x_3) of real numbers (the Cartesian coordinates of the point), a point in n-dimensional Euclidean space is an n-tuple. A geometric object in topology is any arbitrary subset of some n-dimensional Euclidean space.

We do not have a visual picture of geometric objects in higher (>3) dimensional Euclidean spaces, but they do confront us in the physical world. An event for the physicist takes place at a

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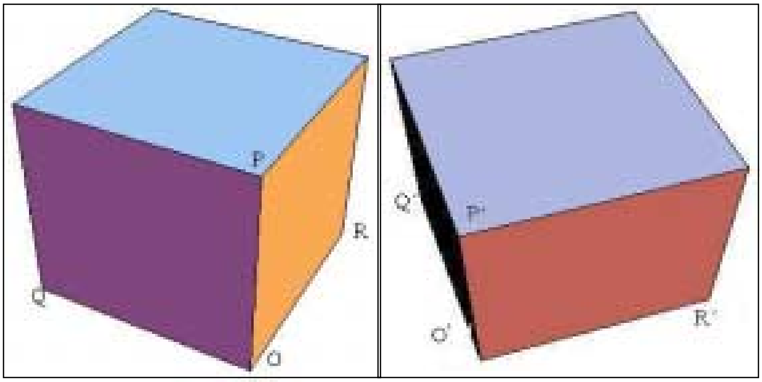

point in 3-dimensional space at a certain time. To specify the event then we need 4 numbers, three to indicate the point in R^3 and the fourth to give the time of the event. Thus we encounter 4-dimensional (Euclidean) space – the space-time continuum in physics.

Another example in physics arises in the study of motion of rigid bodies. If we fix four points O, P, Q and R on the rigid body such that OP, OQ and OR are mutually perpendicular, then the position of the rigid body in space is completely determined if one knows the coordinates of all these four points in 3-dimensional space and these make up four triples of numbers or equivalently twelve numbers.

Thus every possible position that a rigid body occupies in space (one calls this the *configuration space*) is determined by twelve numbers and twelve numbers give a point in R^{12}, the Euclidean space of dimension twelve ; in other words each position of a rigid body in 3-space determines a point in R^{12}. But not every point in R^{12} will correspond to one such position – the four triples cannot be chosen arbitrarily as the distances between any two of O, P, Q and R is fixed. In other words the configuration space is a subset of R^{12} but not all of it. Well, all this goes to show that there are reasons other than the mathematician's curiosity and imagination to study geometric objects in Euclidean spaces of higher dimensions.

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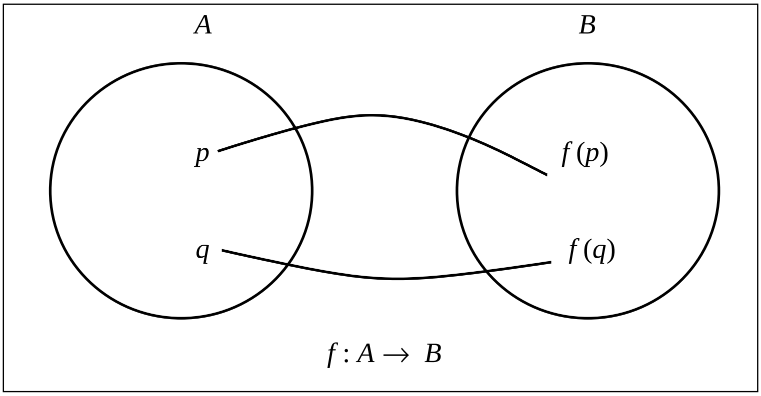
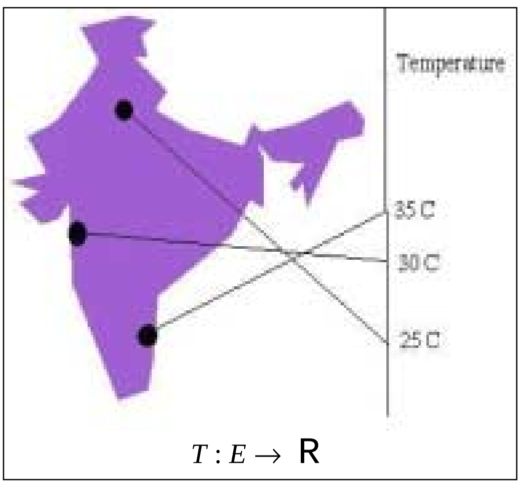


need to explain what that is before I tell you what a topological transformation is. Suppose that A and B are two geometric objects. A *function* f or a *map* from A to B – one writes also $f: A \to B$ – is an assignment of a point $f(p)$ in B for each point p in A.

