Arithmetic lattices in semisimple groups*

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1. Introduction

Borel [1] showed that given a (connected) real semisimple Lie group $G$, it admits a discrete (arithmetic) subgroup $\Gamma$ such that $G/\Gamma$ is compact. In this paper we will establish the following refinement of that theorem.

**Theorem.** Let $G$ be a connected linear semisimple Lie group and $\mathcal{A}$ a commutative group consisting of involutive automorphisms of $G$. Then $G$ admits a discrete (arithmetic) subgroup $\Gamma$ such that $G^a/\Gamma \cap G^a$ is compact for each $a \in A$, $G^a$ being the fixed point set of $a$ in $G$ and $A$ is an abelian group of involutive automorphisms of $G$ containing $A$ and a cartan involution of $G$.

As was the case with Borel's proof, the theorem can be deduced from a result on Lie algebras. We omit the details of this deduction.

**Theorem.** Let $g$ be a semisimple Lie algebra and $\mathcal{A}$ a commutative group consisting of involutive automorphisms of $g$. Then there is a $Q$-structure on $g$ such that all elements of $\mathcal{A}$ are $Q$-rational and $g$ admits a cartan involution defined over $Q$ and commuting with $\mathcal{A}$.

The kind of $Q$-structure introduced on $g$ in the special case when $g$ is compact has the additional property that all representations of $g$ defined over $R$ are equivalent to representations defined over $Q$.

The refined version proved here is likely to be of some interest in the context of geometric constructions for homology of compact locally symmetric spaces given by Millson–Raghunathan [4] and Millson [1]; in the special case where $\mathcal{A}$ is trivial, we get Borel's theorem.

Borel's theorem was preceded by results in the case of many classical groups. Siegel [5] initiated the subject by making the first constructions of uniform arithmetic subgroups in classical groups beyond $SL(2,R)$. This was generalised to cover a wilder class of classical groups by Klingen [2]. Ramanathan [3] pointed

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* To Prof. K G Ramanathan on his 60th birthday.
out further examples and raised the question (in oral conversations) whether any semisimple Lie groups admits a uniform lattice.

2. The standard Q-form of a compact Lie algebra

Let \( k \) be a compact semisimple Lie algebra and \( k = k \oplus_C C \). Let \( t \subseteq k \) be a cartan subalgebra and \( t = C\text{-span of } t \). Let \( \Phi \) be the root system of \( k \) with respect to \( t \) and for \( \alpha \in \Phi \), let \( k(\alpha) \) denote the root space of \( \alpha \). As is well-known there exists a Chevalley basis of \( k \) viz., we have \( \{ H_\varphi | \varphi \in \Phi \} \subseteq \) it and \( E(\varphi) \in k(\varphi), \varphi \in \Phi \) such that

(i) \[ [H_\varphi, E(\psi)] = 2 \langle \varphi, \psi \rangle \langle \psi, \psi \rangle \cdot E(\psi) \]

(ii) \[ [H(\varphi), E(\psi)] = N_{\varphi, \psi} E_{\varphi + \psi} \] with \( N_{\varphi, \psi} \in \mathbb{Z}, \varphi + \psi \in \Phi \)

(iii) \[ [E(\varphi), E(-\varphi)] = H_\varphi. \]

