Proc. Indian Acad. Sci. (Math. Sci.), Vol. 99, No. 2, August 1989, pp. 127–146.
© Printed in India.

Discrete subgroups of algebraic groups over local fields of positive characteristics

M S RAGHUNATHAN
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

MS received 7 September 1988; revised 20 March 1989

Abstract. It is shown in this paper that if G is the group of k-points of a semisimple algebraic group G over a local field k of positive characteristic such that all its k-simple factors are of k-rank 1 and $\Gamma \subset G$ is a non-cocompact irreducible lattice then Γ admits a fundamental domain which is a union of translates of Siegel domains. As a consequence we deduce that if G has more than one simple factor, then Γ is finitely generated and by a theorem due to Venkataramana, it is arithmetic.

Keywords. Discrete subgroups; algebraic groups; local fields; Siegel domains; fundamental domains; positive characteristics.

1. Introduction

Let k_i, $i \in I$, be local fields of characteristic $p > 0$ with I a finite set. For $i \in I$, let $\mathbf{G}_i$ be an algebraic group defined, absolutely almost simple and of *rank* 1 over k_i. Let $\mathbf{G}_i(k) = G_i$ and $G = \prod_{i\in I} G_i$. Let $\Gamma \subset G$ be a discrete subgroup such that the volume of G/Γ with respect to the measure on G/Γ induced by a (bi-invariant) Haar measure μ on G is finite. Our aim in this paper is to exhibit for such a Γ a "good" fundamental domain in G. The existence of the good fundamental domain leads to the following result: if $|I| \geqslant 2$ and Γ is irreducible, then Γ is finitely generated. (Γ is irreducible if for every $i \in I$, the projection of G on G_i is injective when restricted to Γ). The existence of the fundamental domain is proved along the same lines as in characteristic 0 (see Raghunathan [6]) but this requires that we extend some results of Kazdan-Margulis [5]. These last results cannot be extended in the form given in [5] unless p is a good characteristic for G_i for all $i \in I$. We prove a modified version of the main lemma of that paper a version which is weaker but somewhat more delicate to prove. (The stronger statement that holds in characteristic 0 appears to be false in bad positive characteristics.) The modified version being weaker necessitates somewhat more subtle arguments than those given in [5] to draw further conclusions towards the construction of the fundamental domain. However one obtains as a straightforward consequence the following: There is a positive constant $C = C(\mu)$ such that $\mathrm{vol}(G/\Gamma) > C$ (in the measure determined by a Haar measure on G—the constant is independent of Γ).

The fundamental domain constructed is a union of a finite number of translates of a "Siegel domain" and is thus an extension of the corresponding result for arithmetic groups (Behr [1], Harder [4]). The standard properties that hold for these fundamental domains in the case of arithmetic groups carries over to the possibly non-arithmetic case as well but we have not elaborated on this here. We note that our construction is available only when G is a product of groups of rational points of rank 1 groups over local fields. However when G has even one factor of rank $\geqslant 2$, a theorem due to Venkataramana [9] assures us that an irreducible Γ is necessarily arithmetic (and we can appeal to Harder [4] for the existence of good fundamental domains). Actually Venkataramana proves that an irreducible Γ is arithmetic even when all the factors are of rank 1 if $|I| \geqslant 2$ and provided that Γ *is finitely generated.* Our results guarantee that Γ is indeed finitely generated and thus the assumption needed for proving arithmeticity is indeed always valid. It must however be noted that the finite generation of Γ is deduced as a consequence of the existence of a good fundamental domain and thus the proof of the arithmeticity uses the existence of a good fundamental domain.

Throughout this paper we deal only with the case when all the G_i are *adjoint groups.* It is easy to deduce the general case by using the isogeny of any group onto its adjoint group. We have not spelt out the details; even the formulations are confined to the adjoint group.

Although our main interest is in lattices we formulate our results in an apparently more general context. A discrete subgroup Γ of a locally compact group F is a *L-subgroup* if for every neighbourhood V of 1 in F, there is a compact set $K(V) \subset F$ such that $K(V)\Gamma$ contains $\{g \in F | g\Gamma g^{-1} \cap V = \{1\}\}$. Any lattice in a locally compact group is a L-subgroup (chapter I, [6]). We prove our results first for *irreducible* L-subgroups in G: $\Gamma \subset G$ is *irreducible* if the restriction to Γ of the cartesian projection of G on G_i is injective for every i. We then show that if $\Gamma \subset G$ is an irreducible L-subgroup it has a suitable density property. From this density property, we deduce the following. If $\Gamma \subset G$ is any L-subgroup, there is a partition $I = \bigcup_{1 \leqslant \alpha \leqslant r} I_\alpha$ of I such that the following holds: let $H_\alpha = \prod_{i \in I_\alpha} G_i$ and Γ_α the projection of Γ on H, then Γ_α is an irreducible L-subgroup and Γ has finite index in $\prod_{1 \leqslant \alpha \leqslant r} \Gamma_\alpha$. (It turns out after the construction of the fundamental domain that any L-subgroup in G is indeed a lattice. It is of some mild interest to know if this is true even when G has factors which are of higher rank.)

I would like to thank Venkataramana in collaboration with whom I had earlier made some progress on the questions treated here. My thanks are also due to Margulis for his interest in this work and to Harder for making it possible for me to meet and talk to Margulis in *Bonn.*

We use results from the theory of algebraic groups freely without citing references. Much of the background material needed is to be found in Borel-Tits [2] and some use is made also of the classification results due to Tits [8].

2. Some lemmas on k-rank 1 algebraic groups

2.1 Let k be a local field of characteristic $p > 0$ and $\mathbf{G}$ a connected absolutely simple k-algebraic group of adjoint type of k-rank 1. We denote by G the locally compact group $\mathbf{G}(k)$ of k points of $\mathbf{G}$. In the sequel algebraic subgroups of $\mathbf{G}$ are denoted by Roman capital letters in bold face their k-points by corresponding plain types, their Lie algebras by lower case gothic letters in bold face while the k-points of these

Lie algebras are denoted by the gothic letters in plain type. In particular the Lie algebra of **G** is denoted $\mathfrak{g}$. We also identify Lie algebras of algebraic subgroups of **G** with the corresponding Lie subalgebras of $\mathfrak{g}$. Let **T** be a maximal k-split torus in **G**; by assumption $\dim \mathbf{T} = 1$ so that the character group $X(\mathbf{T})$ of **T** is isomorphic to $\mathbb{Z}$. We fix a generator α of $X(\mathbf{T})$ and define a character $\phi \in X(\mathbf{T})$ to be *positive* if $\phi = r\alpha$ with $r > 0$; also $\phi < 0$ if $-\phi > 0$. Let Φ denote the k-root system of **G** with respect to **T**, $\Phi^+ = \{\phi \in \Phi \mid \phi > 0\}$ and $\Phi^- = \{\phi \in \Phi \mid \phi < 0\}$. Then Φ^+ is of the form $\{\alpha\}$ or $\{\alpha, 2\alpha\}$ for a unique root $\alpha \in \Phi$. The root α is the unique simple root of **G** with respect to **T** and the ordering defined above. Let **Z** (resp. **N**) denote the centraliser (resp. normaliser) of **T** in **G**. We denote by **U** (resp. **V**) the unique k-split maximal unipotent subgroup of **G** normalized by T and having for its Lie algebra u(resp $\mathfrak{v}$) the sum $\bigsqcup_{\phi \in \Phi^+} \mathfrak{g}^{\phi}$ (resp $\bigsqcup_{\phi \in \Phi^-} \mathfrak{g}^{\phi}$) of the root-spaces of **G** (w.r.t. **T**) corresponding to the positive roots. We have then $\mathfrak{g} = \mathfrak{u} + \mathfrak{z} + \mathfrak{v}$. Let **P** (resp. **Q**) be the k-group $\mathbf{Z}\cdot\mathbf{U}$ (resp. $\mathbf{Z}\cdot\mathbf{V}$); then **P** and **Q** are minimal k-parabolic subgroups of **G**. If $2\alpha \in \Phi^+$, the centre U'(resp. V') is a **Z**-stable k-split subgroup with Lie algebra $\mathfrak{u}' = \mathfrak{g}^{2\alpha}$(resp. $\mathfrak{v}' = \mathfrak{g}^{-2\alpha}$). Moreover U' (resp. V') is isomorphic to $\mathfrak{g}^{2\alpha}$ (resp. $\mathfrak{g}^{-2\alpha}$) as a k-vector space by an isomorphism compatible with the action of **Z**. The group U/U' (resp. V/V') is also k-isomorphic to a vector space viz $\mathfrak{u}/\mathfrak{u}' \simeq \mathfrak{g}^{\alpha}$ (resp $\mathfrak{v}/\mathfrak{v}'$ $\mathfrak{g}^{-\alpha}$) through an isomorphism again compatible with **Z**-action. Since **U**/**U**' is affine the natural map: $\mathbf{U} \to \mathbf{U}/\mathbf{U}'$ a U'-fibration is necessarily trivial. Hence we can find a section θ: $\mathbf{U}/\mathbf{U}' \to \mathbf{U}$ to ω defined over k. θ can in fact be chosen to be compatible with the action of **T** on **U**/**U**' and **U***. We assume that θ is chosen in this fashion; in particular this means that $\theta(1) = 1$. Let $\sigma: U \to U$ be the morphism $\theta \cdot \omega$ and define $\tau: \mathbf{U} \to \mathbf{U}'$ by setting $u = \sigma(u).\,\tau(u)$. Note that if $2\alpha \notin \Phi$, $\tau(u) = u$. We denote by G' the group generated by U' and $\mathbf{V}'$. Then **G**' is an absolutely simple k-group which contains **T**, is of k-rank 1 and is *simply connected* if $2\,\alpha \in \Phi^+$. Let ρ: $\mathbf{G} \to GL(\mathfrak{g})$ (resp.) denote the adjoint representation of **G** (resp. $\mathfrak{g}$) on $\mathfrak{g}$. Then $\dot{\rho}(\mathfrak{z})$ leaves $\mathfrak{u}$ stable and we denote by $\dot{\rho}^+$ the representation of $\mathfrak{z}$ on $\mathfrak{u}$ obtained by restring $\dot{\rho}$. It is known – and not difficult to show – that $\dot{\rho}^+$ is *faithful*. The product map β:$\mathbf{U} \times \mathbf{Z} \times \mathbf{V} \to \mathbf{G}$ defined by $\beta(u, z, v) = u \cdot z \cdot v$, $u \in \mathbf{U}$, $z \in \mathbf{Z}$, $v \in \mathbf{V}$ is a k-isomorphism of $U \times Z \times V$ onto an open subset $\mathbf{\Omega}$ of **G**. We define morphisms u:$\mathbf{\Omega} \to \mathbf{U}$, z:$\mathbf{\Omega} \to \mathbf{Z}$, v:$\mathbf{\Omega} \to \mathbf{V}$ by setting $\beta^{-1}(x) = (u(x), z(x), v(x))$ for $x \in \mathbf{\Omega}$. Evidently one has $x = u(x)\cdot z(x) v(x)$ for $x \in \mathbf{\Omega}$. We set $\Omega = \mathbf{\Omega}(\mathbf{k})$; then Ω is an open (dense) subset of G. The results summarized and notations introduced above will be used freely in the sequel.

2.2 The group **Z** is reductive and its commutator subgroup **M** is *anisotropic* over k. It follows that M $(=\mathbf{M}(k))$ is *compact* and (hence) Z has a *unique* maximal compact (open) subgroup Z. Now it is well-known that Z has index 2 in N (and $N/Z \simeq \mathbf{N}/\mathbf{Z}$). Let ν be any element in $N \backslash Z$ fixed once and for all. Evidently ν normalizes Z so that $\mathsf{N} = \mathsf{Z}\,U\nu\mathsf{Z}$ is a maximal compact subgroup in N. Let Λ be the maximal compact subring in k. Then $\mathfrak{g}$ admits a Λ-*free* submodule L with the following properties:

(i) $\mathfrak{g}$ is the k-span of L
(ii) L is N-stable
(iii) $L = L \cap \mathfrak{u} + L \cap \mathfrak{z} + L \cap \mathfrak{v}$.