Let me give some examples:

1. Suppose that we are following the path of an aircraft in space during a certain interval $[t_1, t_2]$ of time; then the assignment which associates to t in $[t_1, t_2]$, the position $f(t)$ of the aircraft at time t is a function f from the interval $[t_1, t_2]$ (a geometric object in R^1, the 1-dimensional Euclidean space) into R^3.

2. We can assign to each point X in the waters of the Arabian sea the pressure $P(X)$ at that point; P is a function from W to R where W denotes the aggregate of all points of the waters in the Arabian Sea.



$T : E \to \mathsf{R}$

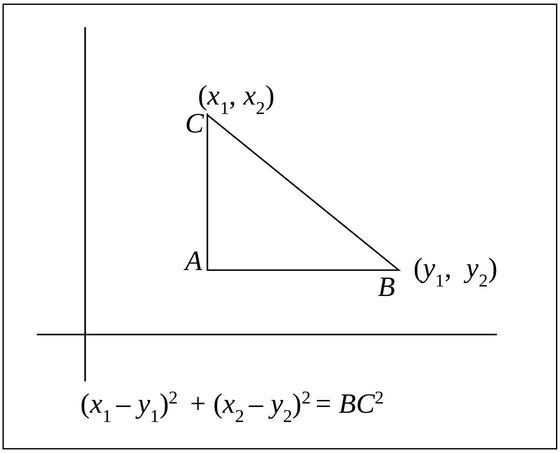
The temperature on the surface of the earth; as the point p on earth varies the temperature $T(p)$ will also vary; T is a function on E (= earth's surface) to R (= real numbers)

Suppose now A is a geometric object – A is a subset of R^n for some n. We then have a notion of distance between two points p with co-ordinates $(x_1, ..., x_n)$ and q with coordinates

$(y_1, ..., y_n)$; it is the number

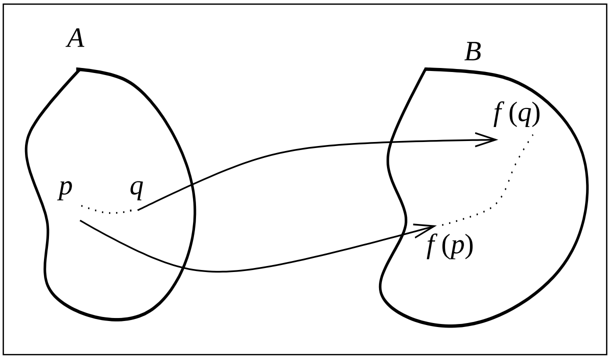
$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}.$$

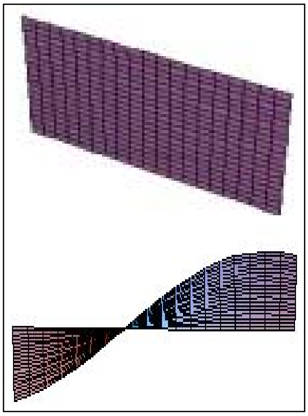
When $n = 3$ this is the familiar formula for the distance in 3-dimensional space. When $n = 2$, this is just the Pythagoras theorem.



With this definition of distance between two points in a geometric object, we can speak of continuous functions (or maps) from a geometric object A to a geometric object B. A function f from A to B is *continuous* if the following holds: if a point p of A moves in A towards a point q also in A, then $f(p)$ moves towards $f(q)$ – in other words as the distance between p and q (in A) shrinks to 0 so does the distance between $f(p)$ and $f(q)$ (in B).

