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Arithmetic lattices in semisimple groups*

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1. Introduction

Borel [1] showed that given a (connected) real semisimple Lie group G, it admits a discrete (arithmetic) subgroup Γ such that G/Γ is compact. In this paper we will establish the following refinement of that theorem.

Theorem. Let G be a connected linear semisimple Lie group and A a commutative group consisting of involutive automorphisms of G. Then G admits a discrete (arithmetic) subgroup Γ such that $G^a/\Gamma \cap G^a$ is compact for each $a \in \tilde{A}$, G^a being the fixed point set of a in G and $\tilde{A}$ is an abelian group of involutive automorphisms of G containing A and a cartan involution of G.

As was the case with Borel's proof, the theorem can be deduced from a result on Lie algebras. We omit the details of this deduction.

Theorem. Let g be a semisimple Lie algebra and A a commutative group consisting of involutive automorphisms of g. Then there is a Q-structure on g such that all elements of A are Q-rational and g admits a cartan involution defined over Q and commuting with A.

The kind of Q-structure introduced on g in the special case when g is compact has the additional property that all representations of g defined over R are equivalent to representations defined over Q.

The refined version proved here is likely to be of some interest in the context of geometric constructions for homology of compact locally symmetric spaces given by Millson–Raghunathan [4] and Millson [1]; in the special case where A is trivial, we get Borel's theorem.

Borel's theorem was preceded by results in the case of many classical groups. Siegel [5] initiated the subject by making the first constructions of uniform arithmetic subgroups in classical groups beyond $SL(2, R)$. This was generalised to cover a wilder class of classical groups by Klingen [2]. Ramanathan [3] pointed

* To Prof. K G Ramanathan on his 60th birthday.

out further examples and raised the question (in oral conversations) whether any semisimple Lie groups admits a uniform lattice.

2. The standard Q-form of a compact Lie algebra

Let k be a compact semisimple Lie algebra and $\boldsymbol{k} = k \otimes_R C$. Let $t \subset k$ be a cartan subalgebra and $\boldsymbol{t} = C$-span of t. Let Φ be the root system of $\boldsymbol{k}$ with respect to $\boldsymbol{t}$ and for $\alpha \in \Phi$, let $\boldsymbol{k}(\alpha)$ denote the root space of α. As is well-known there exists a Chevalley basis of $\boldsymbol{k}$ viz., we have $\{H_\phi / \varphi \in \Phi\} \subset it$ and $E(\varphi) \in \boldsymbol{k}(\varphi)$, $\varphi \in \Phi$ such that

(i) $[H_\phi, E(\psi)] = 2\langle\varphi, \psi\rangle / \langle\psi, \psi\rangle \cdot E(\psi)$

(ii) $[H(\varphi), E(\psi)] = N_{\Phi,\psi} E_{\phi+\psi}$ with $N_{\phi,\psi} \in Z$, $\varphi + \psi \in \Phi$

(iii) $[E(\varphi), E(-\varphi)] = H\varphi$.