---

* See Appendix

(To secure the last condition – in case the residue field of k is small – one argues by adding onto N the group of elements of finite order in $\mathbf{T}(k')$ for an unramified extension k' and looking at the action of this bigger compact group on $\mathfrak{g}(k')$.) The Λ-module L enables us to define a compact open subgroup G of G: we set $\mathrm{G} = \{x \in G \mid x(L) = L\}$ (here and often in the sequel we identify x in G with its image $\rho(x) \in GL(\mathrm{g})$). Clearly $\mathrm{G} = G \cap GL(L)$ (where $GL(L) = \{x \in GL(V) \mid x(L) = L\}$; $GL(L)$ is a compact open subgroups of $GL(\mathrm{g})$). For an integer $r > 0$, let

$$GL(L)(r) = \{x \in GL(L) \mid (x-1)(L) \subset \pi^r L\},$$

where π is a uniformising parameter. If $e_1, \ldots, e_n$ is a Λ-basis of L and we identify $GL(\mathrm{g})$ with $GL(n, k)$ through this basis, one has for an integer $r \geq 0$,

$$GL(L)(r) = \{x \in GL(n, \Lambda) \mid x \equiv 1 \,(\mathrm{mod}\, \pi^r)\}.$$

Let $|\ \ |: k \to \mathbb{R}^+$ be the absolute value on k given by $|x| = p^{-r}$ where $x = \pi^r \cdot u$, u a unit in Λ and $r \in \mathbb{Z}$ and $|0| = 0$. As usual we define $\| X \|$ for a matrix $X \in M(n, k)$ by setting $\| X \| = \max(|X_{ij}|, 1 \leq i, j \leq n)$. Using this norm we define a metric on $GL(\mathrm{g})$ which is left-translation invariant as follows: let $g, h \in GL(\mathrm{g})$ then $d(g, h) = p$ if $g^{-1}h \notin GL(L)$; if $g^{-1}h \in GL(L)$, we set $d(g, h) = \inf\{p^{-r} \mid g^{-1}h \in GL(L)(r)\}$. The family $GL(L)(r)$, $r \in \mathbb{Z}^+$ is a fundamental system of neighbourhoods of 1 in $GL(\mathrm{g})$ so that the metric d defined above is compatible with the topology on $GL(\mathrm{g})$. We obtain a left translation invariant metric on G by restricting this metric to it. We also set for $g \in GL(g)$, $|g| = d(1, g)$; it is easy to see then that if $g \in GL(L)(r) \backslash GL(L)(r+1)$ for an integer $r \geq 0$, then $|g| = p^{-r}$. In other words $GL(L)(r) = \{g \in GL(\mathrm{g}) \mid |g| \leq p^{-r}\}$. Also for $g \in GL(L)$, $|g| = \| g - 1 \| (= \| g^{-1} - 1 \|)$. As is well known we have $[GL(L)(r), GL(L)(s)] \subset GL(L)(r+s)$ for integers $r, s \geq 0$. In particular $GL(L)(r)$ is normal in $GL(L)$. Let $\mathrm{G}(r) = GL(L)(r) \cap G$, $r \in \mathbb{Z}^+$; then $\mathrm{G}(r)$ are compact open normal subgroups of G and $[\mathrm{G}(r), \mathrm{G}(s)] \subset \mathrm{G}(r+s)$ and for $g \in \mathrm{G}$, $|g| = \inf\{p^{-r} \mid g \in \mathrm{G}(r)\}$. We will now establish a series of lemmas using the notions in 2·1 and 2·2.

Lemma 2.3. $\mathrm{G}(1) \subset \Omega$ *and for* $x \in \mathrm{G}(1)$, *we have* $|x| = \max(|u(x)|, |z(x), |v(x)|)$. (*Consequently*) *if* $x, y \in \mathrm{G}(1)$ *and* $x = y (\mathrm{mod}\ \mathrm{G}(r))$ *for some* $r \geq 0$, $u(x) = u(y) (\mathrm{mod}\ \mathrm{G}(r))$; $z(x) = z(y) (\mathrm{mod}\ \mathrm{G}(r))$ *and* $v(x) = v(y) (\mathrm{mod}\ \mathrm{G}(r))$.

Proof. One has a more general fact. Let $\tilde{\mathbf{U}}$ (resp. $\tilde{\mathbf{V}}$) denote the subgroup of $GL(\mathfrak{g})$ stabilising $\mathfrak{u}$ and $\mathfrak{u} + \mathfrak{z}$ (resp. $\mathfrak{v}$ and $\mathfrak{z} + \mathfrak{v}$) and acting trivially on $\mathfrak{u}$ (resp. $\mathfrak{v}$), $(\mathfrak{u} + \mathfrak{z})/\mathfrak{u}$ (resp. $(\mathfrak{z} + \mathfrak{v})/\mathfrak{v}$) and $\mathfrak{g}/(\mathfrak{u} + \mathfrak{z})$ (resp. $\mathfrak{g}/(\mathfrak{z} + \mathfrak{v})$). Let $\tilde{\mathbf{Z}}$ be the group $\{g \in GL(\mathfrak{g}) \mid g(\mathfrak{u}) = \mathfrak{u}, g(\mathfrak{z}) = \mathfrak{z} \text{ and } g(\mathfrak{v}) = \mathfrak{v}\}$. (Then $\mathbf{U} \subset \tilde{\mathbf{U}}$, $\mathbf{V} \subset \tilde{\mathbf{V}}$ and $\mathbf{Z} \subset \tilde{\mathbf{Z}}$.) The morphism $(u, z, v) \to u.z.v.$ of $\tilde{\mathbf{U}} \times \tilde{\mathbf{Z}} \times \tilde{\mathbf{V}}$ in $GL(\mathfrak{g})$ is an isomorphism onto a Zariski open set $\tilde{\mathbf{\Omega}}$ and $GL(L)(1) \subset \tilde{\Omega} = \tilde{\mathbf{\Omega}}(k)$; and if $x = u.z.v$, $u \in \tilde{U} = \tilde{\mathbf{U}}(k)$, $z \in \tilde{Z} = \tilde{Z}(k)$ and $v \in \tilde{V} = \tilde{\mathbf{V}}(k)$ with $x \in GL(L)(1)$, then one has $|x| = \max(|u|, |z|, |v|)$; this can be checked by explicit matrix computations using the representation of matrices in End $(\mathfrak{g})$ by blocks corresponding to the direct-sum decomposition $\mathfrak{g} = \mathfrak{u} + \mathfrak{z} + \mathfrak{v}$ (note that our choice of L ensured that $L = L \cap \mathfrak{u} + L \cap \mathfrak{z} + L \cap \mathfrak{v}$).

Lemma 2.4. (i) *If* $g \in \mathrm{G}(1)$ *is such that* $|u(g)| \geq \max(|z(g)|, |v(g)|)$ *and* x *is any element of* $\mathrm{G}(1)$, *then* $|u(xgx^{-1})| \geq \max(|z(ugx^{-1})|, |v(xgx^{-1})|)$ *and the latter inequality is strict if and only if the former is.*

(ii) *If* $g \in \mathrm{G}\ (1)$ *and* $x \in \mathrm{G}\ (1) \cap P$, $|v(xgx^{-1})| = |v(x)|$; *if* $|z(g)| \geqslant |v(g)|$ *one has also* $|z(xgx^{-1})| = |z(x)|$. *If* $x \in \mathrm{G}\ (1) \cap U$ *and* $|u(g)| \geqslant |z(g)| \geqslant v(g)$, *then* $|u(x\hat{g}x^{-1})| = |u(g)|$, $|z(xgx^{-1})| = |z(g)|$ *and* $|v|xgx^{-1})| = |v(g)|$.

Proof. We have $xgx^{-1} = u(g) \cdot u(g)^{-1} xu(g)x^{-1} \cdot x\xi x^{-1}\xi^{-1} \cdot \xi$ where $\xi = z(g) \cdot v(g) \in Q$. Now $|g| = p^{-r}$ we have $u(g) \in \mathrm{G}(r) \backslash G(r+1)$ while $\xi \in \mathrm{G}(r)$. It follows that $u(g)^{-1} xu(g) \cdot x^{-1} \cdot x\xi x^{-1}\xi^{-1} \in \mathrm{G}(r+1)$. Since $\xi \in Q$, $u(xgx^{-1}) = u(u(g) \cdot u(g)^{-1} xu(g)x^{-1} \cdot x\xi x^{-1}\xi^{-1}) = u(g) (\mathrm{mod}\ \mathrm{G}\ (r+1))$. Hence $u(g) = u(xgx^{-1})$. It is also clear that since $|\xi| \leqslant p^{-r}$ we have $|z(xgx^{-1})| \leqslant p^{-r}$ and $|v(xgx^{-1})| \leqslant p^{-r}$ the inequality being strict if $|u(g)| > \max(|z(g)|, |v(g)|)$. Hence the first assertion. To prove the second assertion, we argue as follows: $xgx^{-1} = x(u(g) \cdot z(g))x^{-1} \cdot xv(g)x^{-1}v(g)^{-1}$. Since $xu(g)z(g)x^{-1}$ is in P, $v(xgx^{-1}) = v(xv(g)x^{-1}v(g)^{-1}) \cdot v(g)$; as $|xv(g)x^{-1}v(g)^{-1}| < |v(g)|$ we conclude that $|v(xgx^{-1})| = |v(g)|$. If $|z(g)| \geqslant |v(g)|$, $|xv(g)x^{-1}v(g)^{-1}| < |z(g)|$ while $z(x \cdot u(g)\ z(g)x^{-1}) = z(g)$ leading to $|z(xgx^{-1})| = |z(g)|$. The last assertion is proved along similar lines.

2.5 The group G is a Lie subgroup of $GL(\mathfrak{g})$ (based on k). Consequently we can find an integer $r > 0$, a neighbourhood W of 0 in $\mathfrak{g}$ and an analytic diffeomorphism $e: W \to \mathrm{G}\ (r)$ with the following properties:

(i) Treating $\mathfrak{g}$ and G as subsets of End $\mathfrak{g}$ and $GL(\mathfrak{g})$ respectively, e may be considerd as a map of W in End $(\mathfrak{g})$. The Taylor series of e converges in W and has the following form:

$$e(X) = 1 + X + \sum_{m \geqslant 1} e_m(X), \tag{*}$$

where e_m for an integer $m > 2$ is a End $\mathfrak{g}$-valued homogeneous polynomial on $\mathfrak{g}$ of degree m.
(ii) e maps $W \cap \mathfrak{u}'$ (resp. $W \cap \mathfrak{u}$, $W \cap \mathfrak{z}$, $W \cap \mathfrak{v}$ and $W \cap \mathfrak{v}'$) analytically isomorphically onto $\mathrm{G}\ (r) \cap U'$ (resp. $\mathrm{G}\ (r) \cap U$, $\mathrm{G}\ (r) \cap Z$, $\mathrm{G}\ (r) \cap V$ and $\mathrm{G}\ (r) \cap V'$)
(iii) Let $C: W \times W \to \mathfrak{g}$ be the (analytic) map
$C(X, Y) = e^{-1}(e(X) \cdot e(Y) \cdot e(X)^{-1} \cdot e(Y)^{-1})$, $X, Y \in W$. Then c admits a convergent Taylor expansion in $W \times W$ of the form

$$C(X, Y) = [X, Y] + \sum_{r > 0, s > 0, r+s > 2} C_{rs}(X, Y), \tag{**}$$

where for integers $r, s > 0$, $C_{rs}: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is $\mathfrak{g}$-valued bihomogeneous polynomial on $\mathfrak{g} \times \mathfrak{g}$ of bidegree r, s.

Lemma 2.6. The Lie bracket operation on $\mathfrak{g}$ *has the following properties*:

(i) *There is a constant* $A > 0$ *such that for any* $X \in \mathfrak{z}$ *we can find* $Y \in \mathfrak{u} \cap W$ *such that* $\| X \| \leqslant A \| [X, Y] \|$
(ii) *There is a constant* $B > 0$ *such that for any* $X \in \mathfrak{u}$ *we can find* $Y \in \mathfrak{u} \cap W$ *such that* $\| \bar{X} \| \leqslant B\ \| [X, Y] \|$ *where* $\bar{X} \in \mathfrak{u}/\mathfrak{u}'$ *is the image of* X *in* $\mathfrak{u}/\mathfrak{u}'$ *under the natural map* $\omega: \mathfrak{u} \to \mathfrak{u}/\mathfrak{u}'$ *and* $\mathfrak{u}/\mathfrak{u}'$ *is identified with* $\mathfrak{g}^\alpha$ *through the isomorphism* $\omega|_{\mathfrak{g}^\alpha}: \mathfrak{g}^\alpha \xrightarrow{\sim} \mathfrak{u}/\mathfrak{u}'$ (*if* $2\alpha \notin \Phi^+$ we set $\mathfrak{u}' = 0$).
(iii) *If* $2\alpha \in \Phi^+$ *there is a constant* $C > 0$ *such that for any* $X \in \mathfrak{u}'$ *and any* $Y \in \mathfrak{v}$ *we have* $\| X \| \cdot \| Y \| \leqslant C\ \| [X, Y] \|$.

The lemma is an easy consequence of the known facts about the structure of k-rank 1 groups: the first assertion is just a reformulation of the fact that $\hat{\rho}^+$ is faithful; the second is a consequence of the fact that the Lie bracket in $\mathfrak{u}$ has the property that, given any $X \in \mathfrak{u}$ with $\bar{X} \neq 0$, there exists $Y \in \mathfrak{u}$ with $[X, Y] \neq 0$; the last assertion is readily deduced from the fact that $\mathbf{G}'$ contains a k-subgroup $\mathbf{H}$ k-isomorphic to $SL(2)$ containing $\mathbf{T}$ and such that $X \in \mathfrak{h}$ (the Lie algebra of $\mathbf{H}$).