The three examples of functions from physics described above are tacitly assumed to be continuous. In fact continuity is an essential philosophic underpinning in experimental science. An experiment has an input that consists of some l measured quantities and an output which again consists of some m measured quantities. An input may be regarded as a point in $\mathbb{R}^l$, but not all points may be possible inputs: in other words the possible inputs is a subset, say I of $\mathbb{R}^l$. Similarly the possible outputs can be thought of as a subset O of $\mathbb{R}^m$. The experiment e associates to an input i an output $e(i)$: e is a function from I to O and performing the experiment amounts to finding the value of e at various inputs (= points of I). Now one generally expects that if the experiment is repeated with the same inputs, the outputs will also be the same. Here by 'same' input or output we mean only that the input or output in the repeated experiment are approximately the same as those of the first experiment; all measurements are only approximate, there



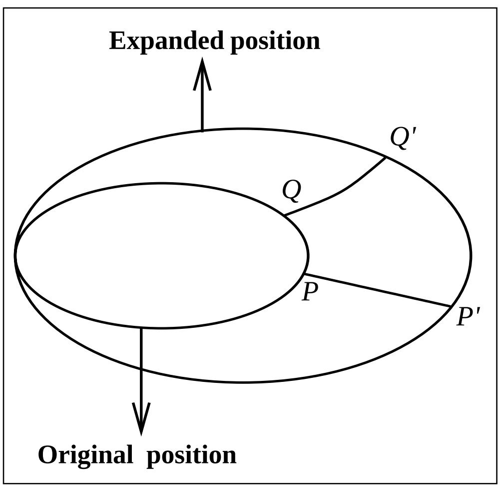


being limits to the accuracy of measurements. So it is being tacitly expected that if the input that is fed in the repeated experiment is close to the input in the first experiment, then the outputs in the two experiments too will be close to each other. This is nothing but the assumption that the function $e:I \to \mathsf{R}^m$ is continuous.

Let me give two further examples of continuous maps but now of a purely geometric character.

Example 1. Consider a rectangular strip of paper. It occupies a certain portion of space, a subset of R^3 which we call A, say. Now twist the paper along its length. After the twist the paper occupies another portion of space which we will call B. Each point p of A determines a point on the paper strip in its first position and when we twist the paper this point on the paper moves to occupy a new place – a point in B which we will call $f(p)$. We obtain thus a function f from A to B. This function is continuous. If a point p on the paper strip when in A moves to a point q, then $f(p)$ moves to $f(q)$.

Example 2. Similarly, when a balloon is inflated, we get a continuous map from the set occupied by the balloon at one point of time to the set occupied by the balloon at another point of time.



Observe that in these examples the distance between two points in position A gets modified when we move to position B but the notion of the point p moving closer and closer to a point q remains unchanged when we effect the transformation f.

In these last two examples one also has a map g from B to A: in the first example untwisting the strip gets us to B from A restoring the point $f(p)$ back to p. Deflating the balloon gives the reverse map g in the second case. These are examples of what the topologists call homeomorphisms or topological transformations.

A map f from a geometric object A to another geometric object B is a *topological transformation* or *homeomorphism* if it satisfies the following conditions.

(i) f is one to one: this means f takes (assigns) distinct points of A to distinct points of B; equivalently, whenever $f(p) = f(p')$ for p and p' in A, then $p = p'$.

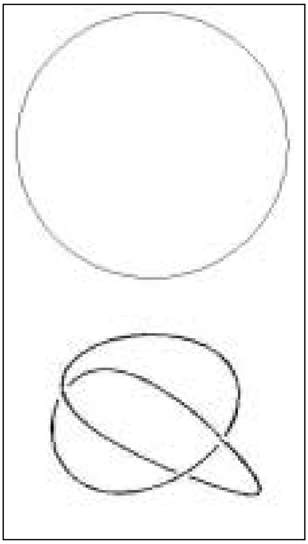
(ii) f is onto: this means that every point q of B is of the form $f(p)$ for some point of A.

(iii) f is continuous.

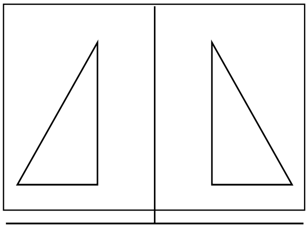
(iv) Let g be the map of B in A defined as follows: g assigns to each point q in B the unique point p of A such that $f(p) = q$. Note that condition (ii) ensures that there is such a point in A for every q in B while (i) ensures there is only one such point. It is clear then that $f(g(q)) = q$ for all q in B and $g(f(p)) = p$ for all p in A. The map g is said to be the inverse of f. We demand that $g{:}B \to A$ be also continuous.