The complex conjugation in \( k \) takes each \( k(\varphi) \) into \( k(-\varphi) \) so that for \( \varphi \in \Phi \), \( E(\varphi) = \lambda(\varphi) \). \( E(-\varphi) \) for some \( \lambda(\varphi) \in C^\ast \). Since \( (E(\varphi), E(\varphi)) > 0 \), we conclude that \( \lambda(\varphi) > 0 \). Let \( x \in T \) the adjoint torus of \( t \) be chosen such that \( \alpha(x) = \lambda(\alpha)^{1/2} > 0 \) for \( \alpha \in \Delta \), a simple system of roots of \( k \). If we set \( E'(\varphi) = \lambda(\varphi)^{-1/2} E(\varphi) = Ad_x E(\varphi) \), we see that for simple \( \varphi \in \Delta \), \( E'(\varphi) = \lambda(\varphi)^{1/2} E(-\varphi) = E'(-\varphi) \) so that the complex conjugation takes \( E'(\varphi) \) into \( E'(-\varphi) \) for all \( \varphi \in \Delta \). It follows immediately that \( E'(\varphi) = \pm E(-\varphi) \) for all \( \varphi \in \Phi \) as well. The \( E'(\varphi), \varphi \in \Phi \) together with the \( \{ H_\alpha | \alpha \in \Delta \} \) constitute again a Chevalley basis. Let \( k_0 \) be the \( Q(i) \)-span of the \( \{ E'(\varphi) | \varphi \in \Phi \} \) and the \( \{ H_\alpha | \alpha \in \Delta \} \). Then \( k_0 \) is a \( Q(i) \)-split form of \( k \). Let \( k_0 \) be the fixed points in \( k_0 \) of the complex conjugation: this is an antilinear involution over \( Q(i) \). Then \( k_0 \) is a \( Q \)-form of \( k \). For each \( \varphi > 0 \), it is easily seen that the Lie algebra \( a_0(\varphi) \) spanned by \( E'(\varphi) \) and \( H(\varphi) \) over \( Q(i) \) is \( Q(i) \)-isomorphic to \( SL(2) \), is stable under the conjugation with fixed algebra \( a_0(\varphi) \) isomorphic over \( Q \) to \( SU(2) \) the standard special unitary group over \( Q(i) \) given by the hermitian form \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). An immediate consequence is that the reflection \( s_\varphi \) corresponding to \( \varphi \) in the Weyl group \( W \) of the adjoint algebraic group \( K \) with \( k \) as Lie algebra has a \( Q \)-rational representative in \( N(T) \) the normaliser of \( T \) in \( K \) (for the natural \( Q \)-structure on \( k \) defined above).

In particular the unique element \( w_0 \in W \) which takes all of \( \Delta \) into negative roots has a \( Q \)-rational representative \( w_0 \in N(T)(Q) \). Let \( S \) be the identity component of the group \( \{ x \in T | w_0 x w_0^{-1} = x \} \). Then on \( T/S \), \( w_0 \) acts by \( w_0(x) = x^{-1} \). Further in \( N(T)/S \) we have \( w_0^2 \in T/S \) so that \( w_0 w_0^2 w_0^{-1} = w_0^{-2} = w_0^2 \) leading to the conclusion that \( w_0^2 \) is an element of order 2 modulo \( S \). Note that \( S \) is defined over \( Q \).

**Definition.** The \( Q \)-structure defined above will be called a Standard \( Q \) structure on the pair \( (K, T) \).

**Proposition.** Let \( G \) be a \( Q \)-algebraic group such that the identity component \( G^o \) of \( G \) is a torus and \( G/G^o \) is abelian with every element of order 2. Suppose that \( G(Q) \rightarrow (G/G^o)(Q) = G^o/G^o \) is onto and the sequence.

\[
(*) \quad 1 \to G^o \to G \to G/G^o \to 1
\]
admits a splitting $\gamma$ over $R$ and that the torus $G^0$ is anisotropic over $R$ and splits over $Q(i)$. Then ($^*$) splits over $Q$ as well and the $Q$-splitting can be chosen to be conjugate to $\gamma$ by an element of $G^0(R)$.

Proof. We argue by induction on $\dim G$. We note first that every subtorus of $G^0$ defined over $R$ is automatically defined over $Q$. Let $X(G^0)$ be the abelian group of $1$ parameter subgroups of $G^0$. The Galois group $\text{Gal}(Q(i)/Q) \cong \text{Gal}(C/R)$ operates on this by $\chi \to -\chi$. The group $G/G^0$ acts on $X(G^0)$ as well and has an eigen vector in $X(G^0) \otimes Q$ hence in $X(G^0)$. Let $S$ denote the corresponding torus in $T$. $S$ is evidently defined over $Q$ and normal in $G$. Let $G' = G/S$. Then by induction hypothesis we can find $u \in G^0(R)$ such that $\tilde{\rho} = \tilde{u}(\pi \circ \rho)\tilde{u}^{-1}$ is defined over $Q$ where $r : G/G^0 \to G$ is the given $R$-splitting, and $\pi : G \to G/S$ is the natural map and $\tilde{u} = \pi(u)$. If we now set $H = \pi^{-1}(\rho(G/G^0))$, $H$ is defined over $Q$ and its identity component $H^0 = S$. We are thus reduced to the case when $\dim G = 1$. First consider the action of the group $G/G^0$ on $G_0$. Since $\dim G = 1$, the automorphism group of $G$ is of order 2; it follows that $G/G^0$ has a subgroup $B$ of index almost $2$ which acts trivially on $G^0$. If $p : G \to G/G^0$ is the natural map $p^{-1}(B)$ is abelian—note that we have a splitting over $R$ and hence diagonalizable. Now we have the exact sequence