Lemma 2.7. *Let $\theta: \mathbf{U}/\mathbf{U}' \to \mathbf{U}$ be the section to $\omega: \mathbf{U} \to \mathbf{U}/\mathbf{U}'$ defined in* 2.1 *and (as in* 2.1) *let $\sigma = \theta \cdot \omega$ and $\tau: \mathbf{U} \to \mathbf{U}'$ be defined by $x = \sigma(x) \cdot \tau(x)$ for all $x \in \mathbf{U}$. There is an integer $d > 0$ such that we have the following inequalities: $p^d \| \omega(x) \| \geqslant |\sigma(x)| \geqslant p^{-d} \| \omega(x) \|$ and $p^d |x| \geqslant \max(|\sigma(x)|, |\tau(x)|) \geqslant p^{-d}|x|$ for all $x \in \mathrm{G}(1) \cap U$. Also if we set $\tau(xy) = \tau(x) \cdot \tau(y) \cdot \psi(x, y)$ we have for $x, y \in \mathrm{G}(1) \cap U$, $|\psi(x, y)| \leqslant p^d |x| |y|$.*

Proof. Since $\sigma(1) = 1$ and $\sigma(x) = \sigma(\omega(x))$ is analytic the first inequality is immediate from the Taylor expansion of σ for x sufficiently close to 1. Since $\mathrm{G}(1) \cap U$ is compct and one has $|x| > \varepsilon > 0$ for all x outside a neighbourhood of 1, the inequality extends to all of $\mathrm{G}(1) \cap U$. For reasons similar to those given above, we need only prove the second inequality for x and y close to 1. Now $\psi(x, y) = \tau(xy)\tau(y)^{-1} \quad \tau(x)^{-1} = xy\sigma(xy)^{-1}\sigma(y)y^{-1}\sigma(x)x^{-1} = \sigma(xy)^{-1}xy \cdot y^{-1}\sigma(y)\sigma(x)x^{-1} = \sigma(xy)^{-1}x \cdot x^{-1}\sigma(x) \cdot (y) = \sigma(xy)^{-1}\sigma(x)\sigma(y)$. Thus $\psi(1, y) = \psi(x, 1) = 1$ for all $x, y \in U$. Taylor expansion of ψ near $(1, 1)$ now gives the desired result.

Lemma 2.8. *There exist positive integers a, b, c such that the following hold.*

(i) *Given $x \in \mathrm{G}(1) \cap Z$, we can find $y \in \mathrm{G}(1) \cap U$ such that*

$$|xyx^{-1}y^{-1}| \geqslant p^{-a}|x|.$$

(ii) *Given $x \in \mathrm{G}(1) \cap U$ we can find $y \in \mathrm{G}(1) \cap U$ such that*

$$|\tau(yxy^{-1})| \geqslant p^{-b}|x| (= p^{-b}|yxy^{-1}|).$$

(iii) *If $2\alpha \in \Phi$ there is an integer $c > 0$ such that for $x \in \mathrm{G}(1) \cap U'$ and $y \in \mathrm{G}(1) \cap V$,*

$$\min(|u(xyx^{-1}y^{-1})|, \max(|z(xyx^{-1}y^{-1})|, |\sigma(u(xyx^{-1}y^{-1}))|)) > p^{-c}|x||y|.$$

Proof. This lemma follows immediately from Lemmas 2.6 and 2.7 and the Taylor expansion (**) of 1.5 for the commutator map. Note that $\tau(yxy^{-1}) = \tau(yxy^{-1}x^{-1}x) = (yxy^{-1}x^{-1})\tau(x)$ for $x, y \in U$ so that in proving (ii) we have only to choose y so that $\max(|yxy^{-1}x^{-1}|, |\tau(x)|) \geqslant p^{-b}|x|$ for a preassigned integer $b \geqslant 0$. From Lemma 2.6 and 2.7 and the Taylor expansion of C we see that we can find $y \in \mathrm{G}(1) \cap U$ such that $|yxy^{-1}x^{-1}| \geqslant p^{-b}|\sigma(x)|$ for a suitable integer $b > 0$ (independent of x). If $|\tau(x)| \geqslant p^{-b}|x|$ we can take $y = 1$; if not $|\sigma(x)| = |x|$ and we can take y to satisfy the inequality above and then $|yxy^{-1}x^{-1}| > |\tau(x)|$ so that $|\tau(yxy^{-1})| = |yxy^{-1}x^{-1}|$. (i) and (iii) are straightforward consequences of the two lemmas and the Taylor expansion for C cited above and the fact that satisfies (ii) of 2.5 above.

2.9 For the concise formulation of later results we now introduce further notations and definitions. We say that an element $x \in \mathrm{G}(1)$ is *P-adapted* if we have

$$|v(x)| \leqslant \max(|u(x)|, |z(x)|) (= |x|). \qquad (*)$$

It is *U-adapted* if we have

$$|u(x)| \geqslant \max(|z(x)|, |v(x)|). \tag{**}$$

Evidently if x is *U-adapted* it is affortiori *P-adapted.* The set $E = \{x \in G(1) | x$ is *P*-adapated$\}$ can also be describe as follows: Let $h \in T$ be any element with $|\alpha(h)| > 1$; then $E = \{x \in G(1) | |hxh^{-1}| \geqslant |x|\}$. Note that for any $x \in G(1)$ either $x \in E$ or $v(x) \in E$. Let E^* be the set of U-adapted elements in $G(1)$. Then one has for $h \in T$ as above and $x \in E^*$, $|hxh^{-1}| \geqslant p|x|$. Finally an element $x \in G(1)$ is *special* if every element of G(1) centralising x belongs to E. We denote by S the set of all special elements in G(1).

If $2\alpha \in \Phi$ let c, d be as in lemmas 2.8 and 2.7. When $2\alpha \notin \Phi$ we set $c = 2d$ and define d as follows: the group **G** is the adjoint group of either $SL_{2,D}$ D a central division algebra over k or $U(h)$ the unitary group of a non-degenerate, isotropic antihermitian form over a quaternion division algebra. Let $\tilde{G} = GL(2, D)$ in the former case and $\tilde{G} =$ the group of similitudes of h in the latter case: note that $\tilde{G} \subset GL(2, D)$ in both cases with D a suitable division algebra. The natural map $\tilde{G} \xrightarrow{q} G$ is a surjection of maximal rank. Moreover we may assume q so chosen that the group $\tilde{U}$ (resp. $\tilde{V}$) of upper triangular (resp. of lower triangular) matrices in $\tilde{G}$ maps onto U (resp. V) isomorphically while $q|_{\tilde{D}(=\text{diagonal matrices in } \tilde{G})}$ maps $\tilde{D}$ onto Z and is of maximal rank. For $g \in \tilde{G}$, $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ set $|g| = \max_{1 \leqslant i,j \leqslant 2}(|g_{ij} - \delta_{ij}|)$ and $|g|_0 = \operatorname{Inf}_{x \in k^*} |gx|$; then there is an integer $d > 0$ such that for $g \in \tilde{G}$ with $q(g) \in G(1)$,

$$p^d |g|_0 \geqslant \max(|u(g)|, |z(g)|, |v(g)|) \geqslant p^{-d} |g|_0.$$

With the above definitions of c and d we define a unipotent element $x \in G(1)$ to be *hyperspecial* if the following inequality holds.

$$|\tau(u(x))| > p^{c+2d} \max(|\sigma(u(x))|, |z(x)|, |v(x)|). \tag{*}$$

This the terminology is justified by

Lemma 2.10. *Any hyperspecial element is special.*

Proof. We deal with the case $2\alpha \in \Phi$ first. Let x be a hyperspecial element and $y \in G(1)$ an element commuting with x. Let $\tau(u(x)) = \rho$, $\sigma(u(x)) \cdot z(x) \cdot v(x) = \xi$, $u(y) \cdot z(y) = \eta$ and $v(y) = \alpha$. Assume that $|\alpha| > \max(|u(y)|, |z(y)|) = |\eta|$: we show that this leads to a contradiction. For $g, h \in G$, we denote $ghg^{-1}h^{-1}$ by $[g, h]$. With this notation we have

$$[x, y] = \rho[\xi, \eta]^{-1} \cdot [\rho, \eta] \cdot \eta\rho[\xi, \alpha]^{-1} \eta^{-1} \cdot \eta[\rho, \alpha]\eta^{-1}.$$

Now let $|\rho| = p^{-l}$ and $|\alpha| = p^{-m}$ and set $n = l + m + c + 2d + 1$. Since $|\alpha| > |\eta|$ and $|\xi| < p^{-(c+2d)} |\rho|$ we see that we have

$$1 = [x, y] = [\rho, \eta] \cdot \eta[\rho, \alpha]\eta^{-1} \pmod{G(n)}.$$

Now as $\rho \in U'$ and $\eta \in P$, $[\rho, \eta] \in U'$. Thus we see that $u([x, y]) = [\rho, \eta]\ u([\eta, \alpha]\eta)^{-1}$ (mod G (n)), $z([x, y]) = z(\eta[\rho, \alpha]^{-1}$ (max G (n)) and $v([x, y]) = v(\eta[\rho, \alpha]\eta^{-1})$ (mod G (n)). Now if $v([\rho, \alpha]) \notin G(n)$, $v(\eta[\rho, \alpha]\eta^{-1}) \notin G(n)$ (Lemma 1.4) so that $v([x, y]) \notin G(n)$, a contradiction. Thus we may assume that $v([\rho, \alpha]) \in G(n)$. Again appealing to Lemma 1.4 we see that $z([x, y]) = z([\rho, \alpha])$ (mod G (n)). This means that we have necessarily $z([\rho,\alpha])\in G\,(n)$. In other words $|z([\rho,\alpha])|<p^{l+m+c+2d}$. We also have

$$1=[x,y]=[\rho,\eta]\cdot u(\eta[\rho,\alpha]^{-1}) \bmod G\,(n)$$

and since $\eta[\rho,\alpha]\eta^{-1}=u([\rho,\alpha])\eta^{-1}\ (\mathrm{mod}\ G\,(n))$ we have in fact

$$1=[x,y]=[\rho,\eta]\cdot\eta\cdot u([\rho,\alpha])\eta^{-1}\ (\mathrm{mod}\ G\,(n)).$$

Now $|u([\rho,\alpha])|=|\eta u([\rho,\alpha])\eta^{-1}|$. It follows that $|\sigma(\eta\cdot u([\rho,\alpha])\cdot\eta^{-1})|\geqslant p^{-d}|\eta u([\rho,\alpha])\eta^{-1}|=p^{-d}|u([\rho,\alpha])|$. It follows from Lemma 1.8 – since $|z([\rho,\alpha])|\leqslant p^{-n}$ – that $|u([\rho,\alpha])|\geqslant p^{-l-m-c}$ so that $\sigma(\eta\cdot u([\rho,\alpha])\eta^{-1})\geqslant p^{-l-m-c-d}$. On the other hand we have $[\rho,\eta]\cdot(\eta u([\rho,\alpha])\eta^{-1})\in G(n)$ and hence by Lemma 1.7 once again (since $\sigma([\rho,\eta]\cdot\eta u([\rho,\alpha])\eta^{-1})=\sigma(\eta u([\rho,\alpha])\eta^{-1})$) we see that we have $|\sigma(\eta u[\rho,\alpha]\eta^{-1})|\leqslant p^{d}\cdot p^{-n}<p^{-l-m-c-d}$, a contradiction. This proves the lemma in the case when $2\alpha\in\Phi$. Assume now that $2\alpha\notin\Phi$ and $\tilde{G}$ be the group introduced in 2.9. Let x be hyperspecial so that

$$|u(x)|=|\tau(u(x))|>p^{4d}\max(|z(x)|,|v(x)|).$$

Let $\tilde{z}(x)\in\tilde{D}$ be any lift of $z(x)$; set $\tilde{x}=\tilde{u}(x)\,\tilde{z}(x)\,\tilde{v}(x)$ where $\tilde{u}(x)$ (resp. $\tilde{v}(x)$) is the unique lift of $u(x)$ (resp. $v(x)$) to $\tilde{U}$ (resp. $\tilde{V}$). In view of the definition of d, one has a lift $\tilde{z}(x)$ of $z(x)$ to D such that

$$|\tilde{u}(x)|>p^{2d}\max(|\tilde{z}(x)|,|\tilde{v}(x)|).$$

Suppose now that $y\in Z(x)$ (= centraliser of x) and $\tilde{y}$ is any lift of y in $\tilde{G}$; then the map $x^{r}\to x^{-r}\tilde{y}\tilde{x}^{-r}\tilde{y}^{-1}=\psi_y(x^{r})$ depends only on x^{r} and y (and not on the lifts $\tilde{x}$ and $\tilde{y}$ chosen) and is a homomorphism of the cyclic p-group generated by x into k^* (since $y\in Z(x)$). Since char $k=p$, ψ_y is trivial i.e. $\tilde{x}$ and $\tilde{y}$ commute. If we set $\tilde{y}=\tilde{u}(y)\cdot\tilde{z}(y)\,\tilde{u}(y)$ with $\tilde{u}(y)$ (resp. $\tilde{z}(y)$, resp. $\tilde{v}(y)$) in $\tilde{U}$ (resp. $\tilde{D}$, resp. $\tilde{V}$), a simple matrix computation shows that

$$|\tilde{v}(y)|\leqslant p^{-2d}\max(|\tilde{z}|y)|,|\tilde{u}(y)|).$$

As this holds for any lift $\tilde{y}$ of y to G we conclude that

$$|\tilde{v}(y)|_0\leqslant p^{-2d}\max(|\tilde{z}(y)|_0,|u(y)|_0).$$

Once again from the definition of d we have

$$|v(y)|\leqslant\max(|z(y)|,|u(y)|).$$

Hence the lemma.

