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A Note on Quotients of Real Algebraic Groups by Arithmetic Subgroups

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Introduction

Let G be a connected semi-simple algebraic group defined over $\boldsymbol{Q}$. Let Γ be an arithmetic subgroup of G, i.e., a subgroup of G such that for some (and therefore any) faithful rational representation $\rho: G \to GL(N, C)$ defined over $\boldsymbol{Q}$, $\Gamma \cap \rho^{-1}(SL(N, \boldsymbol{Z}))$ is of finite index in both Γ and $\rho^{-1}(SL(N, \boldsymbol{Z}))$. Let $K \subset G$ be a maximal compact subgroup of $G_{\boldsymbol{R}}$, the set of real points of G. With this notation, we can state the main result of this note.

Theorem. *Let $\Gamma \subset G_{\boldsymbol{R}}$. There exists a smooth function $f: G_{\boldsymbol{R}}/\Gamma \to \boldsymbol{R}^+$ such that*

i) *$f^{-1}(0, r]$ is compact for all $r > 0$.*

ii) *There exists $r_0 > 0$ such that f has no critical points outside $f^{-1}(0, r_0]$ and*

iii) *f is invariant under the action of K on the left.*

If in addition Γ has no non-trivial elements of finite order, $K \backslash G_{\boldsymbol{R}}/\Gamma$ is a smooth manifold and f defines a smooth function f_1 on this manifold satisfying (i) *and* (ii) *with f replaced by f_1.*

Corollary 1. *$G_{\boldsymbol{R}}/\Gamma$ is homeomorphic to the interior of a smooth compact manifold with boundary; if Γ contains no element of finite order other than the identity, $K \backslash G_{\boldsymbol{R}}/\Gamma$ is homeomorphic to the interior of a compact smooth manifold with boundary.*

We now drop the hypothesis that $\Gamma \subset G_{\boldsymbol{R}}$.

Corollary 2. *Γ is finitely presentable.*

Corollary 3. *If M is any Γ-module finitely generated over $\boldsymbol{Z}$, $H^*(\Gamma, M)$ is finitely generated.*

Corollary 4. *The functor $M \rightsquigarrow H^*(\Gamma, M)$ on the category of Γ-modules commutes with the formation of inductive limits.*

We now deduce the corollaries from the main theorem.

Corollary 1 is a consequence of elementary facts from Morse theory. For $\Gamma \subset G_{\boldsymbol{R}}$ Corollary 2 follows from the fact that Γ is the quotient by

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a finitely generated central subgroup H of the fundamental group Γ' of $G_{\boldsymbol{R}}/\Gamma$ which is finitely presented since the space $G_{\boldsymbol{R}}/\Gamma$ is of the same homotopy type as a finite simplicial complex. The general case follows from the fact that $\Gamma/\Gamma \cap G_{\boldsymbol{R}}$ is finite. Corollary 1 implies that it $\Gamma \subset G_{\boldsymbol{R}}$ has no non-trivial elements of finite order the trivial Γ-module $\boldsymbol{Z}$ admits a free resolution

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to \boldsymbol{Z}$$

where each C_i is a finitely generated free-module over Γ; in fact, in this case, $K \backslash G_{\boldsymbol{R}}/\Gamma$ has the homotopy type of a finite complex L and its universal covering $\tilde{L}$ being of the homotopy type of $K \backslash G_{\boldsymbol{R}}$ is contractible. If we then take the induced triangulation of $\tilde{L}$, the associated chain-complex gives the resolution we are looking for. Corollaries 3 and 4 are then immediate consequences of this fact (when $\Gamma \subset G_{\boldsymbol{R}}$ and has no elements of finite order other than identity). The general case then follows from the Hochschild-Serre spectral sequence and the following fact due to SELBERG [4]. Any arithmetic group Γ admits a subgroup Γ' of finite index contained in $G_{\boldsymbol{R}}$ and such that no element of Γ' other than the identity has finite order.