The complex conjugation in $\boldsymbol{k}$ takes each $k(\varphi)$ into $k(-\varphi)$ so that for $\varphi \in \Phi$, $\bar{E}(\varphi) = \lambda(\varphi)$. $E(-\varphi)$ for some $\lambda(\varphi) \in C^*$. Since $\langle E(\varphi), \bar{E}(\varphi)\rangle > 0$, we conclude that $\lambda(\varphi) > 0$. Let $x \in \boldsymbol{T}$ the adjoint torus of $\boldsymbol{t}$ be chosen such that $\alpha(x) = \lambda(\alpha)^{-1/2} > 0$ for $\alpha \in \triangle$, a simple system of roots of k. If we set $E'(\varphi) = \lambda(\varphi)^{-1/2} E(\varphi) = Ad\, x\, E(\varphi)$, we see that for simple $\varphi \in \triangle$, $\bar{E}'(\varphi) = \lambda(\varphi)^{1/2} E(-\varphi) = E'(-\varphi)$ so that the complex conjugation takes $E'(\varphi)$ into $E'(-\varphi)$ for all $\varphi \in \triangle$. It follows immediately that $\bar{E}'(\varphi) = \pm E(-\varphi)$ for all $\varphi \in \Phi$ as well. The $E'(\varphi)$, $\varphi \in \Phi$ together with the $\{H_\alpha \mid \alpha \in \triangle\}$ constitute again a Chevalley basis. Let $\boldsymbol{k}_0$ be the $\boldsymbol{Q}(i)$-span of the $\{E'(\varphi) \mid \varphi \in \Phi\}$ and the $\{H_\alpha \mid \alpha \in \triangle\}$. Then $\boldsymbol{k}_0$ is a $\boldsymbol{Q}(i)$—split form of $\boldsymbol{k}$. Let k_0 be the fixed points in $\boldsymbol{k}_0$ of the complex conjugation: this is an antilinear involution over $\boldsymbol{Q}(i)$. Then k_0 is a $\boldsymbol{Q}$-form of k. For each $\varphi > 0$, it is easily seen that the Lie algebra $\boldsymbol{a}_0(\varphi)$ spanned by $E'(\pm\varphi)$ and $H(\varphi)$ over $\boldsymbol{Q}(i)$ is $\boldsymbol{Q}(i)$—isomorphic to $SL(2)$, is stable under the conjugation with fixed algebra $a_0(\varphi)$ isomorphic over $\boldsymbol{Q}$ to $SU(2)$ the standard special unitary group over $\boldsymbol{Q}(i)$ given by the hermitian form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. An immediate consequence is that the reflection s_ϕ corresponding to φ in the Weyl group W of the adjoint algebraic group $\boldsymbol{K}$ with k as Lie algebra has a $\boldsymbol{Q}$-rational representative in $N(\boldsymbol{T})$ the normaliser of $\boldsymbol{T}$ in $\boldsymbol{K}$ (for the natural $\boldsymbol{Q}$-structure on k defined above).

In particular the unique element $w_0 \in W$ which takes all of $\triangle$ into negative roots has a $\boldsymbol{Q}$-rational representative $w_0 \in N(\boldsymbol{T})(\boldsymbol{Q})$. Let $\boldsymbol{S}$ be the identity component of the group $\{x \in \boldsymbol{T} \mid w_0 x w_0^{-1} = x\}$. Then on T/S, w_0 acts by $w_0(x) = x^{-1}$. Further in $N(T)/S$ we have $w_0^2 \in T/S$ so that $w_0 w_0^2 w_0^{-1} = w_0^{-2} = w_0^2$ leading to the conclusion that w_0^2 is an element of order 2 modulo S. Note that $\boldsymbol{S}$ is defined over $\boldsymbol{Q}$.

Definition. The $\boldsymbol{Q}$-structure defined above will be called a Standard $\boldsymbol{Q}$ structure on the pair $(\boldsymbol{K}, \boldsymbol{T})$.

Proposition. Let $\boldsymbol{G}$ be a $\boldsymbol{Q}$-algebraic group such that the identity component $\boldsymbol{G}^0$ of $\boldsymbol{G}$ is a torus and $\boldsymbol{G}/\boldsymbol{G}^0$ is abelian with every element of order 2. Suppose that $\boldsymbol{G}(\boldsymbol{Q}) \to (\boldsymbol{G}/\boldsymbol{G}^0)(\boldsymbol{Q}) = \boldsymbol{G}/\boldsymbol{G}_0$ is onto and the sequence.

$$(*)\ 1 \to \boldsymbol{G}^0 \to \boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{G}^0 \to 1$$

admits a splitting γ over R and that the torus G^0 is anisotropic over R and splits over $Q(i)$. Then (*) splits over Q as well and the Q-splitting can be chosen to be conjugate to γ by an element of $G^0(R)$.