PROPOSITION 2.11

Let $N=a+2b+c+6d$ where a,b,c are as in Lemma 2.8 and d is as in Lemma 2.7 and $t\in T$ the unique element such that $\alpha(t)=\pi^{-N}$. Let $C\subset P$ be the compact set $t\cdot(G(1)\cap U)\cdot t\cdot(G(1)\cap U).\ t$. Then given any $x\in E$ with $|x|<p^{-6N}$ there exists $g\in C$ such that the following holds

$|gyg^{-1}|\geqslant|y|$ for all $y\in E$

) $|gyg^{-1}|\geqslant\min(p^{3N}|y|,p)$ for all $y\in E^*$

(iii) $|gyg^{-1}|, |g^{-1}yg| \leqslant \min(p^{6N}|y|, p)$ for all $y \in E$

(iv) $|gxg^{-1}| > p^N|x|$

(v) gxg^{-1} is hyperspecial.

Proof. Observe that for any $h \in T$ with $|\alpha(h)| > 1$, one has $|hyh^{-1}| \geqslant |y$ $y \in E$ while $|hyh^{-1}| \geqslant \min(p, |\alpha(h)| \cdot |y|)$ if $y \in E^*$. It is also clear that $|hyh^{-1}| \leqslant \min(p, |\alpha(h)^2||y|)$. Since $|gyg^{-1}| = |y|$ for all $g \in G$ while E and E^* are stable under linear conjugation by $U \cap \mathrm{G}(1)$, we see that the first three inequalities hold for *any* $g \in C$. Thus we have only to find $g \in C$ to satisfy (iv) and (v). In the sequel we will define the elements x_i, $1 \leqslant i \leqslant 4$ and x' in $\mathrm{G}(1)$ and we will set $u(x_1) = u_i$, $z(x_i) = z_i$ $v(x_i) = v_i$, $u(x') = u'$, $z(x') = z'$, $v(x') = v'$, $u(x) = u$, $z(x) = z$ and $v(x) = v$. We define $x_1 = txt^{-1}$. Then $u_1 = tut^{-1}$, $z_1 = tzt^{-1} = z$ and $v_1 = tvt^{-1}$ so that $\max(|z_1|, |u_1|) \geqslant p^N|v_1|$. In particular $|x_1| = \max(|z_1|, |u_1|) \geqslant |x|$. Now choose an element $\xi \in \mathrm{G}(1) \cap U$ as follows: if $|u_1| \geqslant p^{-a} \cdot |z_1|$, $\xi = 1$; if not ξ is such that $|\xi z_1 \xi^{-1} z_1^{-1}| \geqslant p^{-a}|z_1|$. (Lemma 2.8). Let $x_2 = \xi x_1 \xi^{-1}$; then $|v_2| = |v_1|$, $|z_2| = |z_1|$ while $|u_2| \geqslant p^{-a}|z_2|$ and also $|x_2| = |x_1| = \max(|u_2|, |z_2|) \geqslant p^N|v_2|$. Next let $x_3 = tx_2t^{-1}$; then we have $u_3 = tu_2t^{-1}$ so that $|u_3| \geqslant p^N|u_2| \geqslant p^{N-a}|z_2| = p^{N-a}|z_3|$ (since $z_3 = tz_2t^{-1} = z_2$) and $|v_3| \leqslant p^{-N}|v_2| = \leqslant p^{-2N}\max(|u_2|, |z_2|) \leqslant \max(p^{-3N}|u_3|, p^{-2N}|z_3|)$ $(\leqslant p^{-3N+a}|u_3|)$. It is also clear now that $|x_3| \geqslant p^{N-a}|x_2|$. We now choose an element $\eta \in \mathrm{G}(1) \cap U$ as follows: if $|\tau(u_3)| \geqslant p^{-b}|u_3|$ we set $\eta = 1$; if not choose η such that $\tau(\eta u_3 \eta^{-1}) \geqslant p^{-b}|u_3|$ (Lemma 2.8). Let $x_4 = \eta x_3 \eta^{-1}$. Then $|x_4| = |x_3|$. We claim that $|\tau(u_4)| \geqslant p^{-b}|u_4| = p^{-b}|u_3|$. When $\eta = 1$, this is evident. Hence we assume that $\eta \neq 1$ i.e. $|\tau(u_3)| < p^{-b}|u_3|$. Since we have $|u_3| > |z_3| > |v_3|$ and $\eta \in \mathrm{G}(1) \cap U$ it follows from Lemma 2.4 that $|u_4| = |u_3|$, $|z_4| = |z_3|$ and $|v_4| = |v_3|$; now $x_4 = \eta \cdot x_3 \eta^{-1} = \eta u_3 \eta^{-1} \cdot \eta z_3 v_3 \eta^{-1}(z_3 v_3)^{-1} \cdot z_3 v_3$ and as $z_3 v_3 \in P^-$, we have $u_4 = \eta u_3 \eta^{-1} \cdot u([\eta, z_3 v_3]) = [\eta, u_3] \cdot u_3 \cdot u([\eta, z_3 v_3])$. Thus we have

$$\tau(u_4) = [\eta, u_3] \cdot \tau(u_3 \cdot u([\eta, z_3 v_3]))$$

Now $|\tau(u_3)| < p^{-b}|u_3|$ while $|[\eta, z_3 v_3]| < p^{-N+a}|u_3|$ so that $|u([\eta, z_3 v_3])| < p^{-N+a}|u_3|$ as well. By Lemma 2.7, one has $\tau(u_3 \cdot u([\eta, z_3 v_3])) = \tau(u_3) \cdot \tau(u([\eta, z_3 v_3])) \psi(u_3, u([\eta, z_3 v_3])$. With $|\psi(u_3, u([\eta, z_3 u_3]))| \leqslant p^d$ $|u_3| \cdot u([\eta, z_3 v_3] < p^{-N+a+d}|u_3|$. On the other hand $|\tau(u_3)| < p^{-b}|u_3|$ and $\tau(u([\eta, z_3 v_3])) \leqslant p^d \cdot |u([\eta, z_3 v_3]) > p^{-N+a+d}|u_3|$. Since $N - a - d > b$ and $|[\eta, u_3]| \geqslant p^{-b}$ $|u_3|$, we can now conclude that $\tau(u_4) \geqslant p^{-b}$ $|u_3| = p^{-b}|u_4|$. We have also $|\sigma(u_4)| \leqslant p^d|u_4|$. Finally let $x' = tx_4t^{-1}$. Then we have $u' = tu_4t^{-1}$ $z' = tz_4t^{-1} = z_4$ and $v' = tv_4t^{-1}$. Thus $|v'| \leqslant p^{-N}$ $|v_4| < p^{-N}|z_4| = p^{-N}|z'|$. Next observe that σ and τ are compatible with the action of T so that $\sigma(u') = t\sigma(u_4)t^{-1}$ and $\tau(u') = t$ $\tau(u_4)t^{-1}$. It follows that $|\tau(u')| = |\alpha(t)|^2 \cdot |\tau(u_4)| = p^{2N}|\tau(u_4)| \geqslant p^{2N-b}$ $|u_4|$. On the other hand we have $|\sigma(u')| \leqslant p^d \| \omega\sigma(u') \| = p^{d+N} \| \omega\sigma(u_4) \| \leqslant p^{2d+N}|\sigma(u_4)| \leqslant p^{3d+N}$ $|u_4|$. Thus $|\sigma(u')| \leqslant p^{3d+N}|u_4| \leqslant p^{3d+N-2N+b}|\tau(u')|$; and $N - b - 3d \geqslant c + 2d$. Also, $|u_4| = |u_3| \geqslant p^{N-1}$ $|z_3| = p^{N-a}|z_4|$ leading to $|z'| = |z_4| \leqslant p^{-N+a}|u_4| \leqslant p^{-N+a}$ p^{-2N+b} $|u'|$; and $-3N + a + b > c + 2d$. Thus x' is hyperspecial. If we set $g = t\eta t\xi t$, $g \in C$ and $x' = gxg^{-1}$. From our definitions it is easy to see that $|x'| > p^N|x|$. Thus the proposition is proved.

Lemma 2.12. There is a compact set $K \subset G$ such that $K \supset \mathrm{G}$ and for any $x \in G$ contained in the unipotent radical of a k-parabolic subgroup of **G**, *we can find $g \in K$ such that $gxg^{-1} \in U$. Also if $x \in \mathrm{G}(1)$, we can find $g \in K$ such that $gxg^{-1} \in E$.*

Proof. Since G/P is compact and any two minimal k parabolic subgroups of G are conjugate by an element of G and for any $x \in \mathrm{G}$ (1) either $x \in E$ or $vxv^{-1} \in E$, the lemma follows.

3. Bound on "covolumes" and existence of good unipotents

3.1 *Notation*

We now deviate somewhat from the notation of §2. We will denote by G a direct product of groups of the form $G_i = \mathbf{G}_i(k_i)$ $1 \leqslant i \leqslant q$ where for each i, k_i is a local field of characteristic $p > 0$ (independent of i) and $\mathbf{G}_i$ is a connected absolutely almost simple k-algebraic group of adjoint type of k-rank 1. We fix in each $\mathbf{G}_i$ a split torus $\mathbf{T}_i$ and let $\mathbf{U}_i$, $\mathbf{V}_i$ denote the two maximal k_i-split unipotent subgroups normalised by $\mathbf{T}_i$. Also $\mathbf{Z}_i$ and $\mathbf{N}_i$ denote respectively the centraliser and normaliser of $\mathbf{T}_i$ in $\mathbf{G}_i$ and set $\mathbf{\Omega}_i = \mathbf{U}_i\mathbf{Z}_i\mathbf{V}_i$. Applying the results of §2 to G_i we obtain a family of compact open subgroups $\mathrm{G}_i(r)$, r an integer $\geqslant 0$, in G_i such that for $r \geqslant 1$, $\mathrm{G}_i(r)$ is a pro-p group, $\mathrm{G}_i(r) \supset \mathrm{G}_i(r+1)$ and $[\mathrm{G}_i(r), \mathrm{G}_i(s)]$ is contained in $\mathrm{G}_i(r+s)$. This family of groups define a metric d_i on G_i if we set

$$d_i(g,h) = \begin{cases} p & \text{if } g^{-1}h \notin \mathrm{G}_i = \mathrm{G}_i(0) \\ \inf\{p^{-r} \mid g^{-1}h \in \mathrm{G}_i(r)\} & \text{if } g^{-1}h \in \mathrm{G}_i \end{cases}$$

and let $|g|_i = d_i(1,g)$ for all $g \in G_i$. We assume $\mathrm{G}_i(1)$ to be so chosen that it is contained in $\Omega_i = \mathbf{\Omega}_i(k_i)$ and denote by E_i (resp. E_i^*) the set $x \in \mathrm{G}_i(1)$, $x = uzv$, $u \in U_i = \mathbf{U}_i(k_i)$ $z \in Z_i = \mathbf{Z}_i(k_i)$ and $v \in V_i = \mathbf{V}_i(k_i)$ with $|v|_i \leqslant \max(|u|_i, |z|_i)$ (resp. $|u_i| > \max(|z_i|, |v_i|)\}$. We also denote by S_i the set $\{x \in E_i^* \mid \text{centraliser of } x \text{ in } G_i \text{ is contained in } E_i\}$ and call the elements of S_i special elements in G_i. Finally let U (resp. $T, V, Z, N, \Omega, \mathrm{G}_i(r), E, E^*, S$) denote the product of the U_i (resp. $T_i, V_i, Z_i, N_i, \Omega_i, \mathrm{G}_i(r), E_i, E_i^*, S_i$) considered as a subset of G. Also we set for $g, h \in G$, $d(g,h) = \max\{d_i(g_i, h_i) \mid 1 \leqslant i \leqslant q\}$ where $g_i h_i$ are the components of g, h in the factor G_i and denote by $|g|$ the distance $d(1,g)$ of g from 1 in the metric d. With these new notations we have the following result which is essentially a reformulation of Proposition 2.11 and Lemma 2.12 in the new notation.

PROPOSITION 3.2

There is an integer $N > 0$ and an element $t \in T$ such that the following holds. Let $C = t(\mathrm{G}(1) \cap U)t(\mathrm{G}(1) \cap U) \cdot t$. Then E, E^* and S are stable under inner conjugation by elements of C. Further we have

(i) $|gyg^{-1}| \geqslant y$ *for all* $y \in E$ *and* $g \in C$

(ii) $|gyg^{-1}| \geqslant \min(p^{3N}|y|, p)$ *for all* $y \in E^*$ *and* $g \in C$

(iii) $|gyg^{-1}| \leqslant \min(p^{6N}|y|, p)$ *for all* $y \in \mathrm{G}(1)$ *and* $g \in C$ *or* $g^{-1} \in C$.