$$O \to X^*(B) \to X^*(p^{-1}(B)) \to X^*(G^0) \to 0$$

of the character groups. These are modules over $\text{Gal}(C/R) \cong \text{Gal}(Q(i)/Q)$ and by assumption the sequence is split as modules over $\text{Gal}(C/R)$ hence also over $\text{Gal}(Q(i)/Q)$. Moreover any $R$-splitting is a $Q$-splitting $(X(B)$ is a trivial Galois-module). Thus we conclude that $p^{-1}(B)$ admits a $Q$-splitting of the form $B \cdot G^0$. The character group is a direct sum $X^*(B) \oplus X^*(G^0)$ with the action $G/G^0$ trivial on $X(B)$ and non trivial on $X(G^0) \cong Z$; if $B \neq G/G^0$, $X(B)$ then can be characterised as those elements which are fixed by $G^0$ as well as $G/G^0$. It is immediate now that $B$ is normal in $G$. Consider then the quotient $H = G/B$. $H^0$ is isomorphic to $G^0$ and is hence 1-dimensional. The sequence

$$1 \to H^0 \to H \to H/H_0 \to \mathbb{Z}/2 \to 1$$

is assumed to be split over $R$. Let $\tau \in H/H^0$ be the non trivial element. Then $q^{-1}(\tau)$ is a principal homogeneous space over $Q$; it has a rational point over $Q$ by assumption ($G(Q) \to G/G^0$ was assumed surjective). Now let $\tau_0$ be the lift of $\tau$ given by the splitting over $R$ and $\tau_0'$ a lift over $Q$. Then we have $\tau_0' = \tau_0 \cdot x$, $x \in H^0(R)$ so that

$$(\tau_0')^2 = \tau_0 \cdot \tau_0 \cdot x = \tau_0^2 = 1$$

Thus $\tau_0'$ also gives a splitting of (**) in order to assert that $\tau_0'$ is a conjugate of $\tau_0$ we need only have that $x$ is a square of an element $y$ in $H^0(R)$: for then

$$y^{-1} \tau_0 y = \tau_0 \cdot y^{-1} \tau_0 \cdot y = \tau_0 \cdot x.$$ 

Now $H^0(R)$ is isomorphic to the circle group, hence each $x \in H^0(R)$ is a square.

We obtain the required $Q$-splitting by taking the inverse image under $f : G \to G/B$ of the group $(\tau_0', 1)$. This completes the proof of the proposition.
Corollary. Let $K$ be a compact (connected semisimple) group and $A \subset \text{Aut} \ K$ be an abelian subgroup consisting entirely of elements of order 2. Then there is a $A$-stable torus $T$ in $K$ and a "standard" $Q$-structure on $(k, t)$ with $A$ consisting entirely of $Q$-rational automorphisms of $k$.

Proof. We assume $K = (\text{Aut} \ K)^0$. We fix a maximal subgroup $A_1$ of $A$ which is contained in some maximal torus. Let $z(A_1)$ be the fixed point set of $A_1$ in $k$. Then $z(A_1)$ is $A$-stable. Moreover a maximal abelian subalgbra of $z(A_1)$ is maximal abelian in $k$ as well. Since $A$ consists of elements of order 2, $A$ has a common eigen vector $X \in k$. The corresponding torus in $K$ is evidently $A$-stable. Hence there is among abelian subalgebras of $z(A_1)$, a maximal non zero one say $b$ which is $A$-stable. Since $b$ is $A$-stable so is $z_1(b)$ the centraliser of $b$ in $z(A_1)$. If $b$ is not maximal abelian its orthogonal complement in $z_1(b)$ will contain a 1-dimensional $A$-stable subspace leading to a contradiction. Thus $b$ is a $A$-stable cartan subalgebra of $k$. We denote the corresponding torus by $T$. Take now any standard $Q$-structure on $(k, t)$. The group $A$ is a direct product $A_1 \times A_1$ where $A_1 \cap T = \{1\}$ and $A_1 \subset T$. $A_1$ consists of elements of order 2 and these are easily seen to be $Q$-rational. By Proposition we can find $x \in T(R)$ which conjugates $A_1$ into $Q$-rational points. Replacing the Chevalley basis we started out with for defining the standard structure by their transforms under $Ad x^{-1}$ we obtain all the requisite properties. Observe that as $x \in T(R)$ the $Q$-structure on $T$ remains unchanged. The $Q$-structure on $k$ remains isomorphic to the original one as well as is easily seen. If $N(T) = \text{normaliser} T \in \text{Aut}(k)$, $N(T)/(Q) \rightarrow N(T)/T = [N(T)/T](Q)$ gives surjection at the $Q$-rational level as the Dynkin automorphisms fixing $T$ is also $Q$-rational (all the hypothesis of the proposition are satisfied by $G = \pi^{-1} \pi(A)$ and $G^1 = T$).