§1. A Lemma on Root Systems

By a root system we mean as usual a set $\alpha_1, \ldots, \alpha_l$ of l linearly independent vectors in $\boldsymbol{R}^l$ (with the usual scalar product) such that (i) $\langle\alpha_i, \alpha_j\rangle \leqq 0$ for $i \neq j$ and (ii) $2\langle\alpha_i, \alpha_j\rangle/\langle\alpha_i, \alpha_i\rangle$ is an integer. (In the sequel we make no use of (ii).) Let λ_i be the unique vector in $\boldsymbol{R}^l$ such that $\langle\lambda_i, \alpha_j\rangle = \delta_{ij}$. We have then

Lemma 1.1. *If we set*

$$\lambda_k = \sum_{i \in I} a_i^{Ik} \alpha_i + \sum_{i \notin I} b_j^{Ik} \lambda_j$$

where I is any subset of $[1, \ldots, l]$, then b_j^{Ik}, a_i^{Ik} are all greater than or equal to zero.

Proof. Clearly, if $k \notin I$, $a_i^{Ik} = 0$ for all $i \in I$ and $b_j^{Ik} = \delta_{jk}$. Hence we can assume that $k \in I$. Let then λ_k' be the unique vector in the subspace generated by $\{\alpha_i\}_{i \in I}$ such that $\langle\lambda_k', \alpha_j\rangle = \delta_{kj}$ for all $j \in I$. We then assert that

$$\lambda_k' = \sum_{i \in I} m_i \alpha_i \quad \text{with} \quad m_i \geqq 0.$$

If not, in fact, let

$$\lambda_k' = \sum_{i \in I_1} m_i \alpha_i - \sum_{\gamma \in I - I_1} n_j \alpha_j$$

with $n_j>0$ for all $j\in I-I_1$ and $m_i\geqq 0$ for all $i\in I_1$. We then have,

$$0\leqq\sum n_j\langle\lambda'_k,\alpha_j\rangle=\sum_{ij} m_i n_j\langle\alpha_i,\alpha_j\rangle-\|\sum n_j\alpha_j\|^2$$

a contradiction, since $m_i n_j\geqq 0$ and $\langle\alpha_i,\alpha_j\rangle\leqq 0$ for $i\neq j$. Hence

$$\lambda'_k=\sum_{i\in I} m_i\alpha_i \quad \text{with} \quad m_i\geqq 0.$$

Now consider $\lambda_k-\lambda'_k$. Clearly $\langle\lambda_k-\lambda'_k,\alpha_i\rangle=0$ if $i\in I$ and for $i\notin I$,

$$\langle\lambda_k-\lambda'_k,\alpha_i\rangle=\langle-\lambda'_k,\alpha_i\rangle=-\langle\sum_{j\in I} m_j\alpha_j,\alpha_i\rangle\geqq 0.$$

Since if $i\notin I$, $i\neq j$ for any $j\in I$, in particular for $j=k$. It follows that

$$\lambda_k-\lambda'_k=\sum_{j\notin I} b_j\lambda_j \quad \text{where} \quad b_j=\langle\lambda_k-\lambda'_k,\alpha_j\rangle\geqq 0.$$

It follows that

$$\lambda_k=\sum_{i\in I} m_i\alpha_i+\sum_{i\notin I} b_j\lambda_j$$

where $m_i\geqq 0$, $b_j\geqq 0$. Hence the lemma.

§2. A Lemma on Siegel Domains

Let G be a connected semisimple algebraic group defined over $\boldsymbol{Q}$. Let T be a maximal $\boldsymbol{Q}$ split torus of G. For a subgroup H of G we denote by $H_{\boldsymbol{R}}$, the group $H\cap G_{\boldsymbol{R}}$ where $G_{\boldsymbol{R}}$ is the set of real points of G. Let A be the connected component of the identity of $T_{\boldsymbol{R}}$. Let $X(T)$ denote the lattice of rational characters on T. Then for $a\in A$ and $\chi\in X(T)$, $\chi(a)>0$. Let $\mathfrak{g}$ be the Lie algebra G and for $\chi\in X(T)$, let

$$\mathfrak{g}^\chi=\{v/v\in\mathfrak{g}, \operatorname{Ad} t(v)=\chi(t)\,v \text{ for all } t\in T\}$$