Further given any $x \in E$ *we can choose a* $g \in C$ *such that*

(iv) $|gxg^{-1}| \geqslant p^N|x|$

(v) $gxg^{-1} \in S$.

There is a compact subset K of G containing 1 with the following property. Given any $x \in G$ whose components x_i are contained in the unipotent radicals of k_i-parabolic subgroups of the $\mathbf{G}_i$, $1 \leqslant i \leqslant q$, there is a $g \in K$ such that $gxg^{-1} \in U$. Given any $x \in \mathrm{G}$ (1) there is a $g \in K$ such that $gxg^{-1} \in E$. (and hence) an element $g' \in C \cdot K$ such that $g'xg'^{-1}$ is special. Also for $g \in K$ and $x \in \mathrm{G}$ (1), we have $|gxg^{-1}| < p^N |x|$.

DEFINITIONS 3.3

A subgroup U' of G is a *horocycle* if it is conjugate to U in G. A subgroup P' of G is a *parabolic subgroup* if it is conjugate to $P(=$normaliser of U in $G)$. Such a P' contains a unique conjugate U' of U. The subgroup U' of P' is normal in P' and will be referred to as the *nilradical* of P'. An element $g \in G$ is *unipotent* if $g^{p^r} = 1$ for some integer $r \geqslant 0$. A subgroup of G is unipotent if every element in it is unipotent. A unipotent element is *good* if it has a conjugate in U (equivalently if it belongs to a horocycle).

Lemma 3.4. *Any unipotent element x in $[G, G]$ is good.*

Proof. It is clear from the definition that $x \in G$ is a good unipotent if and only if every i, $1 \leqslant i \leqslant k$, the component x_i of x in G_i is the unipotent radical of a k_i-parabolic subgroup of $\mathbf{G}_i$ i.e. x_i is good in the sense of Borel-Tits [3]. Let $\tilde{\mathbf{G}}_i$ be the simply connected covering of $\mathbf{G}_i$. Then according to Borel-Tits [3] every unipotent in $\tilde{\mathbf{G}}_i(k_i)$ is good. The lemma follows from the fact that the cokernel of the natural map $\tilde{\mathbf{G}}_i(k_i) \to G_i$ is *abelian.*

Lemma 3.5. *Let $x \in G$ be a good unipotent element such that $x_i \neq 1$ for all i, $1 \leqslant i \leqslant q$ where x_i is the component of x in G_i. Then x is contained in a unique parabolic subgroup $P(x)$ of G. Further x belongs to the nilradical $U(x)$ of $P(x)$. If $g \in G$ is such that x and gxg^{-1} generate a unipotent subgroup of g, then $g \in P(x)$.*

Proof. One reduces the problem to the case when $q = 1$ by looking at the projections into the various factors. Clearly, then assuming that $G = \mathbf{G}(k)$, $\mathbf{G}$ a connected absolutely simple adjoint group over k of k-rank 1, it suffices to prove the following assertion: if $x \in P$ is good and $y^{-1}xy \in P$, then $y \in P$. To see this we make use of Bruhat-decomposition. If $y \notin P$, $y = unu'$ where u, $u' \in U$ and $n \in N \backslash Z$ uniquely. Let $x' = u^{-1}xu$. Then since $u'Pu'^{-1} \subset P$ we conclude that $n^{-1}x'n \in P$. But for any $n \in N - Z$, $n^{-1}Un \subset V$ and $x' \neq 1$ belongs to U, a contradiction. The uniqueness of the parabolic subgroup containing x is thus proved. We denote this subgroup by $P(x)$. Suppose now that $x \in U$ and P' is a parabolic subgroup containing x. Since $P' = g^{-1}Pg$ for some $g \in G$, we have $gxg^{-1} \in P$ so that $g \in P$ and hence $gxg^{-1} \in U =$ nilradical of P'. Thus if a good unipotent belongs to a parabolic subgroup it belongs to its nilradical and hence the second statement. Now let $y = gxg^{-1}$ and Ψ the (unipotent) subgroup generated by x and y. Let $z \in \Psi$ be a nontrivial good unipotent centralising x and y: if Ψ is abelian then we can take $y = z$; if not $[\Psi, \Psi]$ consists entirely of good unipotents and we can take for z any nontrivial element in the last term of the descending central series of (the nilpotent group). Thus it suffices to prove the following: if g and h are commuting nontrivial good unipotents, then $P(g) = P(h)$. Now $gP(h)g^{-1} = P(ghg^{-1}) = P(h)$ so that $g \in P(h)$ leading to $P(g) = P(h)$.

The following well-known lemma is recorded for future use.

Lemma 3.6. *For any parabolic subgroup P' G, G/P' is compact.*

Lemma 3.7. *Let $\Gamma \subset G$ be a discrete subgroup. Then we have*

(i) $\Gamma \cap \mathrm{G}(1)$ *is a finite unipotent group*
(ii) *If* $\Gamma \cap \mathrm{G}(1)$ *contains a good unipotent* x *all of whose components in the* G_i *are non-trivial, then* $\Gamma \cap \mathrm{G}(1) \subset P(x)$
(iii) *If* $\Gamma \cap \mathrm{G}$ *does not contain a non-trivial good unipotent it is abelian.*

Proof. G(1) is a pro-p group. The first assertion follows from this. If $x, g \in \Gamma \cap \mathrm{G}(1)$ and x is good unipotent x and gxg^{-1} generate a unipotent group. By Lemma 3.5 $g \in P(x)$ if all the components of x are non-trivial; hence the second assertion. The third follows from Lemma 3.4.

DEFINITIONS 3.8

A discrete subgroup $\Gamma \subset G$ is *irreducible* if the restriction to Γ of the Cartesian projection of G on G_i is injective for all i with $1 \leqslant i \leqslant q$. An *irreducible* discrete subgroup $\Gamma \subset G$ is in *good position* if either $\Gamma \cap \mathrm{G}(1) \cap U^+$ is non-trivial or $\Gamma \cap \mathrm{G}(1)$ contains an element of S in its centre and $\Gamma \cap \mathrm{G}(6N+1)$ does not contain any nontrivial good unipotents.

Lemma 3.9. *If* Γ *is irreducible and in good position,* $\Gamma \cap \mathrm{G}(1) \subset E$.

Proof. If $1 \neq x \in \Gamma \cap \mathrm{G}(1) \cap U$ one has, $\Gamma \cap \mathrm{G}(1) \subset P(x) = P \subset E$ (Lemma 3.7). If on the other hand $\Gamma \cap \mathrm{G}(1) \cap U = \{1\}$, by definition $\Gamma \cap \mathrm{G}(1)$ contains an element x of S in its centre. Thus $\Gamma \subset \mathrm{G}(1) \subset$ centraliser x; and centraliser $x \subset E$.

Lemma 3.10. *Let* C *and* K *be compact sets as in Proposition* 3.2. *Then given any irreducible discrete subgroup* $\Gamma \subset G$ *with* $\Gamma \cap \mathrm{G}(8N) \neq 1$ *we can find* $g \in C'K \cup CK \cup K.C.K$ *such that* $g\Gamma g^{-1}$ *is in good position.*

Proof. We assume that Γ is not in good position. Suppose now that $\Gamma \cap \mathrm{G}(N+1)$ contains a nontrivial good unipotent x. Choose $g \in K$ such that $gxg^{-1} \in U$; since $|gxg^{-1}| \leqslant p^N |x|$, $|gxg^{-1}| \leqslant p^{-1}$ so that $g\Gamma g^{-1}$ is in good position. Consider now the case when $\Gamma \cap \mathrm{G}(N+1)$ does not contain any nontrivial unipotent. In this case pick an element $x \neq 1$ in $\Gamma \cap \mathrm{G}(8N)$ and an element $g \in C \cdot K$ such that $x' = gxg^{-1}$ is in S. Let $\Gamma' = g\Gamma g^{-1}$; if $\Gamma' \cap G(N+1)$ contains a nontrivial good unipotent y' we can find $h \in K$ such that $hy'h^{-1} \in \Gamma'' \cap \mathrm{G}(1) \cap U$ where $\Gamma'' = h\Gamma h^{-1} = hg\Gamma g^{-1}h^{-1}$. Since $hg \in K \cdot CK$ and Γ'' are in good position we need only deal with the case when Γ' is such that $\Gamma' \cap U \cap \mathrm{G}(1) = \{1\}$ and $\Gamma' \cap \mathrm{G}(N+1)$ does not contain any nontrivial good unipotents. In this case we claim that Γ' is in good position. To see this, observe that we have only to prove that the element $x' (\in \Gamma' \cap \mathrm{G}(N) \cap S)$ is central in $\Gamma' \cap \mathrm{G}(1)$. Now $x' \in \Gamma' \cap \mathrm{G}(N)$ and if $y \in \Gamma' \cap \mathrm{G}(1)$, $xyx^{-1}y^{-1}$ is a good unipotent contained in $\Gamma' \cap \mathrm{G}(N+1)$ and must hence be trivial. Thus x' is central in $\Gamma' \cap \mathrm{G}(1)$.

PROPOSITION 3.11

Let $\Gamma \subset G$ *be an irreducible descrete subgroup in good position with* $\Gamma \cap \mathrm{G}(12N) \neq 1$. *Let* $1 \neq x \in \Gamma \cap \mathrm{G}(12N)$. *Then there exists* $g \in C$ *such that*

(i) $|gyg^{-1}| \geqslant |y|$ *for all* $y \in \Gamma \cap \mathrm{G}(1)$
(ii) $|gyg^{-1}| \geqslant \min(p^{3N}|y|, p)$ *for all* $y \in \Gamma \cap \mathrm{G}(1) \cap E^*$
(iii) $|gxg^{-1}| \geqslant p^N |x|$

(iv) gxg^{-1} *is special.*

(v) $g\Gamma g^{-1}$ *is in good position.*

(vi) $d(g\Gamma g^{-1}) < d(\Gamma)$.

(Here for a discrete subgroup $\Psi \subset G$, $d(\Psi) = \sum_{x \in \Psi \cap G(12N) \setminus \{1\}} -\log x$.

Proof. By Lemma 3.9, $\Gamma \cap G\,(1) \subset E$. Choose now g as in Proposition 3.11. Then evidently g satisfies all the requirements (i)–(iii) above. We will now show that $g\Gamma g^{-1}$ is in good position as well. Let $x' = gxg^{-1}$. $\Gamma' = g\Gamma g^{-1}$. If $\Gamma' \cap G(1) \cap U \neq \{1\}$, there is nothing to prove. We assume then that $\Gamma' \cap G\,(1) \cap U = \{1\}$. Suppose now that $\Gamma \cap G\,(6N+1)$ contains a nontrivial good unipotent y'; then $y' = gyg^{-1}$ with $y \in \Gamma \cap G\,(1)$. But in view of (i) we have $y \in \Gamma \cap G\,(6N+1)$. As Γ is in good position and y is a good unipotent, $y \in U$. Now $g \in C \subset P$ so that $y' = gyg^{-1} \in U$, a contradiction to our assumption that $\Gamma' \cap G\,(1) \cap U = \{1\}$. We conclude thus that $\Gamma' \cap G\,(6N+1)$ contains no nontrivial good unipotents. Finally as x is in $G\,(12N)$, $x' \in G\,(6N)$ so that for $\xi \in \Gamma \cap G(1)$ $x'\xi x'\xi^{-1}$ is a good unipotent contained in $\Gamma' \cap G\,(N+1)$; thus x' is central in $\Gamma' \cap G(1)$. It follows that Γ' is in good position. To prove (vi) we note that if $y' = gyg^{-1}$, $y \in \Gamma$, belongs to $G\,(12N)$, $y \in G\,(6N)$; but then $|y'| \geqslant |y|$ so that $y \in G\,(12N)$. It is thus clear that we have $F' = \{y \in \Gamma \mid gyg^{-1} \in G\,(12N)\} \subset \{y \in \Gamma \mid y \in G\,(12N)\} = F$ and that for every $y \in F'$, $|gyg^{-1}| \geqslant |y|$ with strict inequality for at least one element if $F' = F$. It is clear from this that we have $d(g\Gamma g^{-1}) < d(\Gamma)$.

Theorem 3.12. *Given any irreducible discrete subgroup $\Gamma \subset G$ there is an element $g \in G$ such that $g\Gamma g^{-1} \cap G\,(12N) = \{1\}$. (Consequently) if μ is a Haar measure on G, for any discrete subgroup $\Gamma \subset G$, the volume of G/Γ for the measure derived from μ is bounded below by a constant $C > 0$ depending only G (and μ).*

Proof. The first assertion is an immediate consequence of the Proposition. For irreducible Γ, the second assertion follows from the first since $G\,(12N)$ maps injectively into $G/g\Gamma g^{-1}$. For general Γ we need only prove the assertion under the assumption that volume (G/Γ) is finite; and in that case we can decompose G into a direct product so that Γ is contained in a corresponding product of irreducible discrete subgroups in the different factors.