Lemma. Let $G$ be a connected linear semisimple Lie group and $A \subset \text{Aut} \ G$ a finite abelian group consisting of involutions. Then $G$ admits a cartan involution commuting with $A$.

Proof. Let $K$ be a maximal compact subgroup of $\text{Aut} \ G$ containing $A$. $K$ defines a cartan involution of $G$ which evidently commutes with all the elements of $A$.

Theorem. Let $G$ be a connected linear semisimple Lie group and $g$ its Lie algebra. Let $A \subset \text{Aut} \ G$ be any group of commuting involutions of $G$. Then $g$ admits a $Q$-structure such that all $a \in A$ are $Q$-rational and there is a $Q$-rational certain-involution commuting with $A$ as well.

Proof. Enlarge $A$ to include a cartan involution $\theta$ (cf. Lemma above). Let $g = u + p$ be the cartan-decomposition with $u$ compact. Then $u$ and $p$ are $A$-stable as all of $A$ commute with $\theta$. Let $k = u + ip$. Then $k$ is a compact Lie algebra. By proposition we can find a $A$-stable torus $t$ in $k$ such that $(k, t)$ admits a standard $Q$-structure with $A \subset K(Q)$. Since $\theta$ is $Q$-rational $u$ and $ip$ are defined over $Q$ for this $Q$-structure. This immediately gives a $Q$-structure on $u + p = g$ as well. Next since each $a \in A$ acts $Q$-rationally on $u$ as well as $ip$ and hence on $p$, each $a \in A$ is $Q$-rational for this $Q$-structure on $g$. 


3. Representations of the standard $Q$-form

The following property of the standard $Q$-form of $k$ seems to be of some interest.

Theorem. Let $k_Q$ be a standard $Q$-form of $(k, r)$ with $k$ a compact semisimple Lie algebra. Then every representation of $k_Q$ defined over $R$ is equivalent to a unique one defined over $Q$.

In view of complete reducibility, it suffices to show that each irreducible $R$-representation of $k_Q$ is equivalent to one defined over $Q$. (The uniqueness part of the statement is easy to prove: one way is to use the Zariski density of $K(Q)$ in $K$ ($K$ = simply connected $Q$ algebraic group determined by $k_Q$) and use the fact that representations of $K(Q)$ are characterised by their characters: see for instance Van der Waarden [6, exercise, p. 175]. In fact it suffices to show that each irreducible representation over $Q$ of $k_Q$ remains irreducible over $R$. To see this observe that if $\sigma$ is an irreducible representation of $k_Q$ defined over $R$, $\sigma$ may be assumed to be defined over some number field; the set of all representations of $k_Q$ on a fixed finite dimensional vector space is a variety $V$ defined over $Q$ and $\sigma \in V(R)$. The orbit of $\sigma$ under $K$ is open in $V(R)$ in view of the Whitehead lemma and hence contains $Q$-rational points. We may thus assume $\sigma$ to be defined over a real number field $L \supset Q$, with $L$ of minimal possible degree. Consider now the underlying $L$ vector space as a $Q$ vector space and denote the corresponding representation by $\tau$. Since $L$ commutes with the action of $K(Q)$ and $L$-span of any non zero $K(Q)$-irreducible $Q$-subspace of $W(\sigma)$ (= representation space of $\sigma$) is all of $W(\sigma)$, we conclude that $\tau$ is isotypical of fixed type $\tau_0$. Evidently, $W(\sigma)$ is a quotient of $W(\tau_0) \otimes \kappa L$. Since $L \subset R$, this last tensor product is irreducible so that $W(\sigma) \simeq W(\tau_0) \otimes \kappa L$ leading to the conclusion $L = Q$. We have thus to prove.

Proposition. Let $\rho$ be an irreducible representation of $k_Q$ over $Q$. Then $\rho \otimes \kappa R$ is irreducible.

