FUNDAMENTAL DOMAINS FOR LATTICES IN RANK ONE SEMISIMPLE LIE GROUPS

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Abstract.—We construct a fundamental domain Ω for an arbitrary lattice Γ in a real rank one, real simple Lie group, where Ω has finitely many cusps (i.e., is a finite union of Siegel sets) and has the Siegel property (i.e., the set $\{\gamma \in \Gamma | \Omega\gamma \cap \Omega \neq \phi\}$ is finite). From the existence of Ω we derive a number of consequences. In particular, we show that Γ is finitely presentable and is almost always rigid.

0. *Introduction.*—In this note we announce an extension of Borel's reduction theory (cf. ref. 1) to an arbitrary lattice in a connected, semisimple Lie group which is simple and of rank one over the real numbers **R** (see Theorem 1.2, below). In section 2 we describe some applications of this extended reduction theory, and in section 3 we give an indication of the proof of Theorem 1.2. Our proof of Theorem 1.2 relies on the results of D. A. Kazdan and G. A. Margolies (cf. ref. 4).

After we obtained these results, we learned that in the cases $SO(n,1)$ and $SU(n,1)$ A. Selberg had also proved Theorem 1.2, his methods being somewhat different from ours (it seems probable that his methods also work for general **R**-rank one groups). A few years ago we had many stimulating conversations with Professor Selberg, and in these conversations he was kind enough to show us his early results on the existence of unipotent elements in nonuniform lattices. It gives us great pleasure to extend to him our hearty thanks.

1. *Statement of the Main Theorem.*—*Throughout this paper* G will denote a linear, connected, semisimple group which is *simple and of rank one over* **R**. $\Gamma \subset G$ will denote a *lattice*, i.e., a discrete subgroup of G such that G/Γ has finite invariant volume. Moreover, Γ will be called a *uniform lattice* (resp. nonuniform) in case G/Γ is compact (resp. G/Γ is noncompact). Let $\mathfrak{g}$ denote the Lie algebra of G, and $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to a maximal compact subgroup K of G, which we fix once and for all. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$ be the Cartan decomposition corresponding to $\mathfrak{k}$ and for the whole paper we fix a nonzero vector $Y_0 \in \mathfrak{b}$. Let $\mathfrak{a}$ be the **R**-span of Y_0 and let A be the analytic subgroup of G corresponding to $\mathfrak{a}$. $\mathfrak{g}$ then decomposes into simultaneous eigenspaces relative to $Ad\ A$.

In fact there is a unique character α of A, so that $(\exp Y_0)^\alpha > 1$ and so that

$$\mathfrak{g} = \mathfrak{n}^{2\alpha} \oplus \mathfrak{n}^{\alpha} \oplus \mathfrak{z}(\mathfrak{a}) \oplus \mathfrak{n}^{-\alpha} \oplus \mathfrak{n}^{-2\alpha}, \tag{1.1}$$

where

$$\mathfrak{n}^{i\alpha} = \{v \in \mathfrak{g} | Ad\ a(v) = a^{i\alpha}\, v,\ a \in A\}, \qquad i = \pm 1,\ \pm 2,$$

and $\mathfrak{z}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$. The existence of the decomposition (1.1) follows from our assumption that G has **R**-rank one. $\mathfrak{n}^{\pm} = \mathfrak{n}^{\pm 2\alpha} \oplus \mathfrak{n}^{\pm\alpha}$ is a subalgebra of $\mathfrak{g}$, and we let $N^{\pm}$ denote the corresponding analytic subgroup. Recall that G

is a linear group; then it is known that $N^{\pm}$ is a maximal unipotent subgroup. We also have the Iwasawa decomposition $G = KAN^{+}$. We let $Z(A)$ denote the centralizer of A in G and we let $P = Z(A)N^{+}$. For $t > 0$ and an open, relatively compact subset $\eta \subset N^{+}$, we set

$$A_t = \{a \in A \mid a^{\alpha} < t\},$$

and we let $\mathfrak{S}_{t,\eta}$ denote the Siegel set $KA_t\eta$. We can now state our main theorem.