3.13 Proposition 3.11 carries much more information than we have used in the proof of Theorem 3.12. Suppose that C and C' are compact subsets of G as in Lemma 3.10 and Γ_0 is an irreducible *L-subgroup* in G. Then there is a *compact* set $B_0 \subset G$ such that

$$B_0\Gamma_0 \supset \{g \in G \mid g\Gamma_0 g^{-1} \cap G(12N) = \{1\}\}.$$

Let $A_0 = \{ghxh^{-1}g^{-1} \mid x \in G\,(12N), g^{-1} \in CUC' \text{ and } h \in B_0\}$. Clearly A_0 is compact. Let

$$\Delta = \{\theta \in A \cap \Gamma_0 \mid \theta \neq 1 \text{ and } g\theta g^{-1} \in G\,(12N) \text{ for some } g \in G\}.$$

Evidently Δ is finite. Let Δ_0 be the subset of all good unipotents in Δ and Δ_0' its complement in Δ. Now $x \in G$ is a good unipotent if and only if G-orbit of x under inner conjugation contains 1 in its closure. Thus there is an integer $N' > 0$ such that we have for all $\theta \in \Delta_0'$ and $g \in G$,

$$|g\theta g^{-1}| \geqslant p^{-N'}$$

Let $e = |\Delta'_0|$, $f = \max(1, [(N' - 12N)/3] + 1)$ and $N_1 = 12N + 6N(e + f)$. Let $B_1 \subset G$ be a compact set containing B_0 such that $B_1\Gamma_0 \supset \{g \in G | g_0 g^{-1} \cap \mathrm{G}\ (N_1)$ is trivial$\}$. Let $A_1 = \{ghxh^{-1}g^{-1} | x \in \mathrm{G}\ (12N),\ g^{-1} \in C \cup C',\ h \in B_1\}$ and $\Delta_1 = \{\delta \in \Gamma_0 \cap A_1 | \delta$ a nontrivial good unipotent$\}$.

Let $g \in G$ be any element and consider the discrete subgroup $g\Gamma_0 g^{-1}$. Choose an element $h_0 \in C'$ such that $h_0 g \Gamma_0 g^{-1} h_0^{-1} = \Gamma_1$ is in good position. We will define inductively conjugates $\Gamma_2, \Gamma_3 \ldots$ of Γ in G as follows. Assume Γ_i, $1 \leqslant i \leqslant l$ is defined to satisfy the following conditions. There are elements $h_i \in C \cup \{1\}$, $1 \leqslant i < l$ such that

(1°) $\Gamma_{i+1} = h_i \Gamma_i h_i^{-1}$ for $1 \leqslant i < l$

(2°) If $\Gamma_i \cap \mathrm{G}\ (12N) = \{1\}$, $h_i = 1$.

(3°) If $\Gamma_i \cap \mathrm{G}\ (12N) \neq \{1\}$, then (h_i, Γ_i) fulfils all the conditions on (g, Γ) in Proposition 3.2 with the element $x_i \neq 1$ in $\Gamma_i \cap \mathrm{G}\ (12N)$ taking the place of x in Proposition 3.2 chosen in the following fashion: let $b_i = h_i h_{i-1} \cdots h_0 g$ so that $\Gamma_{i+1} = b_i \Gamma_0 b_i^{-1}$; then if $(\Gamma_i \cap \mathrm{G}(12N)) \cap E^*$ contains a Γ_i-conjugate of some element of $b_i \Delta'_0 b_i^{-1}$ we take x_i to be such a conjugate; otherwise x_i is taken to be any nontrivial element of $\Gamma_i \cap G(12N)$.

Now if $\Gamma_l \cap \mathrm{G}\ (12N) = \{1\}$ we set $h_l = 1$; if $\Gamma_l \cap \mathrm{G}\ (12N) \neq \{1\}$, choose $h_l \in C$ so that (h_l, Γ_l) satisfy the conditions on (g, Γ) in Proposition 3.11 taking in place of x an element x_l in $\Gamma_l \cap \mathrm{G}\ (12N)$, $x_l \neq 1$ which is a conjugate in Γ_l of an element $b_l \delta'_0 b_l^{-1}$, $\delta'_0 \in \Delta'_0$, if such a conjugate exists in $\Gamma_l \cap \mathrm{G}\ (12N)$. Note that all the Γ_i constructed are in good position and there is an integer $r > 0$ such that $b_r \Gamma b_r^{-1} \cap \mathrm{G}\ (12N) = \{1\}$ while (if $r > 1$), $b_{r-1} \Gamma b_{r-1}^{-1} \cap \mathrm{G}\ (12N) \neq \{1\}$.

3.14 Let $\Psi_i = \{x \in \Gamma | b_{i-1} x b_{i-1}^{-1} \in \mathrm{G}\ (12N)\}$. Then one has

$$\Psi_{r+1} = \{1\} \subset \Psi_{i+1} \subset \Psi_i \subset \Psi_1$$

for $1 \leqslant i \leqslant r$. Also, clearly for $1 \leqslant i \leqslant r$ we have

$$d(\Gamma_i) = -\sum_{x \in \Psi_i} \log |b_{i-1} x b_{i-1}^{-1}| > 0$$

while $d(\Gamma_{r+1}) = 0$. For $\gamma \in \Gamma_0$, let $\langle \gamma \rangle$ denote the Γ-conjugacy class of γ in Γ. We then claim that if $\theta \in \Psi_r \backslash \{1\}$, then $\theta \in \langle \delta \rangle$ for some $\delta \in \Delta$. We have in fact $b_r \theta b_r^{-1} \in \mathrm{G}\ (12N)$. While $b_{r+1} \Gamma_0 b_{r+1}^{-1} \cap \mathrm{G}\ (12N) = \{1\}$; the second fact shows that $b_{r+1} = x\gamma$ with $x \in B_0$ and $\gamma \in \Gamma$. Further $b_r = h_r^{-1} b_{r+1}$ so that $\delta = \gamma \theta \gamma^{-1} \in \Delta$. Now if $\delta \in \Delta_0$, i.e. if δ is *a good unipotent* it follows that $b_i \theta b_i^{-1} \in U \backslash \{1\} \subset E^*$ and we conclude that

$$p^{-12} \geqslant |b_r \theta b_r^{-1}| \geqslant p^{3N(r-1)} |b_1 \theta b_1^{-1}| \geqslant p^{3N(r-2)} |g\theta g^{-1}|$$

leading to

$$|g\theta g^{-1}| \leqslant p^{-3N(r+2)}.$$

Suppose then that for every $\theta \in \Psi_r \backslash \{1\}$, $\theta \in \langle \delta \rangle$ for $\delta \in \Delta'_0$. In this case let s be the minimal positive integer such that $\Psi_{s+1} \cap \langle \delta \rangle$ has cardinality at most one for any $\delta \in \Delta'_0$. In view of condition 3^0 (in our choice of the h_i) in 3.13, we see that there is an integer $a \geqslant 0$ with $a \leqslant l$ such that $b_{s+1+a}(\Psi_{s+1} \cap \langle \delta \rangle) b_{s+1+a}^{-1} \subset E^*$ for all $\delta \in \Delta'_0$. Let $r - s - 1 - a = t$; then one has for $\theta \in \Psi_r \cap \langle \delta \rangle$ ($\delta \in \Delta'_0$ necessarily) we have

$$p^{-12} \geqslant |b_r \theta b_r^{-1}| \geqslant p^{3Nt} |b_{s+1+a} \theta b_{s+1+a}^{-1}| \geqslant p^{3Nt - N'}$$

by the definition of N'. Thus $N' \geqslant 3Nt + 12$ so that $t < f$. It follows now that $r - s \leqslant e + f$. On the other hand Ψ_s contains two distinct element θ, θ' both belonging to $\langle \delta \rangle$ for some $\delta \in \Delta_0'$. It follows that $\phi = \theta'\theta^{-1}$ is a commutator (in Γ, hence) in G. Evidently it belongs to Ψ_s. Thus Ψ_s contains a good unipotent ϕ. Let s' be the integer ($\geqslant s$) such that $\phi \in \Psi_{s'}$, $\phi \notin \Psi_{s'+1}$. Now for all $x \in \Psi_{s'+1}$, $x \neq 1$, we have

$$|b_{s'+1} x b_{s'+1}^{-1}| = |h_{s'+1}^{-1} h_{s'+2}^{-1} \cdots h_r^{-1} b_{r+1} x b_{r+1}^{-1} h_r h_{r-1} \cdots h_{s'+1}|$$
$$\geqslant p^{-6N(r-s') - 12N} \geqslant p^{-N_1}.$$

Thus $b_{s'+1} = x\gamma$ with $x \in B_1$ and $\gamma \in \Gamma_0$. As $b_s, x b_{s'}^{-1} \in \mathrm{G}\,(12N)$ while $h_{s'}^{-1} \in C$, we conclude that $\gamma\phi\gamma^{-1} \in \Delta_1$. Since $b_1 \phi b_1^{-1}$ is in G $(12N)$ and Γ_1 is in good position, $b_1 \phi b_1^{-1} \in U \backslash \{1\} \subset E^*$. Thus we find that

$$p^{-12N} \geqslant |b_{s-1} \phi b_{s-1}^{-1}| \geqslant p^{3N(s-2)}| \geqslant b_1 \phi b_1^{-1}| \geqslant p^{3N(s-3)} |g\phi g^{-1}|$$

leading to $|g\phi g^{-1}| \leqslant p^{-3N(s+1)}$ and since $s \geqslant r - e - f$,

$$|g\phi g^{-1}| < p^{-3N(r-e-f+1)}.$$

Finally let $y \in \Gamma - \{1\}$ be such that $p^{-r'} = |gyg^{-1}| \leqslant |gy'g^{-1}|$ for all $y' \in \Gamma - \{1\}$. Then $|b_1 y b^{-1}| \leqslant p^{-r'N}$ and hence $p^{-12N} \leqslant |b_{r+1} y b_{r+1}^{-1}| \leqslant p^{6rN - r' - N}$ leading to the inequality

$$r' \leqslant 6N(r + 2) + N.$$

We have thus proved the following.

Theorem 3.15. *Let $\Gamma \subset G$ be an irreducible L-subgroup. Then there exist integers $l, N_0 > 0$ and a finite set $\Delta_1 \subset \Gamma$ of nontrivial good unipotents such that the following holds. If $g \in G$ is such that $g\Gamma g^{-1} \cap \mathrm{G}(2n + l) \neq \{1\}$ with $n \geqslant N_0$ then there exists $\delta \in \Delta_1$ and $\theta \in \langle \delta \rangle$ such that $|g\theta g^{-1}| < p^{-n}$.*

COROLLARY 3.16

If $P' \subset G$ is a parabolic subgroup of G such that $U' \cap \Gamma \neq \{1\}$, U' being the nilradical of P', then there is a $\delta \in \Delta_1$ and $\gamma \in \Gamma$ such that $P' = \gamma P(\delta)\gamma^{-1}$.

4. Fundamental domains

We will prove the existence of a good fundamental domain for an irreducible L-subgroup $\Gamma \subset G$ in this section. The first step towards this is

Theorem 4.1. *Let Γ be an irreducible L-subgroup of G and $\theta \in \Gamma \backslash \{1\}$ a good unipotent. Let $P(\theta)$ (resp. $U(\theta)$) be the unique minimal parabolic subgroup (resp. horocyclic subgroup) containing θ and ${}^0P(\theta) = \{g \in P \mid g$ normalizes U and preserves a Haar measure on it$\}$. Then ${}^0P(\theta)/{}^0P(\theta) \cap \Gamma$ is compact.*

4.2 After conjugating Γ by a suitable element of G, we may assume that $\theta \in U$. Clearly then $U(\theta) = U$ and $P(\theta) = P$. We also set ${}^0P = {}^0P(\theta)$. Let $q: P \to P/U$ be the natural map. Then q maps Z isomorphically onto P/U. Let D be the *unique* maximal pro-p

subgroup of Z. (Such a subgroup exists: this follows from the fact that $Z = \prod_{i \in I} Z_i$ with $Z_i = \mathbf{Z}_i(k_i)$ where $\mathbf{Z}_i$ is a reductive k_i-algebraic group whose commutator subgroup is anisotropic over k_i). Now U admits a family U_n, $n \in \mathbb{Z}^+$, of open compact D-stable subgroups such that $\bigcup_{n \in \mathbb{Z}^+} U_n = U$. The U_n, $n \in \mathbb{Z}^+$ are all pro-p groups and hence so are the DU_n. It follows that $DU_n \cap \Gamma$ is a finite unipotent group for every n and hence $(DU \cap \Gamma)$ is a unipotent group which we denote Λ in the sequel. Let L denote the "Zariski closure" of Λ in G. Here and in the sequel we mean by the *Zariski closure* of a subgroup Γ' of G, the subgroup G' of G is obtained as follows: let $\Gamma_i'(\subset G_i)$ be the projection of Γ' in G_i and $\mathbf{G}_i'$ be the Zariski closure of Γ_i' in $\mathbf{G}_i$; let $G_i' = \mathbf{G}_i'(k_i)$, then $G' = \prod_{i \in I} G_i'$ (*Note that all Zariski closures in our definition decompose as products of k_i-points of algebraic subgroups in G_i*). We observe that if $A \subset G$ is any set, the centraliser $Z(A)$ of A is its own Zariski closure; also A admits a *finite* subset A' such that $Z(A) = Z(A')$. Yet another observation needed repeatedly in the sequel is the following: *if $B \subset G$ is any subgroup, then $Z(B) = Z(\bar{B})$ where $\bar{B}$ is the Zariski closure of B*. We record for future use the following well known result.