Proof. We argue by induction on dim G. We note first that every subtorus of G^0 defined over R is automatically defined over Q. Let $X(G^0)$ be the abelian group of 1 parameter subgroups of G^0. The Galois group Gal $(Q(i)/Q) \simeq$ Gal (C/R) operates on this by $\chi \to -\chi$. The group G/G^0 acts on $X(G^0)$ as well and has an eigen vector in $X(G^0) \underset{Z}{\otimes} Q$ hence in $X(G^0)$. Let S denote the corresponding torus in T. S is evidently defined over Q and normal in G. Let $G' = G/S$. Then by induction hypothesis we can find $u \in G^0(R)$ such that $\bar{\rho} = \bar{u}(\pi \circ r)\bar{u}^{-1}$ is defined over Q where $r : G/G^0 \to G$ is the given R-splitting, and $\pi : G \to G/S$ is the natural map and $\bar{u} = \pi(u)$. If we now set $H = \pi^{-1}(\rho(G/G^0))$, H is defined over Q and its identity component $H^0 = S$. We are thus reduced to the case when dim $G = 1$. First consider the action of the group G/G^0 on G_0. Since dim $G = 1$, the automorphism group of G is of order 2; it follows that G/G^0 has a subgroup B of index almost 2 which acts trivially on G^0. If $p : G \to G/G_0$ is the natural map $p^{-1}(B)$ is abelian—note that we have a splitting over R- and hence diagonalisable. Now we have the exact sequence

$$O \to X^*(B) \to X^*(p^{-1}(B)) \to X^*(G^0) \to 0$$

of the character groups. These are modules over Gal $(C/R) \cong$ Gal $Q(i)/R$ and by assumption the sequence is split as modules over Gal C/R hence also over Gal $Q(i)/Q$. Moreover *any* R-splitting is a Q-splitting ($X(B)$ is a trivial Galois-module). Thus we conclude that $p^{-1}(B)$ admits a Q-splitting of the form $B \cdot G^0$. The character group is a direct sum $X^*(B) \oplus X^*(G^0)$ with the action G/G^0 trivial on $X(B)$ and nontrivial on $X(G^0) \simeq Z$; if $B \neq G/G^0$, $X(B)$ then can be characterised as those elements which are fixed by G^0 as well as G/G^0. It is immediate now that B is normal in G. Consider then the quotient $H = G/B$. H^0 is isomorphic to G^0 and is hence 1-dimensional. The sequence

$$** \qquad 1 \to H^0 \to H \overset{a}{\to} H/H_0 \to Z/2 \to 1$$

is assumed to be split over R. Let $\tau \in H/H^0$ be the nontrivial element. Then $q^{-1}(\tau)$ is a principal homogeneous space over Q; it has a rational point over Q by assumption ($G(Q) \to G/G^0$ was assumed surjective). Now let τ_0 be the lift of τ given by the splitting over R and τ_0' a lift over Q. Then we have $\tau_0' = \tau_0 \cdot x$, $x \in H^0(R)$ so that

$$(\tau_0')^2 = \tau_0 x \cdot \tau_0 x = \tau_0^2 = 1$$

Thus τ_0' also gives a splitting of (**); in order to assert that τ_0' is a conjugate of τ_0 we need only have that x is a square of an element y in $H^0(R)$: for then

$$y^{-1}\tau_0 y = \tau_0 \cdot \tau_0^{-1} y^{-1} \tau_0 y = \tau_0 \cdot x.$$

Now $H^0(R)$ is isomorphic to the circle group, hence each $x \in H^0(R)$ is a square.

We obtain the required Q-splitting by taking the inverse image under $f: G \to G/B$ of the group $(\tau_0', 1)$. This completes the proof of the proposition.

Corollary. Let K be a compact (connected semisimple) group and $A \subset \text{Aut } K$ be an abelian subgroup consisting entirely of elements of order 2. Then there is a A-stable torus T in K and a "standard" Q-structure on (k, t) with A consisting entirely of Q-rational automorphisms of k.