A homeomorphism or a topological transformation from A to B is a continuous one-to-one, onto map from A to B such that the inverse $g{:}B \to A$ is also continuous. Two geometric objects A and A' are said to be *homeomorphic* or *topologically equivalent* if there is a topological transformation taking one to the other.

Loosely speaking in 3-dimensional space, bending, stretching, contracting or very generally any gradual change that does not break, tear or squash the object is a topological transformation of an object. But topological transformations are even more general: for example the circle and the knotted string are topologically equivalent to each other:



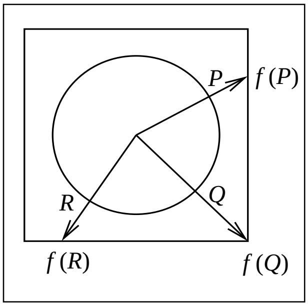
But one cannot deform a circular piece of string gradually into the knotted string in 3-dimensional space though it can be done in 4 dimensions. This is somewhat like our not being able to move a triangle into its mirror image continuously staying in the plane though we can do it by rotating around in 3 dimensions.



Examples

(1) There is a topological transformation *f* that takes the circumference of a circle into the perimeter of a square. Similarly the surfaces of a sphere and the cube are topologically equivalent.

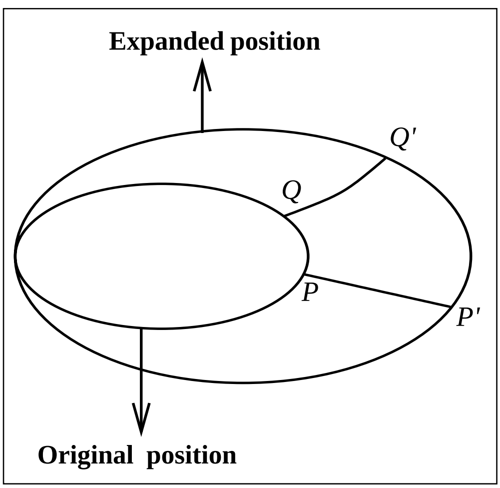
(2) When one inflates a balloon, the various shapes it acquires are all naturally homeomorphic to one another.

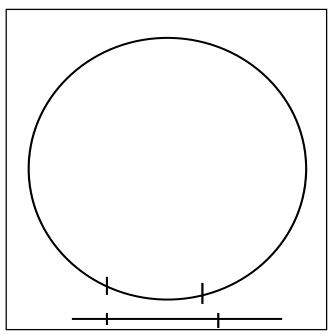


As the balloon expands each point on it moves to a new position. Two points P and Q move under the expansion of the balloon to points P' and Q' which are further apart than P and Q; nevertheless as P moves to Q, P' moves to Q' although more slowly. The reverse process of deflation gets the expanded balloon back to its original shape and thus we have a topological transformation.

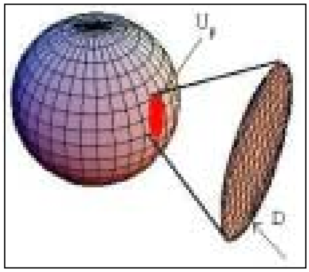
Of special interest to us are geometric objects called *manifolds*. These are objects that 'locally' resemble Euclidean space. For example, the circle. If you take a small piece of an arc near a point we can flatten it out into an interval on a line.

Similarly a portion of the sphere can be flattened out to a disc on the plane. An observer on the sphere with limited observational capabilities is likely to think that he is on a flat surface of infinite extent – like primitive man did of the earth's surface. An object in 3-dimensional space that looks locally like Euclidean plane or a closed disc of unit radius in the plane is called a *surface* or a 2-*manifold*. We can now generalise the notion to any dimension *n*. An *n-manifold* (*n*, a natural number) is a geometric object *M*





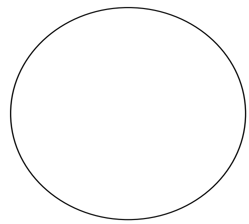
with the following property: given any point p in M there is a topological transformation f_p of the unit disc D in R^n (D is the set of all points within and up to unit distance from the origin in R^n) on a subset U_p in M with U_p containing all points within some distance δ (depending on p) of p; in particular U_p contains p.



A manifold M is *compact* if a finite number of such U_p cover all of M. The Euclidean n-space is evidently a manifold but it is not compact.