Lemma 4.3. *If $A \subset \Gamma$ is any subset, the natural map $Z(A)/Z(A) \cap \Gamma \to G/\Gamma$ is proper.*

Proof. Using a standard argument involving the Baire-category Theorem, it is easy to see that we need only prove that $Z(A) \cdot \Gamma$ is closed in G. We assume that A is finite. Suppose $g_n \in Z(A)$ and $\gamma_n \in \Gamma$ are sequences such that $g_n \gamma_n$ converges to a limit. Then for $x \in A$, $\gamma_n^{-1} g_n^{-1} x g_n \gamma_n = \gamma_n^{-1} x \gamma_n$ converges to a limit; but $\gamma_n^{-1} x \gamma_n \in \Gamma$. Thus we see that for some integer $m > 0$ we have $\gamma_n^{-1} x_n \gamma_n = \gamma_m^{-1} x \gamma_m$ for all $n \geqslant m$. i.e. $\theta_n = \gamma_n \gamma_m^{-1} \in Z(A)\ \Gamma$ for $n \geqslant m$. Clearly $g_n \theta_n$ converges to a limit. This proves the lemma.

4.4 Now let Φ be a *maximal abelian* subgroup of $U \cap \Gamma$ and set $\Phi' = Z(\Phi) \cap \Gamma$. Evidently $\Phi \subset \Phi'$. Let F' be the Zariski closure of Φ'. We claim that *F'/Φ' is compact*. Since F' is contained in $Z(\Phi)$, it suffices to show that *$Z(\Phi)/Z(\Phi) \cap \Gamma$ is compact*. If this last space is not compact, in view of Lemma 4.3 we can find sequences $g_n \in Z(\Phi)$ and $\gamma_n \in \Gamma \backslash \{1\}$, γ_n a good unipotent such that $g_n \gamma_n g_n^{-1} \to 1$. We assume (as we may after a conjugation by a suitable element of T) that $Z(\Phi \cap \mathrm{G}\,(1)) = Z(\Phi)$. Since $g_n x g_n^{-1} = x \in \mathrm{G}\,(1)$ for all $x \in \Phi \cap \mathrm{G}\,(1)$, we conclude that $\mathrm{G}\,(1) \cap \Phi$ and γ_n generate a unipotent subgroup of Γ. By forming repeated commutations of γ_n with elements of $\Phi \cap \mathrm{G}\,(1)$ we see that we can find $\theta_n \in \Phi$ with $\theta_n = g_n \theta_n g_n^{-1}$ converging to identity, a contradiction. Thus $Z(\Phi)/Z(\Phi) \cap \Gamma$ is compact. One consequence of this is that $Z(\Phi) \cap U \subset F'$. This follows from the following two observations: There is a representation ρ of G on a vector space W and a vector $w_0 \in W$ such that $F' = \{g \in G \mid \rho(g) w_0 = w_0\}$; secondly, the orbit of any $v \in W$ under $Z(\Phi) \cap U$ if relatively compact is trivial (these observations are consequences of standard results about *split* unipotent groups over local fields). Since Φ is *abelian* $Z(\Phi) \cap U$ contains a *maximal abelian* subgroup U_1 of U. From the structure of rank 1 algebraic groups over local fields (using classification, for instance) it is not difficult to see that U_1 is *maximal abelian in* G. Thus one has $Z(F')(= Z(\Phi')) \subset U_1$. Since $\Phi' \subset \Lambda$ one sees without difficulty that $Z(\Phi) \subset MU \subset \mathrm{D}$. $U - Z(\Lambda)$ is contained in U_1. Also since Λ is unipotent $Z(\Lambda)$ is non-trivial, the map $Z(\Lambda)/Z(\Lambda) \cap \Gamma \to G/\Gamma$ is proper and factors through the *compact* space $Z(\Phi)/Z(\Phi) \cap \Gamma$, *$Z(\Lambda)/Z(\Lambda) \cap \Gamma$ is compact*. Let $\Phi_0 = Z(\Lambda)$ and F_0 the Zariski closure of Φ_0. Then $F_0 = Z(\Lambda)$ and F_0/Φ_0 is compact. Also $F_0 \subset U$.

4.5 Consider now the case when $|I| = 1$ i.e. $G = G(k')k'$ *a finite extension of k and G a k-rank* 1 *absolutely simple k′-algebraic group of adjoint type.* In this case Λ has *finite* index $P \cap \Gamma$ as is easily seen. Also ${}^0P = MU$ so that $(P \cap \Gamma)\cdot {}^0P = K \cdot U$ where K is a *compact* subset of P. We claim now that $Z(\Phi_0)/Z(\Phi_0) \cap \Gamma$ is compact. If not we can find $g_n \in G$, $\gamma_n \in \Gamma$ and a nontrivial good unipotent $\delta \in \Gamma$ such that $g_n \gamma_n \delta \gamma_n^{-1} g_n^{-1}$ tends to 1. We may assume that $Z(\Phi_0) = Z(\Phi_0 \cap \mathrm{G}(1))$ so that for $x \in Z(\Phi_0) \cap \mathrm{G}(1)$, $x = g_n x g_n^{-1}$ and $g_n \gamma_n \delta_n \gamma_n^{-1} g_n^{-1}$ are both contained in the same unipotent group. It follows that $\gamma_n \delta \gamma_n^{-1} \in U \cap \Gamma$. Thus replacing δ by $\gamma_1 \delta \gamma_1^{-1}$ and γ_n by $\gamma_n \gamma_1^{-1}$, we find that $g_n \gamma_n \delta \gamma_n^{-1} g_n^{-1}$ tends to 1 with $\gamma_n \in P \cap \Gamma$ and $\delta \in U \cap \Gamma$. Now $g_n \gamma_n = k_n \cdot u_n$, $k_n \in K$, $u_n \in V$; K being compact this means that $u_n \delta u_n^{-1}$ tends to 1. But the inner conjugation orbit of δ under the ("split") unipotent group U is closed, a contradiction. This proves our claim that $Z(\Phi_0)/Z(\Phi_0) \cap \Gamma$ is compact. Since $Z(\Phi_0) \supset L \supset Z(\Phi_0) \cap \Gamma = \Lambda$, we see that L/Λ *is compact.* We assert now that $L \supset U$. To see this let N be the normaliser of L in G. Then N normalises $Z(L) = Z(\Lambda) = F_0$. Let $N_0 = \{g | g \in N, g$ preserves a Haar measure μ on $F_0\}$. If $L \cap U = U' \neq U$, let $U_1 = \{g \in U | gxg^{-1}x^{-1} \in U'$ for all $x \in L\}$; then U_1/U' is a non-compact group. It is clear that U_1 normalises L and – as is easily seen – that $U_1 \subset N_0$. Now the map $N_0/N_0 \cap \Gamma \to G/\Gamma$ *is proper:* this is seen as folows: let $g_n \in N_0$ and $\gamma_n \in \Gamma$ be sequences such that $g_n \gamma_n$ converges to a limit x. We assume (as we may after a conjugation) that $\mu(F_0 \cap \mathrm{G}(1)) > \mu(F_0/\Phi_0)$. Then one can find $\theta_n \in \Phi_0 \backslash \{1\}$ such that $g_n \theta_n g_n^{-1} \in \mathrm{G}(1)$. It follows immediately that $\gamma_n^{-1} \theta_n \gamma_n \in \Gamma \cap A$ where A is a compact set. Thus passing to a subsequence we may assume that $\gamma_n^{-1} \theta_n \gamma_n = \gamma_m^{-1} \theta_m \gamma_m$ for all $n \geq m$ for some integer $m > 0$. *But since* $\theta_n \in U$ for all $n \geq m$, $\gamma_n \in P \cap \Gamma$. Since $(P \cap \Gamma)/\Lambda$ is finite we see that we can find $\gamma'_n \in \Lambda$ such that $g_n \gamma'_n$ converges to a limit. This shows that $N_0/N_0 \cap \Gamma \to G/\Gamma$ is proper. Clearly $U_1 L/L \approx U_1/U'$ is non-compact so that we can find $u_n \in U_1$, $\gamma_n \in \Gamma$ and a nontrivial good unipotent $\delta \in \Gamma$ such that $u_n \gamma_n \delta \gamma_n^{-1} u_n$ tends to 1. Once again since $u_n \in N_0$, we can find $\theta_n \in \Phi_0$ such that $u_n \theta_n u_n^{-1} \in \mathrm{G}(1)$. We conclude that $\gamma_n \delta \gamma_n^{-1}$ and θ_n generate a unipotent group. Replacing δ by $\gamma_1 \delta \gamma_1^{-1}$ and γ_n by $\gamma_n \gamma_1^{-1}$, we see that $\gamma_n \in P \cap \Gamma$. But then $u_n \gamma_n = k_n \xi_n$ with $k_n \in K$ and $\xi_n \in U$; as before this leads to a contradiction since the U-orbit of δ is closed. This proves that $U' = U$. Thus sine L/Λ is compact ${}^0P/{}^0P \cap \Gamma$ is compact when $G = G(k')$ G an absolutely simple k'-rank 1 k'-algebraic group of adjoint type.

4.6 Suppose now $G = \prod_{i \in I} G_i$. Let G_i be a compact open subgroup of G_i. Let $\eta_i: G \to G_i$ be the cartesian projection and $H_i = \bigcap_{j \neq i} \eta_j^{-1}(\mathrm{G}_j)$. Then H_i is an open subgroup of G. Let $\Gamma'_i = H_i \cap \Gamma$ and Γ_i the projection of Γ'_i on G_i. Then Γ_i is evidently a discrete subgroup of G_i. Since $H_i/\Gamma'_i \to G/\Gamma$ and $H_i \to G_i$ are proper, one sees easily that the Γ_i are L-subgroups of the G_i. It is now clear that if $\Gamma \cap U \neq \{1\}$, then $\Gamma_i \cap U_i \neq \{1\}$ for all i. By the results of 4.6 we know that $M_i U_i / M_i U_i \cap \Gamma_i$ is compact ($M_i = \eta_i(M)$). It is immediate from this that $MU/MU \cap \Gamma$ *is compact.* Now $\Gamma \not\subset P$. In fact if $\Gamma \subset P$, $\Gamma \cap P$ normalises $M \cdot U \cap \Gamma$; since $MU/MU \cap \Gamma$ is compact, $\Gamma \cap P$ preserves a Haar measure on MU so that $\Gamma \subset {}^0P$. Now let $t_n \in T$ be a sequence such that $|t_n x t_n^{-1}| \geq p^n |x|$ for $x \in U$. Then $\bar{t}_n =$ image t_n in G/Γ has no convergent subsequence. But then we can find nontrivial good unipotents θ_n such that $t_n \theta_n t_n^{-1}$ tends to 1, a contradiction since $\theta_n \in \Gamma \cap {}^0P$ and hence $\theta_n \in U$. Now let $\theta \in \Gamma \backslash P$. Let $P' = \theta P \theta^{-1}$ and more generally for any subset $A \subset G$, $A' = \theta A \theta^{-1}$. Let $Z^* = P \cap P'$ and T^* the unique conjugate of T in Z_1. From the fact that $MU/MU \cap \Gamma$ is compact, one sees easily that F_0 is contained in the centre of U and is hence T^*-stable. The same applies to F'_0. It is also not difficult to see that if

$t \in T^*$ preserves the Haar measure on F_0 it preserves the Haar measure on F'_0 as well. Let $T_0^* = \{t \in T^* | t$ preserves the Haar measure on $F_0\}$. Now if $t_n \in T_0^*$ is such that image $t_n (= \bar{t}_n)$ in G/Γ has no convergent subsequence we can find $1 \neq \theta_n \in \Gamma \cap U$ such that $t_n \theta_n t_n^{-1}$ tends to 1. On the other hand we can find $\theta'_n \neq 1$ in $F_0 \cap \Gamma$ such that $t_n \theta'_n t_n^{-1}$ is in a fixed compact set. We conclude that θ_n and θ'_n generate a unipotent group, a contradiction since $U' \neq U$. We see thus that $T_0^* \cdot M \cdot U / P \cap \Gamma$ is compact. Thus $T_0^* MU$ is unimodular and one concludes from this that $T_0^* MU = {}^0P$. Thus ${}^0P/{}^0P \cap \Gamma$ is compact. This proves Theorem 4.1.