Proof. We assume $K = (\text{Aut } K)^0$. We fix a maximal subgroup A_1 of A which is contained in some maximal torus. Let $z(A_1)$ be the fixed point set of A_1 in k. Then $z(A_1)$ is A-stable. Moreover a maximal abelian subalgebra of $z(A_1)$ is maximal abelian in k as well. Since A consists of elements of order 2, A has a common eigen vector $X \in k$. The corresponding torus in K is evidently A-stable. Hence there is among abelian subalgebras of $z(A_1)$, a maximal non zero one say b which is A-stable. Since b is A-stable so is $z_1(b)$ the centraliser of b in $z(A_1)$. If b is not maximal abelian its orthogonal complement in $z_1(b)$ will contain a 1-dimensional A-stable subspace leading to a contradiction. Thus b is a A-stable cartan subalgebra of k. We denote the corresponding torus by T. Take now any standard Q-structure on (k, t). The group A is a direct product $A_2 \times A_1$ where $A_2 \cap T = \{1\}$ and $A_1 \subset T$. A_1 consists of elements of order 2 and these are easily seen to be Q-rational. By Proposition we can find $x \in T(R)$ which conjugates A_2 into Q-rational points. Replacing the Chevalley basis we started out with for defining the standard structure by their transforms under $Ad\, x^{-1}$ we obtain all the requisite properties. Observe that as $x \in T(R)$ the Q-structure on T remains unchanged. The Q-structure on k remains isomorphic to the original one as well as is easily seen. If $N(T)$ = normaliser T in $\text{Aut}(k)$, $N(T)(Q) \xrightarrow{\pi} N(T)/T = [N(T)/T](Q)$ gives surjection at the Q-rational level as the Dynkin automorphisms fixing T is also Q-rational (all the hypothesis of the proposition are satisfied by $G = \pi^{-1}\pi(A)$ and $G^0 = T$).

Lemma. Let G be a connected linear semisimple Lie group and $A \subset \text{Aut } G$ a finite abelian group consisting of involutions. Then G admits a cartan involution commuting with A.

Proof. Let K be a maximal compact subgroup of Aut G containing A. K defines a cartan involution of G which evidently commutes with all the elements of A.

Theorem. Let G be a connected linear semisimple Lie group and g its Lie algebra. Let $A \subset \text{Aut } G$ be any group of commuting involutions of G. Then g admits a Q-structure such that all $a \in A$ are Q-rational and there is a Q-rational cartan-involution commuting with A as well.

Proof. Enlarge A to include a cartan involution θ (*cf.* Lemma above). Let $g = u + p$ be the cartan-decomposition with u compact. Then u and p are A-stable as all of A commute with θ. Let $k = u + ip$. Then k is a compact Lie algebra. By proposition we can find a A-stable torus t in k such that (k, t) admits a standard Q-structure with $A \subset K(Q)$. Since θ is Q-rational u and ip are defined over Q for this Q-structure. This immediately gives a Q-structure on $u + p = g$ as well. Next since each $a \in A$ acts Q-rationally on u as well as ip and hence on p, each $a \in A$ is Q-rational for this Q-structure on g.

3. Representations of the standard Q-form

The following property of the standard Q-form of k seems to be of some interest.

Theorem. Let k_Q be a standard Q-form of (k, ι) with k a compact semisimple Lie algebra. Then every representation of k_Q defined over R is equivalent to a unique one defined over Q.

In view of complete reducibility, it suffices to show that each irreducible R-representation of k_Q is equivalent to one defined over Q. (The uniqueness part of the statement is easy to prove: one way is to use the Zariski density of $K(Q)$ in K (K = simply connected Q algebraic group determined by k_Q) and use the fact that representations of $K(Q)$ are characterised by their characters: see for instance Van der Wearden [6, exercise, p. 175]. In fact it suffices to show that each irreducible representation over Q of k_Q remains irreducible over R. To see this observe that if σ is an irreducible representation of k_Q defined over R, σ may be assumed to be defined over some number field; the set of all representations of k_Q on a fixed finite dimensional vector space is a variety V defined over Q and $\sigma \in V(R)$. The orbit of σ under K is open in $V(R)$ in view of the Whitehead lemma and hence contains $\bar{Q}$-rational points. We may thus assume σ to be defined over a real number field $L \supset Q$, with L of minimal possible degree. Consider now the underlying L vector space as a Q vector space and denote the corresponding representation by τ. Since L commutes with the action of $K(Q)$ and L-span of any non zero $K(Q)$-irreducible Q-subspace of $W(\sigma)$ (= representation space of σ) is all of $W(\sigma)$, we conclude that τ is isotypical of fixed type τ_0. Evidently, $W(\sigma)$ is a quotient of $W(\tau_0) \otimes_Q L$. Since $L \subset R$, this last tensor product is irreducible so that $W(\sigma) \simeq W(\tau_0) \otimes_Q L$ leading to the conclusion $L = Q$. We have thus to prove.

Proposition. Let ρ be an irreducible representation of k_Q over Q. Then $\rho \otimes_Q R$ is irreducible.