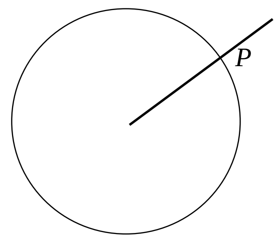
We now give examples of compact manifolds. Compact 2-manifolds are also called *surfaces*.

Circle



Manifold

Circle with line crossing it



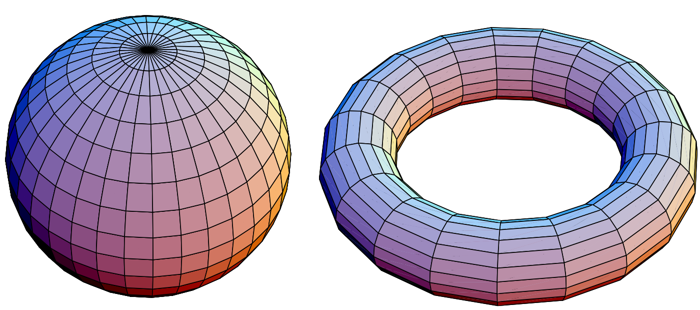
Not a manifold
P is a 'bad' point

The circle is a compact 1-manifold but the other figure in the picture is not a manifold. The point P does not satisfy the condition we want.

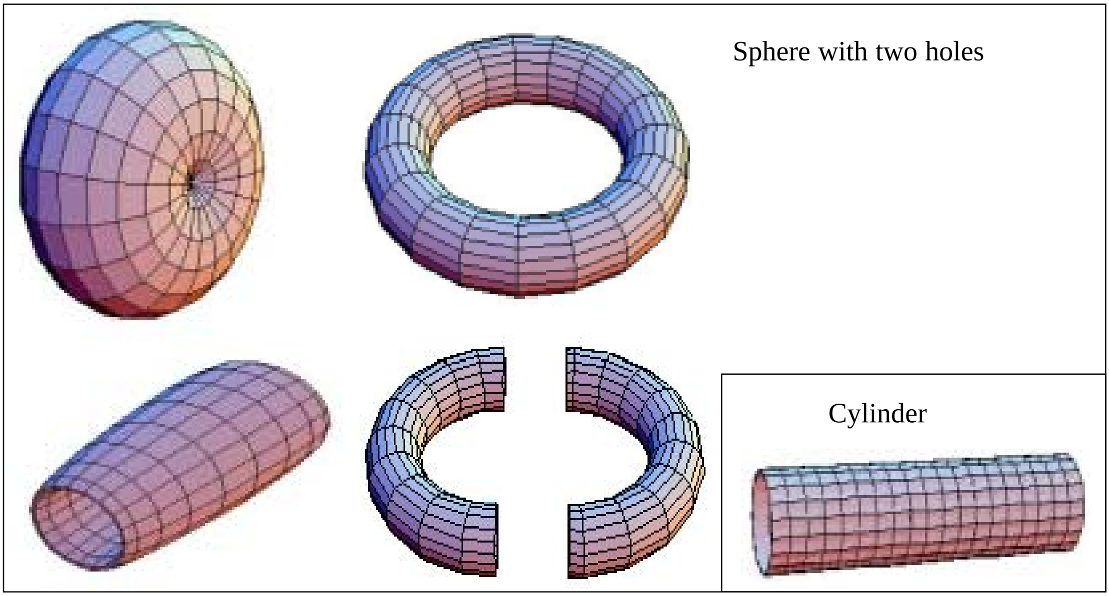
The sphere is a compact 2-manifold. The surface of a bicycle tube or a doughnut is a 2-manifold.

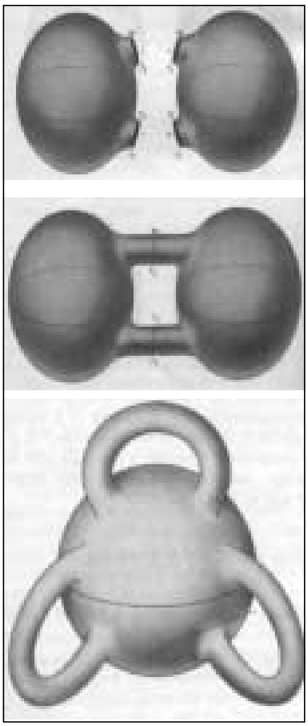
The surface of the doughnut is called a *torus*. It is topologically equivalent to a 'sphere with a handle' . By that we mean the surface obtained in the following manner. Make two circular holes in the sphere. Take a circular cylinder and glue it on to the sphere with the holes so that the circles that are at the ends of the cylinder fall along the boundaries of the two holes.