Proof. The Lie algebra $k_Q$ splits over $Q(i)$. It follows that over $Q(i)$ all representations over $C$ have equivalents. In particular this means that an irreducible representation $\rho$ over $Q$ decomposes over $C$ into at most two representations. If $\rho$ remains irreducible over $Q(i)$ hence over $C$, there is nothing to prove. Assume that $\rho \otimes Q(i) \simeq \rho_1 \oplus \rho_2$ over $Q(i)$. If $\rho_1$ and $\rho_2$ are inequivalent, then the commutant of $\rho$ is an algebra which when tensored with $Q(i)$ is isomorphic to $Q(i) \times Q(i)$. It follows that the commutant of $\rho$ (for instance $\dim W(\rho)$ = representation space for $\rho$) is $Q(i)$. Since $Q(i) \otimes \kappa R \simeq C$ is a field, it follows that in this case too $\rho$ remains irreducible. We have thus to consider now only the case

$$\rho \otimes Q(i) \simeq \sigma \oplus \sigma$$

two copies of the same irreducible representation. Let $A$ be a simple system of $Q(i)$-roots with respect to $T$ fixed as in the beginning of §2 and $w_0$ be the Weyl group element defined there. Let $S \subset T$ be the maximal torus fixed pointwise by $w_0$. Let $\Lambda$ be the highest weight of $\sigma$ and $W(\Lambda) \subset W(\sigma)$ the eigen space corresponding to $\Lambda$. $W(\Lambda)$ is defined over $Q(i)$. Let $\sigma$ be considered as a subrepresenta-
tion through the direct sum decomposition over $Q(i)$ and label the two factors by 1, 2. Then we can choose the components so that we have

$$\overline{W}(\wedge)_1 \subset W(\sigma)_2, W(\rho \otimes Q(i)) = W(\sigma)_1 \oplus W(\sigma)_2 \text{ and } W(\wedge)_1 \subset W(\sigma)_1$$

is the highest weight space: otherwise $W(\sigma)_1$ would be stable under conjugation so that it will be defined over $Q$ contradicting the irreducibility of $\rho$ over $Q$. Similarly $\overline{W}(\wedge)_2 \subset W(\sigma)_1$. Now since complex conjugation takes $t$ to $t^{-1}$ in the torus we have necessarily $\overline{W}(\wedge)_1 = W(\wedge^{-1})_2$. Since $\wedge^{-1}$ is necessarily the least weight of $\sigma$ again, we conclude that $w_0(\wedge) = \wedge^{-1}$. Consider now the representation $\mu$ of the group $B$ generated by $w_0$ and $T$ on $E = W(\wedge)_1 + W(\wedge)_2 + W(\wedge^{-1})_2 + W(\wedge)_2$. We have then for $\mu(w_0)$, $\mu(w_0)^2$ is the unique element of order 2 in the group $\mu(TS)(Q)$. Now $\mu$ is a 4-dimensional real irreducible representation of $B$ as is easily seen. Its commuting algebra is thus a division algebra of degree 2. The restriction of $\text{End}_E(\rho)$ to $E$ is seen to be nontrivial division algebra; since $\dim E = 4$, these commuting algebras must coincide. If $D$ denotes this division algebra $E$ is necessarily a 1-dimensional vector space and the algebra generated by $B(Q)$ is contained in the commutant $H$ of $D$ in $\text{End}_E(E) \subset M_4(Q)$. The last algebra is evidently isomorphic to $D$ (note degree $D = 2$ so that $D \simeq D^0$). We will show that $D$ is the definite quaternion algebra generated by $i, j, k$ with $i^2 = j^2 = k^2 = -1$, $ij = k$, etc. To see this let $L$ be the subfield of $H$ generated by $\mu(i\sigma)$. $L$ is isomorphic $Q(i)$ where we denote by $i$ the square root of $-1$ in $L$. Next set $j = \mu(w_0)$. Now $j^2 = \mu(w_0)^2$; it equals either the unique element of order 2 in $L$, viz., $-1$ or $j = 1$. If $j^2 = 1$, $Q(j)$ contains a zero divisor a contradiction to $j \in H$. Thus $j^2 = -1$. Finally set $k = ij$. Then $(ij) = i^2(j^{-1}ij) = ij^{-1} = j^2 = -1$. Showing that the algebra generated by $\mu(i\sigma)$ and $\mu(w_0)$ is isomorphic to the definite quaternion algebra. This implies that $D$ is a definite quaternion algebra over $Q$. Hence $D \otimes Q R$ remains a division algebra proving that $\rho \otimes Q$ is irreducible.

References