Proof. The Lie algebra k_Q splits over $Q(i)$. It follows that over $Q(i)$ all representations over C have equivalents. In particular this means that an irreducible representation ρ over Q decomposes over C into at most two representations. If ρ remains irreducible over $Q(i)$ hence over C, there is nothing to prove. Assume that $\rho \otimes_Q Q(i) \simeq \rho_1 \oplus \rho_2$ over $Q(i)$. If ρ_1 and ρ_2 are inequivalent, then the commutant of ρ is an algebra which when tensored with $Q(i)$ is isomorphic to $Q(i) \times Q(i)$. It follows that the commutant of $\rho(k_Q)$ in End $W(\rho)$ ($W(\rho)$ = representation space for ρ) is $Q(i)$. Since $Q(i) \otimes R \simeq C$ is a field, it follows that in this case too ρ remains irreducible. We have thus to consider now only the case

$$\rho \otimes Q(i) \simeq \sigma \oplus \sigma$$

two copies of the same irreducible representation. Let Δ be a simple system of $Q(i)$-roots with respect to T fixed as in the beginning of § 2 and w_0 be the Weyl group element defined there. Let $S \subset T$ be the maximal torus fixed pointwise by w_0. Let $\wedge$ be the highest weight of σ and $W(\wedge) \subset W(\sigma)$ the eigen space corresponding to $\wedge$. $W(\wedge)$ is defined over $Q(i)$. Let σ be considered as a subrepresentation through the direct sum decomposition over $Q(i)$ and label the two factors by 1, 2. Then we can choose the components so that we have

$$\overline{W(\wedge)_1} \subset W(\sigma)_2, W(\rho \otimes_Q Q(i)) = W(\sigma)_1 \oplus W(\sigma)_2 \text{ and } W(\wedge)_1 \subset W(\sigma)_1$$

is the highest weight space: otherwise $W(\sigma)_1$ would be stable under conjugation so that it will be defined over Q contradicting the irreducibility of ρ over Q. Similarly $\overline{W(\wedge)_2} \subset W(\sigma)_1$. Now since complex conjugation takes t to t^{-1} in the torus we have necessarily $\overline{W(\wedge)_1} = W(\wedge^{-1})_2$. Since $\wedge^{-1}$ is necessarily the least weight of σ again, we conclude that $w_0(\wedge) = \wedge^{-1}$. Consider now the representation μ of the group B generated by w_0 and T on $E = W(\wedge)_1 + W(\wedge)_2 + W(\wedge^{-1})_1 + W(\wedge)_2$. We have then for $\mu(w_0)$, $\mu(w_0)^2$ is the unique element of order 2 in the group $\mu(T/S)(Q)$. Now μ is a 4-dimensional real irreducible representation of B as is easily seen. Its commuting algebra is thus a division algebra of degree 2. The restriction of $\mathrm{End}_\sigma(\rho)$ to E is seen to be nontrivial division algebra; since dim $E = 4$, these commuting algebras must coincide. If D denotes this division algebra E is necessarily a 1-dimensional vector space and the algebra generated by $B(Q)$ is contained in the commutant H of D in $\mathrm{End}_Q(E) \subset M_4(Q)$. The last algebra is evidently isomorphic to D (note degree $D = 2$ so that $D \simeq D^0$). We will show that D is the definite quaternion algebra generated by i, j, k with $i^2 = j^2 = k^2 = -1$ $ij = k$, etc. To see this let L be the subfield of H generated by $\mu(t_Q)$. L is isomorphic $Q(i)$ where we denote by i the square root of -1 in L. Next set $j = \mu(w_0)$. Now $j^2 = \mu(w_0)^2$; it equals either the unique element of order 2 in L, viz., -1 or $j^2 = 1$. If $j^2 = 1$, $Q[j]$ contains a zero divisor a contradiction to $j \in H$. Thus $j^2 = -1$. Finally set $k = ij$. Then (ij). $(ij) = ij^2(j^{-1}ij) = ij^2 i^{-1} = j^2 = -1$. Showing that the algebra generated by $\mu(t_Q)$ and $\mu(w_0)$ is isomorphic to the definite quaternion algebra. This implies that D is a definite quaternion algebra over Q. Hence $D \otimes_Q R$ remains a division algebra proving that $\rho \otimes_Q R$ is irreducible.

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