This idea leads to constructing more surfaces. One punches $2g$ holes (g a whole number) in a sphere and glues on g cylinders



Sphere Torus

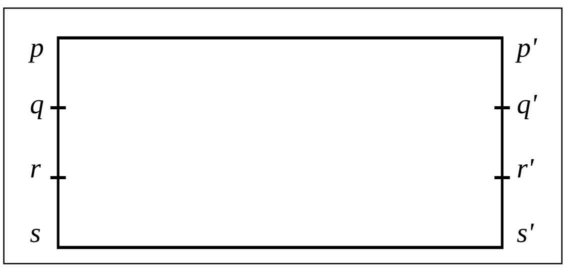


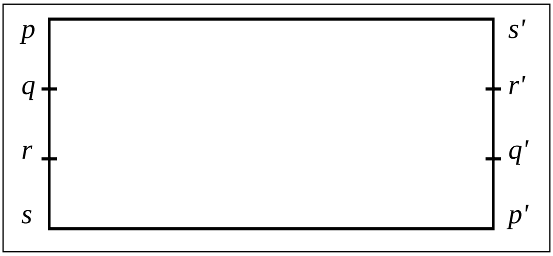


(= handles), each cylinder glued on by attaching its two bounding circles to two of the $2g$ holes.

As the picture shows the sphere with g handles is a surface with g 'holes' . It turns out that the sphere with g handles is not topologically equivalent to the sphere with g' handles if g is different from g'.

There is another more subtle way of getting new surfaces out of the sphere. The cylinder is a surface with the boundary consisting of two circles. There is a curious surface called the Moebius band whose boundary is just one circle. This can be described as follows: The cylinder can be thought of as a rectangular lamina with one pair of opposite sides glued together, each point on one



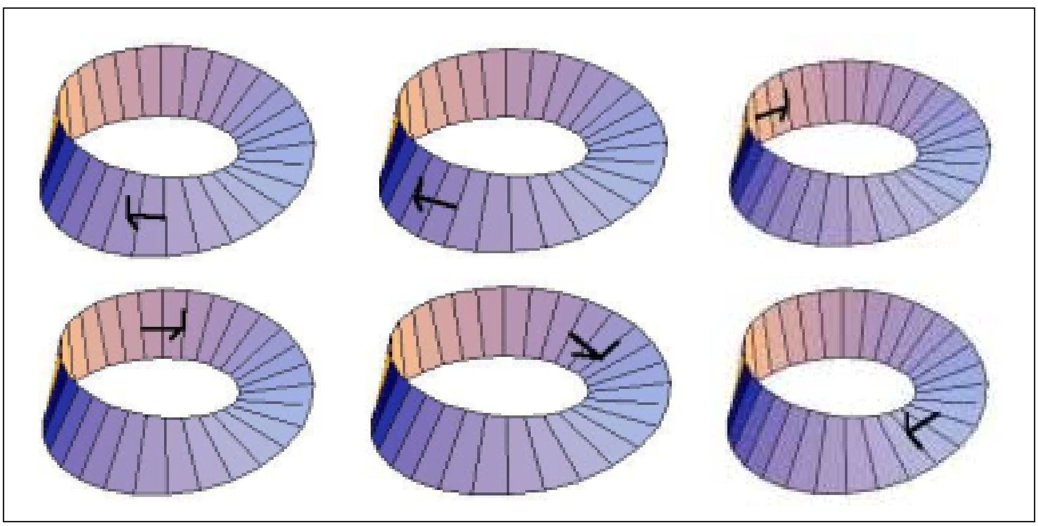


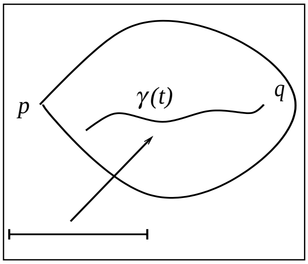
side being identified with the point on the opposite side lying on the line parallel to the other two sides.