THEOREM 1.2. *Let $\Gamma \subset G$ be a nonuniform lattice. Then there exists $t_0 > 0$, an open relatively compact subset $\eta \subset N^{+}$, and a finite subset $\Xi \subset G$, such that:*

(i) *For all $b \in \Xi$, $b^{-1}N^{+}b \cap \Gamma$ is a lattice in $b^{-1}N^{+}b$.*

(ii) *If $\Omega = \bigcup_{b \in \Xi} \mathfrak{S}_{t_0,\eta}\, b$, then $\Omega\Gamma = G$,*

(iii) *the set $\{\gamma \in \Gamma \mid \Omega\gamma \cap \Omega \neq \phi\}$ is finite, and*

(iv) *there exists $t > 0$, so that for all $\gamma \in \Gamma$, $b,b' \in \Xi$ such that $\mathfrak{S}_{t,\eta}\, b\gamma \cap \mathfrak{S}_{t_0,\eta}b' \neq \phi$, we must have $b = b'$, and $b\gamma b'^{-1} \in P$.*

Remark: If Γ is a uniform lattice, then we can, of course, find an open, relatively compact subset Ω of G, so that *(ii)* and *(iii)* hold for this Ω.

2. *Applications of the Main Theorem.*—The following two results are direct consequences of Theorem 1.2.

THEOREM 2.1. *Let Γ be a lattice in G; then G/Γ is diffeomorphic to the interior of a compact differentiable manifold with boundary (the boundary being empty if G/Γ is compact).*

From Theorem 2.1, we obtain

THEOREM 2.2. (i) *Γ is finitely presentable.*

(ii) *If M is any Γ-module which is finitely generated as an abelian group, then the Eilenberg-Mac Lane groups $H^k(\Gamma,M)$ are finitely generated.*

(iii) *If $\{M_i\}_{i \in I}$ is an inductive family of Γ-modules with limit M, then $\{H^k(\Gamma,M_i)\}_{i \in I}$ is an inductive family of abelian groups, with limit $H^k(\Gamma,M)$.*

Kazdan has shown (cf. ref. 3) that lattices in semisimple Lie groups, all of whose **R**-simple factors have **R**-rank greater than one, are finitely generated. Assertion *(i)* of Theorem 2.2 may then be regarded as an extension of Kazdan's result. In the light of Theorem 1.2, the arguments in reference 6 carry over verbatim to give

THEOREM 2.3. *If Ad denotes the adjoint representation of G in $\mathfrak{g}$, then for every lattice $\Gamma \subset G$, we have $H^1(\Gamma,Ad) = 0$,* provided *that G is not locally isomorphic to $SL(2,\mathbf{R})$ or $SL(2,\mathbf{C})$.*

The following result has something to say about the case when G is locally isomorphic to $SL(2,\mathbf{C})$ as well.

THEOREM 2.4. *Assume that G is the topological identity component of the set of* **R***-rational points of an algebraic linear group defined over* **Q**. *If G is not locally isomorphic to $SL(2,R)$, then for any lattice $\Gamma \subset G$, there exists $g \in G$ and a subfield $k \subset R$ of finite degree over* **Q**, *such that $g\Gamma g^{-1}$ is contained in the k-rational points of G.*

When G is not locally isomorphic to $SL(2,\mathbf{C})$, then Theorem 2.4 follows from Theorem 2.3, assertion *(i)* of Theorem 2.2, and a result of A. Weil (cf. ref 7). When G is locally isomorphic to $SL(2,\mathbf{C})$, then an argument of J. P. Serre (originally Serre's argument was for arithmetic groups, but Theorem 1.2 allows one to carry this argument over to arbitrary nonuniform lattices) shows that $H^1(\Gamma,Ad)$ is almost never zero for nonuniform Γ. Nevertheless, one is able to obtain Theorem 2.4 in this case by proving that if Γ is a nonuniform lattice in G, then a deformation of Γ, which takes unipotent elements to unipotent elements, must be a trivial deformation. As a final application of Theorem 1.2, we give the following extension of a theorem of G. D. Mostow (cf. ref. 5).

THEOREM 2.5. *Let $G = SO(n,1)$, $n \geq 6$, and let Γ and Γ' be two isomorphic lattices in G. Let $X = SO(n)\backslash SO(n,1)$ be the symmetric space associated to G. Assume that Γ (and hence Γ') contains no nontrivial elements of finite order and that the C^∞ manifolds X/Γ and X/Γ' which are therefore defined are diffeomorphic. Then there is an automorphism $\phi: G \rightarrow G$, carrying Γ onto Γ'.*

Theorem 2.5 is deduced from Theorem 1.2 by using the S-cobordism theorem, some elementary facts from differential topology, and, finally, Mostow's theorem given in reference 5.