Theorem 4.7. *Let $\Gamma \subset G$ be an irreducible L-subgroup. Then there is a finite set $\sum$, a constant $t_c \geqslant 0$, a maximal compact subgroup G^* and a compact subset $\eta \subset {}^0P$ such that*

$$G = \mathrm{G}^* \cdot A_t \cdot \eta \cdot \sum \Gamma, \quad \text{for all } t \geqslant t_c$$

where for $t > 0$ $A_t = \{|x \in T||\chi(x)| \leqslant t\}$, χ is the character on T given by $\mathrm{Int}\,(t)(\mu) = \chi(t)\mu$, *$\mu$ a Haar measure on U. Moreover we have for every $\xi \in \sum$, $\xi^{-1}\eta\xi \cdot (\Gamma \cap \xi^{-1}{}^0P\xi) = \xi^{-1}{}^0P\xi$ and there exists $t_0 > 0$ such that if $\mathrm{G}^* \cdot A_t \eta \xi \gamma \cap \mathrm{G}^* A_{t_0} \eta \xi' \neq \phi$ then $\xi = \xi'$ and $\gamma \in \xi^{-1}{}^0P\, \xi \cap \Gamma$.*

COROLLARY 4.8

If $|I| \geqslant 2$, Γ is finitely generated. This follows from the fact that for any good unipotent $1 = \theta \in \Gamma$, ${}^0P(\theta)/{}^0P(\theta)\, \Gamma$ is compact and ${}^0P(\theta)$ is compactly generated combined with the theorem above. The theorem is proved by a straightforward imitation of the proof of Theorem 13.12 *of* [6].

4.9 From the structure of local fields, we know that we can find a finite field f (of maximal cardinality such that) each of k_i is a finite unramified extension of $f((X))$, the quotient field of the power series ring $f[[X]]$ in one variable over f. (in other words, replacing k by a suitable field we may assume that all the k_i are unramified extensions of k). With this modification we see that $\mathbf{G} = \prod_{1 \leqslant i \leqslant q} \mathbf{G}_i$ may be regarded as a *semisimple* k-algebraic group of adjoint type and that $G = \mathbf{G}(k)$. With these remarks we have thus following corc'lary.

COROLLARY 4.10

If $\Gamma \subset G$ is irreducible, Γ is Zariski dense in the algebraic group $\mathbf{G}$.

Proof. Let $\mathbf{P} \subset \mathbf{G}$ be any minimal k-parabolic subgroup then $\Gamma \not\subset P = \mathbf{P}(k)$). If G/Γ is compact this is clear from the fact that P is *not* unimodular when G/Γ is not compact, this is shown in 4.6. Thus it suffices to show that for some minimal parabolic subgroup $\mathbf{P} \subset \mathbf{G}$, the unipotent radical $\mathbf{U}$ of $\mathbf{P}$ is in the Zariski closure of Γ in G. Theorem 4.7 assures us of this when G/Γ is not compact. When G/Γ is *compact* we argue as follows. Let $\mathbf{H}$ be the Zariski closure of Γ in $\mathbf{G}$ and $H = \mathbf{H}(k)$. Let $\rho: G \to PGL(E)$ be a k-representation of G in $PGL(E)$, E a k-vector space such that $H = \{g \in G | \rho(g)(p) = p\}$ for a suitable k-point p of the projective space $\mathbb{P}(E)$. The closure of the U-orbit $\Lambda(U = \mathbf{U}(k))$ of p contains a $\mathbf{U}$-fixed point since $\mathbf{U}$ is k-split. But Λ is contained in the G orbit of p and this G-orbit is compact. Thus we find that U has a fixed point in G/H. This means that a conjugate of U is contained in H. This proves the corollary.

COROLLARY 4.11

Let $G = \prod_{i \in I} G_i$ $(I = (1, 2, \ldots q))$ be as above and Γ any L-subgroup of G. Then there is a partition $I = I_1 \, UI_2 \cdots UI_r$ of I into disjoint subsets I_j, $1 \leqslant j \leqslant r$ such that if we set $H_j = \prod_{i \in I_j} G_i$ and $\Gamma_j = H_j = H_j \cap \Gamma$ then Γ_j is an irreducible L-subgroup of H_j and $\prod_{1 \leqslant j \leqslant r} \Gamma_j$ has finite index in Γ.

Proof. We will argue by induction on the number of factors of G. Let $\mathsf{I} = \{J \subsetneqq I | \Gamma \cap \prod_{i \in J} G_i \neq \{1\}\}$. If I is empty Γ is irreducible and there is nothing to prove. Assume that $\mathsf{I} \neq \phi$ and let $I' \in I$ be a minimal element. Let $G' = \prod_{i \in I'} G_i$ and $G'' = \prod_{i \in I \setminus I'} G_i$. Let $\pi: G' \times G''(=G) \to G'$ be the cartesian projection. Let $\Gamma'_0 = \Gamma \cap G'$ and $\tilde{\Gamma}' = \pi(\Gamma)$. Evidently Γ'_0 is a normal subgroup $\tilde{\Gamma}'$ (note that $\Gamma'_0 = \pi(\Gamma'_0)$). Let K'' be a maximal compact open subgroup and $\Gamma' = \pi(\Gamma \cap (G' \times K''))$. Evidently $\Gamma'_0 \subset \Gamma' \subset \tilde{\Gamma}'$ and Γ' is easily seen to be a L-subgroup of G'. We claim that Γ' *is an irreducible L-subgroup in* G'. Suppose that Γ' is not irreducible. Since $|I'| < |I|$, by induction hypothesis, G' decomposes as a product $\prod_{\alpha \in A} H_\alpha$ such that $H_\alpha \cap \Gamma' = \Gamma'_\alpha$ is a irreducible L-subgroup of H_α and $\prod_{\alpha \in A} \Gamma'_\alpha = \Gamma'_1$ is of finite index in Γ'. By Corollary 4.10, Γ'_α is Zariski dense in H'_α. Clearly Γ'_α normalises Γ'_0. Hence the Zariski closure H'_0 of Γ'_0 is normalised by H'_α. As this holds for every α, H'_0 is a normal subgroup of G'. Thus H'_0 is a product of certain of the k-simple factors of G' and from the minimality of I', one sees that $H'_0 = G$. But then $[\Gamma'_\alpha, \Gamma'_0]$ is nontrivial and contained in H_α, a contradiction to the minimality of I' if $H_\alpha \neq G'$. Thus Γ' is an irreducible L-subgroup in G'. Moreover since Γ'_0 is Zariski dense in G', it is immediate that the normaliser of Γ'_0 in G' is discrete. We conclude from this that $\tilde{\Gamma}'$ is discrete and Γ' has finite index in $\tilde{\Gamma}'$. But this means that $\Gamma G''$ is closed in G or in other words, the map $G''/G \cap \Gamma \to G/\Gamma$ is *proper.* If $g_n \in G''$ is a sequence tending to infinity mod $\Gamma''_0 = G'' \cap \Gamma$, it follows that we can find $\theta_n \in \Gamma \setminus \{e\}$ such that $g_n \theta_n g_n^{-1}$ tends to identity. On the other hand for $\rho \in \Gamma'_0$, $g_n \rho g_n^{-1} = \rho$; thus $g_n(\rho\theta_n\rho^{-1}\theta_n^{-1})g_n^{-1}$ tends to the identity; since $\rho\theta_n\rho^{-1}\theta_n^{-1} \in G'$ and G' and G'' commute we see that $\rho\theta_n\rho^{-1}\theta_n^{-1}$ tends to 1 i.e. $\rho\theta_n\rho^{-1}\theta_n^{-1} = 1$ for large n. Varying ρ over a suitable finite set and using the Zariski density of Γ'_0 in G', we conclude that θ_n commute with G' for large n i.e. $\theta_n \in G'' \cap \Gamma = \Gamma''_0$ for large n. Thus Γ''_0 is a L-subgroup of G''. The induction hypothesis combined with Corollary 4.10 now shows that Γ''_0 is Zariski dense in G''. Thus $G' = Z(\Gamma''_0)$, the centraliser of Γ''_0 in G so that $G'/G' \cap \Gamma \to G/\Gamma$ is proper. One now concludes arguing as above (with Γ''_0 in place of Γ'_0) that $\Gamma'_0 = G' \cap \Gamma$ is a L-subgroup of G'. It is now clear that $\Gamma'_0 \times \Gamma''$ is a L-subgroup of G and has finite index in Γ. Using the induction hypothesis on G' and G'', the result now follows for G.

5. Appendix

We will prove the following using the notations of §2.

PROPOSITION 5.1

The natural map ω: $\mathbf{U} \to \mathbf{U}/\mathbf{U}'$ admits a section $\theta: \mathbf{U}/\mathbf{U}' \to \mathbf{U}$ defined over k compatible with the action of $\mathbf{T}$.

Proof. Since $\mathbf{U}/\mathbf{U}'$ is affine and $\mathbf{U}'$ is a vector space over k, the fibration ω is trivial. Thus we can find a section $\rho: \mathbf{U}/\mathbf{U}' \to \mathbf{U}$ *defined over* k. If ρ_1, ρ_2 are two sections to

ω one has a morphism α: $\mathbf{U}/\mathbf{U}' \to \mathbf{U}'$ such that $\rho_1(x) = \rho_2(x)\cdot\alpha(x)$. If ρ_1, ρ_2 are defined over k so is α. Let $t \in T$; then one has $t^{-1}\rho(txt^{-1})t = \rho(x)\cdot\phi(t,x)$ where $\phi(t,-)$ is a morphism of $\mathbf{U}/\mathbf{U}'$ in $\mathbf{U}'$ denoted $\Phi(t)$ in the sequel. It is easy to to see that ϕ:$\mathbf{T} \times \mathbf{U}/\mathbf{U}' \to \mathbf{U}'$ is a k-morphism and that Φ: $\mathbf{T} \to \mathrm{Hom}(\mathbf{U}/\mathbf{U}', \mathbf{U}')$ (where Hom denotes the morphisms in the category of algebraic varieties) is a 1-cocycle on $\mathbf{T}$. Here $\mathfrak{H} = \mathrm{Hom}(\mathbf{U}/\mathbf{U}', \mathbf{U}')$ is given the abelian group structure derived from that on $\mathbf{U}'$ and $t \in \mathbf{T}$ acts on $\mathfrak{H}$ by $f \to \overline{\mathrm{Int}}\, t.f.\mathrm{Int}\, t^{-1}$ where $\mathrm{Int}\, t$ is the inner automorphism of $\mathbf{G}$ induced by t and $\overline{\mathrm{Int}}$ is the natural automorphism of $\mathbf{U}/\mathbf{U}'$ induced by $\mathrm{Int}\, t$. Let $\mathfrak{H}(n) = \{f \in \mathfrak{H} \mid f$ is homogeneous of degree $n\}$ (Note that $\mathbf{U}'$ and $\mathbf{U}/\mathbf{U}'$ are k-vector spaces in a natural fashion). Then $\mathbf{T}$ acts through the character $2\alpha\text{-}n\alpha$ on $\mathfrak{H}(n)$. It follows that $H^1(T, \mathfrak{H}(n)) = 0$ for $n \neq 2$ (see for instance Raghunathan [6, Preliminaries]). If $n = 2$ $H^1(T, \mathfrak{H}(n))$ will consist of abstract group homomorphisms of $\mathbf{T}$ in $H(n)$. Now Φ:$\mathbf{T} \to \mathfrak{H}$ necessarily factors through to an algebraic morphism of $\mathbf{T}$ into a finite dimensional k-vector space $\bigsqcup_{0 \leqslant n \leqslant N} \mathfrak{H}(n)$ for some integer $N \geqslant 2$. Let $\Phi_n = p_n \circ \Phi$ where p_n is the natural projection on $\mathfrak{H}(n)$. Then since Φ_2 is algebraic, it is zero. On the other hand for $n \neq 2$, Φ_n is the coboundary of a *unique* $f_n \in \mathfrak{H}(n)$. Since Φ_n is defined over k, so is f_n as is easily seen. Clearly $\{f_n\}_{0 \leqslant n \leqslant N}$ (with $f_2 = 0$) define a morphism f of $\mathbf{U}/\mathbf{U}'$ in $\mathbf{U}'$. If we now modify ρ by f we obtain a section θ satisfying the requirements of the lemma.

References

[1] Behr H, Eindliche Erzeugbarkeit arithmetischer Gruppen über Funktionenkorper, *Inv. Math.* 7 (1969) 1–32

[2] Borel A and Tits J, Groupes réductifs, *Publ. Math. Inst. Haut. Sci.* **27** (1965) 55–150

[3] Borel A and Tits J, Eléments unipotents et sousgroupes paraboliques de groupes réductifs, *Inv. Math.*12 (1971), 95–104

[4] Harder G, Minkowskische Reduktionstheorie über Funktionenkorper, *Inv. Math.* **7** (1969) 33–54

[5] Kazhdan D A and Margulis G A, Proof of Selberg's hypothesis, *Math. Sbornik* (N.S) **75** (117) (1968), 162–168 (Russian)

[6] Raghunathan M S, Discrete subgroups of Lie Groups, (New York, Heidelberg: Springer Verlag) (1972)

[7] Raghunathan M S, Discrete groups and *Q*-structures on semisimple groups in Discrete Subgroups of Lie Groups and Applications to Moduli, (Bombay: Oxford University Press) (1975) 225–321

[8] Tits J, Classification of algebraic semisimple groups, *Proc. Symp. Pure Math.* **9** (1969) 33–62

[9] Venkataramana T N, On super-rigidity and arithmeticity of lattices in semisimple groups, *Inv. Math.* **92** (1988) 255–306