The Moebius band is also obtained by gluing together opposite sides of a rectangle but now each point on the side is glued to the diametrically opposite point. This amounts to giving a 180 degree twist to the rectangle along its length before gluing the two edges.

The boundary of the surface consists of the lines PS' and SP' with S and S' glued and P and P' glued. So it is the curve going from P to $S'=S$ to $P'=P$, a circle. this has many interesting properties. It does not have two sides like a rectangular lamina or a cylinder. If we imagine a flat 2-dimensional creature swimming along the middle of the band one sees that when it goes round it is on the 'other' side of the surface and that its left and right hands have been interchanged.

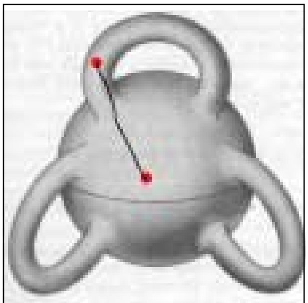
The Moebius band is also obtained by gluing together opposite sides of a rectangle but now each point on the side is glued to the diametrically opposite point.



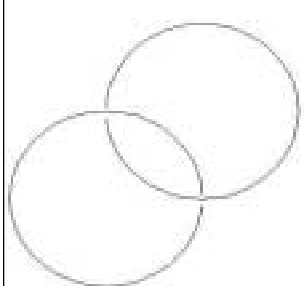


Now one can make a single puncture in the sphere and glue on the Moebius band to the boundary of this puncture along the band's own boundary. This cannot be done in 3 space – in any attempt other parts of the Moebius band away from the boundary curve will come in the way. It can however be carried out in 4-dimensional Euclidean space. Such a construction is called *attaching a cross cap*. We can perform attaching handles as well as cross caps together. Take a sphere with $2g+h$ holes and glue on a cylinder for each pair in the first $2g$ holes and glue on one Moebius band each for the other h holes. We get a surface with g handles and h cross caps.

The surfaces obtained in this way are 'connected'. A geometric object X in Euclidean n space is *connected* if every pair p, q of points of X can be connected by a path lying wholly in X. By a path from P to Q we mean a continuous function γ: $[0,1] \rightarrow X$ with $\gamma(0)=p$ and $\gamma(1)=q$. Connectedness is a property invariant under topological transformations – if A and B are geometric objects that are topologically equivalent then if A is connected so is B and conversely.



Sphere with handle



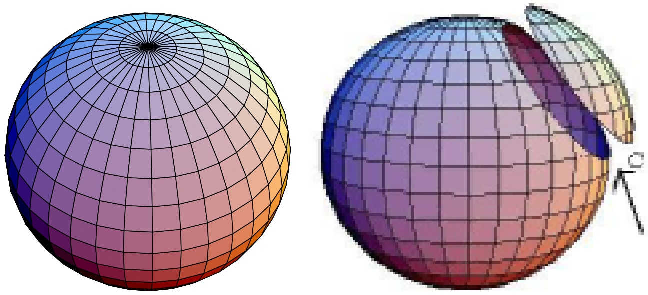
Linked circles

We can 'continuously' move from p to q staying inside X. The spheres with handles and cross caps are all connected.

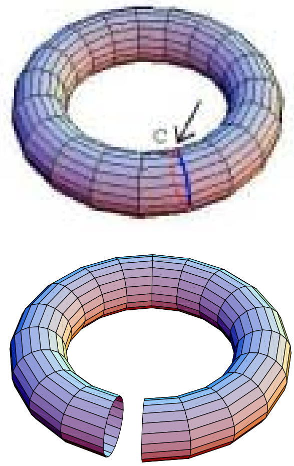
Topology of surfaces is well understood. It has been shown that any compact connected surface is topologically equivalent to a sphere with g handles and h cross caps for some whole numbers g and h. Moreover a sphere with g handles and h cross caps is not topologically equivalent to one with g' handles and h' cross caps unless $g=g'$ and $h=h'$. The numbers g and h remain unchanged under topological transformations.

The concept of connectedness enables one to see why the sphere and the torus are not topologically equivalent. On the torus there is a circle with the following property. If we remove this circle from the torus, the remaining portion of the torus stays connected.

Now if there was a topological transformation that takes the



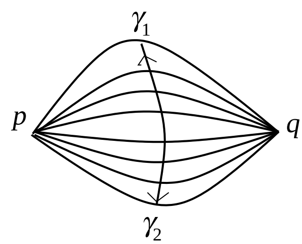
The sphere cut along the circle C gets disconnected.