3. *An Indication of the Proof of the Main Theorem.*—Let X denote the symmetric space $K\backslash G$, and from now on let $\Gamma \subset G$ denote a *nonuniform* lattice. Let $\pi: G \rightarrow K\backslash G = X$ denote the natural projection and let $\bar{e} = \pi(e)$. After conjugating Γ, we can assume that $\bar{e}\gamma \neq \bar{e}$, for all $\gamma \in \Gamma$. Γ acts as a discontinuous group of isometrics on X, and we construct a fundamental domain $\mathcal{E}$ for this action of Γ in the following well-known manner: let $d(\ ,\ )$ denote the distance function on X corresponding to a fixed G-invariant Riemannian metric on X, and set

$$\mathcal{E} = \{x \in X \mid d(\bar{e},x\gamma) \geq d(\bar{e},x), \gamma \in \Gamma\}.$$

Definition 3.1: $Y \in \mathfrak{b}$ is called a *ray*, in case $Y \neq 0$ and $\bar{e} \exp tY \in \mathcal{E}$ for all $t \leq 0$.

It is well known, and not difficult to show, that $X = \mathcal{E}\Gamma$ and that $\mathfrak{b}$ contains at least one ray (recall that G/Γ is now assumed to be noncompact). For $Y \in \mathfrak{b}$, we define a subalgebra

$$\mathfrak{n}_Y = \text{linear span of } \{v \in \mathfrak{g} \mid Ad\ Y(v) = c\cdot v,\ c > 0\},$$

and we let N_Y denote the corresponding analytic subgroup of G. The following result is central for the proof of Theorem 1.2.

LEMMA 3.2 (Main Lemma). *For every ray $Y \in \mathfrak{b}$, $N_Y/N_Y \cap \Gamma$ is compact.*

The proof of this lemma rests on a series of results:

LEMMA 3.3. *For every ray $Y \in \mathfrak{b}$, $N_Y \cap \Gamma$ contains a nontrivial element.*

LEMMA 3.4. *If $\rho \in \Gamma$ is unipotent, if G_ρ is the centralizer of ρ, and if $\Gamma_\rho = \Gamma \cap G_\rho$, then G_ρ/Γ_ρ is compact.*‡

LEMMA 3.5. *If N' is a maximal R-unipotent subgroup, and if $N' \cap \Gamma$ contains a nontrivial element, then $N' \cap \Gamma$ contains a nontrivial element in the center of N'.*

We remark that N_Y is maximal **R**-unipotent, so that Lemmas 3.3 and 3.5 imply that $N_Y \cap \Gamma$ contains a nontrivial element in the center of N_Y. It is not difficult then to deduce Lemma 3.2 from this fact and from Lemma 3.4. The proof of Lemmas 3.3 and 3.5 makes use of the ideas given in reference 4. Though our proof of Theorem 1.2 is entirely free of case-by-case checks, it nevertheless seems curious that a case-by-case method does yield a relatively simple proof of

Lemma 3.5, *except in the low-dimensional cases* $SU(2,1)$, $Sp(2,1)$, and $F_{4(-20)}$, and *it is actually only for these cases that one requires the methods given in reference 4*, in the proof of Lemma 3.5.

We fix a K-invariant norm $\| \ \|$ on $\mathfrak{b}$. The following lemma is needed in order to obtain the finiteness of the set Ξ in Theorem 1.2.

LEMMA 3.6. *$\mathfrak{b}$ contains only finitely many rays of unit norm.*

The proof of Lemma 3.6 is based on Lemma 3.2 and on a further technical lemma, which we now proceed to describe. Let $\mathfrak{a}_Y$, for $Y \in \mathfrak{b}$, be the $\mathbf{R}$-span of Y, and let A_Y denote the analytic subgroup of G corresponding to $\mathfrak{a}_Y$. We then have an Iwasawa decomposition $G = KA_YN_Y$, and for $g \in G$, we let $g = k_Y(g)$-$a_Y(g)n_Y(g)$ denote the representation of g with respect to this Iwasawa decomposition. If $g = \exp tY'$, $Y' \in \mathfrak{b}$, then $a_Y(g) = \exp sY$ for some $s \in \mathbf{R}$, and we set $f_Y'(t) = s$. We can now state the necessary technical lemma:

LEMMA 3.7. *Let $Y \in \mathfrak{b}$ have unit norm. Then given $M > 0$, there exists $\epsilon > 0$ such that if Y' is any element of unit norm in $\mathfrak{b}$ such that $\|Y - Y'\| < \epsilon$, then there is a maximal $t_0(Y') < 0$ such that $f_{Y'}(t_0(Y')) = 0$ (and hence $\exp t_0(Y')Y' = k_Y(\exp t_0(Y')Y')n_Y(\exp t_0(Y')Y'))$, and such that $d(\pi(n_Y(\exp t_0(Y')Y')),\bar{e}) \geq M$. Moreover, as Y' tends to Y, $t_0(Y')$ tends to $-\infty$.*