The torus cut along the circle C remains connected.

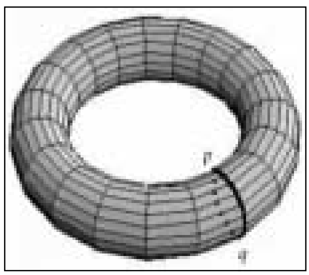
torus into the sphere, that transformation will carry the circle into a circle. The circle so obtained on the sphere when removed from the sphere should leave the remaining part connected. But there is no such circle on the sphere.

We talked about a geometric object X being connected. A connected geometric object X is said to be *simply connected* if any two paths connecting the same two points p, q on X can be deformed continuously into each other. This means that if γ_1 and γ_2 are two paths joining p to q, one can move the path γ_1 gradually into γ_2 all the time keeping p and q as the end points.



γ_1 can be moved to γ_2 gradually

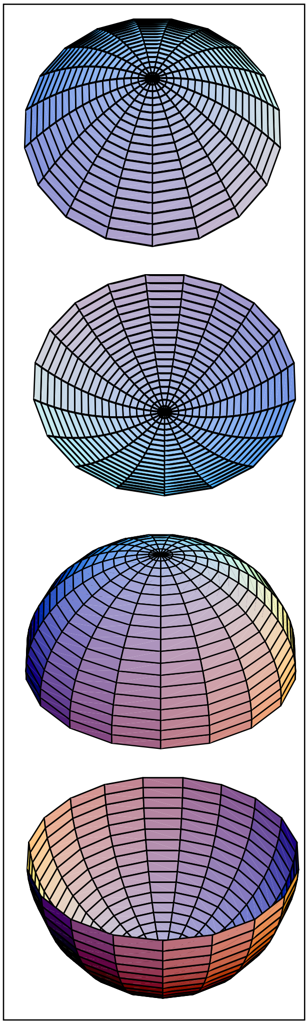
With this concept one can characterise the 2-sphere as the only compact surface which is simply connected. In the figure, the path from p to q indicated by an unbroken line cannot be deformed into the one indicated by the broken segments.



The Poincaré conjecture is about compact 3-manifolds. Examples of compact 3-manifolds are lot more difficult to understand as we cannot visualise them at all. The simplest example of a compact 3-manifold is the 3-sphere and this already cannot be realised as an object in Euclidean 3-space. One can think of this as the collection of points in 4-space R^4 at unit distance from the origin – i.e., it is the set of all points with coordinates x_1, x_2, x_3, x_4 satisfying

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

in analogy with the 2-sphere. Another way to describe the 3-



sphere again in analogy with a description of the 2-sphere is as follows: Let D^3 be the ball of unit radius in $\mathbb{R}^3$. Take two copies of D^3 and paste them along their boundary spheres – the corresponding construction gluing together two discs in the plane along their boundary circles yields the sphere as illustrated below. The disc for this purpose is replaced by the hemisphere to which it is topologically equivalent. The corresponding construction for the 2 dimensional sphere can be visualised.

The universe that surrounds us looks to us like stretching out to infinity in all directions, but this may only be the result of our limitations. Primitive man thought of the surface of the earth as a plane stretching out in all directions, but it turned out to be the 2-sphere. Our universe too may be a compact manifold like the 3-sphere or it could be a more complicated manifold. One can speculate on the possibility of a space traveller returning home after a long journey with his left and right hands interchanged as with the swimmer in the Moebius band. I can now state the million dollar question – the *Poincaré conjecture.*

Is any compact simply connected 3-manifold homeomorphic to the 3-sphere?

This question can be suitably generalised to higher dimensions and curiously it has been settled affirmatively for all higher dimensions – dimensions ≥ 4. In dimensions ≥ 5 it was done in the sixties by S Smale and in dimension 4 by M H Freedman in the eighties. Both Smale and Freedman were awarded Fields Medals essentially for settling this conjecture in higher dimensions.

A million dollars would certainly be welcome to the mathematician. But prize or no prize there were, there are and there will be topologists who would want to know if the Poincaré conjecture is true. To know the truth, not the million dollars, is the magnificient obsession that drives these people.

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