We can now prove Lemma 3.6. Since Lemma 3.2 implies that $N_Y/N_Y \cap \Gamma$ is compact for a ray Y, we know that there is a compact subset $\omega \subset N_Y$, such that $N_Y = \omega(N_Y \cap \Gamma)$. Clearly then, we have a constant $M > 0$ such that

$$d(\pi(n),\bar{e}) < M, \qquad n \in \omega. \tag{3.8}$$

On the other hand, according to Lemma 3.7, if ϵ is chosen sufficiently small, if Y' is of unit norm, and $\|Y - Y'\| < \epsilon$, then there exists $p = \exp tY'$, so that $p = k_0n_0$, $k_0 \in K$, $n_0 \in N_Y$, and so that

$$d(\pi(k_0n_0),\bar{e}) = d(\pi(n_0),\bar{e}) \geq M. \tag{3.9}$$

Clearly, we can find $\gamma \in \Gamma \cap N_Y$ such that $n_0\gamma \in \omega$. But then from (3.8) and (3.9),

$$d(\pi(p),\bar{e}) \geq M > d(\pi(p)\gamma,\bar{e}),$$

so that $\pi(p) \notin \mathcal{E}$. Hence the rays of unit norm form a discrete subset of a compact space, and hence a finite set. This proves Lemma 3.6.

Let

$$S = \{Y \in \mathfrak{b} \mid \|Y\| = 1 \text{ and } Y \text{ is a ray}\}.$$

For $t > 0$, $Y \in \mathfrak{b}$, let $A_{Y,t} = \{a \in A_Y \mid a^\alpha < t\}$, where now we define α with respect to A_Y and Y, just as we did with respect to A and Y_0 in section 1. One can use the preceding results to obtain

LEMMA 3.10. *We can find $t > 0$ and for each $Y \in S$, an open relatively compact subset $\omega_Y \subset N_Y$, so that if $\Omega' = \bigcup_{Y \in S} KA_{Y,t}\,\omega_Y$, then*

(a) $\Omega'\Gamma = G$,

(b) $\{\gamma \in \Gamma \mid \Omega'\gamma \cap \Omega' \neq \phi\}$ *is finite.*

The proof of (*a*) follows from Lemmas 3.2, 3.6, and a fairly straightforward argument again involving Lemma 3.7. Before discussing the proof of (*b*), we note that we can find a finite subset $\Xi \subset G$ (in fact we can take Ξ in K), such that for each $Y \in S$, we can find a unique $b_Y \in \Xi$ such that $Y = Ad\ b_Y^{-1}(Y_0)$ (where we assume that Y_0 was chosen so that $\|Y_0\| = 1$). Moreover, we can choose an open relatively compact set $\eta \subset N^+$, so that $b_Y^{-1}\eta b_Y \supseteq \omega_Y$ for all $Y \in S$. We then have

$$\Omega = \cup_{b \in \Xi} KA_t\, \eta b,$$
$$\supseteq \quad \cup_{Y \in S} KA_{Y,t}\, \omega_Y.$$

Hence, we see that Lemmas 3.10 and 3.2 already imply (*i*) and (*ii*) of Theorem 1.2. On the other hand, (*b*) will follow from

$$\text{For all } b,b' \in \Xi, \text{ the set } \{\gamma \in \Gamma | \mathfrak{S}_{t,\eta}\, b\gamma \cap \mathfrak{S}_{t,\eta} b' \neq \phi\}, \text{ is finite.} \tag{3.11}$$

Now the pair (N^+, A) determines a Bruhat decomposition for G. In fact, let $W \subset K$ be a set of representatives for the Weyl group of G (relative to A). Then each $g \in G$ has a representation

$$g = uwmav,\ u,v \in N^+,\ a \in A,\ m \in Z(A) \cap K,\ w \in W.$$

The following is not difficult to deduce from Lemma 3.2:

LEMMA 3.12. *There exists $C_0 > 0$ such that for all $b,b' \in \Xi$, $\gamma \in \Gamma$, if $b\gamma b'^{-1} = uwmav$, then*

$$a^\alpha \geq C_0. \tag{3.13}$$

The significance of the inequality (3.13) seems to have been fully recognized for the first time by Harish-Chandra. In particular, he has shown that this inequality implies (3.11). (See ref. 2 for a discussion when $G = SL(n,\mathbf{R})$. The proof given there carries over directly to the present case.) Finally, we mention that (*iv*) of Theorem 1.2 follows from our earlier results and standard results in reduction theory, where one might have to shrink the set Ξ (this shrinking is probably not necessary). Details of the arguments described here will appear elsewhere.

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† Partially supported by NSF grant NSF-GP-7131.

‡ The proof of this lemma is based on an argument of A. Selberg, who has himself obtained the lemma in some cases. An alternative proof for $SO(n,1)$ was obtained by one of the authors.

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