and let Φ be the system of roots of G with respect to T i.e. $\Phi=\{\chi\,|\,\chi\in X(T), \chi\neq 0, g^\chi\neq 0\}$. We introduce a lexicographic order on $X(T)$ and denote by Φ^+,Φ^- and Δ the system of positive negative and simple roots of G with respect to this order. Let

$$\mathfrak{n}=\coprod_{\alpha\in\Phi^+}\mathfrak{g}^\alpha;$$

then $\mathfrak{n}$ is a Lie subalgebra and the Lie subgroup N corresponding to it is a unipotent algebraic subgroup of G defined over $\boldsymbol{Q}$ (it is moreover maximal with respect to this property). Let $Z(T)$ be the centralizer of T; then $Z(T)$ is reductive and can be written in the form $M\cdot T$ where M is a reductive algebraic group defined and anisotropic over $\boldsymbol{Q}$. Moreover M normalizes N so that $MN=P^0$ is a subgroup of G. Finally let K be a maximal compact subgroup of $G_{\boldsymbol{R}}$ so chosen that its Lie algebra

$\mathfrak{k}$ is orthogonal to that of A with respect to the Killing form on $\mathfrak{g}$. (Lie algebras of Lie subgroups of G are identified with the corresponding Lie subalgebras.)

Definition 2.1. *For a relatively compact open subset $\eta \subset P_{\mathbf{R}}^0$ and a map $\underline{t}$: $\Delta \to \mathbf{R}^+$ (following* Borel [2]), *we call the set*

$$S_{\underline{t}_\eta} = K \cdot A_{\underline{t}} \cdot \eta$$

where $A_{\underline{t}} = \{a \mid a \in A,\ \alpha(a) \leqq \underline{t}(\alpha)$ for all $\alpha \in \Delta\}$ a Siegel-domain.

The following fundamental theorem is due to Borel [1] (see also [2]).

For a subgroup H of G we denote by $H_{\mathbf{Q}}$ its intersection with $G_{\mathbf{Q}}$ the set of $\mathbf{Q}$-rational points of G. Then we have

Theorem (Borel). (i) *The set of double coset classes $P_{\mathbf{Q}} \backslash G_{\mathbf{Q}} / \Gamma$ is finite.*

(ii) *For any relatively compact set η in $P_{\mathbf{R}}$ and $\underline{t}$: $\Delta \to \mathbf{R}^+$ and any pair q, $q' \in G_{\mathbf{Q}}$, the set*

$$\{\gamma \mid K A_{\underline{t}} \eta\, q\, \gamma \cap K A_{\underline{t}} \eta\, q' \neq \emptyset \text{ and } \gamma \in \Gamma\}$$

is finite.

(iii) *If $q_1, \ldots, q_m$ are representatives in $G_{\mathbf{Q}}$ for the double coset classes $P_{\mathbf{Q}} \backslash G_{\mathbf{Q}} / \Gamma$, then there exists a relatively compact open subset $\eta_1 \subset P_{\mathbf{R}}^0$ and a map $\underline{t}_1$: $\Delta \to \mathbf{R}^+$ such that if $\eta \subset P_{\mathbf{R}}^0$ contains η_1 and $\underline{t}$: $\Delta \to \mathbf{R}^+$ is such that $\underline{t}(\alpha) \geqq \underline{t}_1(\alpha)$ for all $\alpha \in \Delta$,*

$$\bigcup_{i=1}^{m} K A_{\underline{t}} \eta\, q_i \Gamma = G.$$

Now it is known that the Lie algebra $\mathfrak{g}$ of G admits a basis $e_1, \ldots, e_N$ such that

a) the structural constants of $\mathfrak{g}$ with respect to this basis are rational

b) each $\mathfrak{g}^{\alpha}, \alpha \in \Phi$, as also $\mathfrak{z}$ the Lie subalgebra corresponding to $Z(T)$ is spanned by those elements of the basis which belong to it

c) Γ is commensurable with the subgroup of G which under the adjoint action fixes the lattice $\mathscr{L}$ generated by $e_1, \ldots, e_N$ in $\mathfrak{g}$.

In the sequel when we speak of the entries of $\operatorname{Ad} g$ (or simply g) we mean the entries of the matrix of $\operatorname{Ad} g$ referred to the basis $e_1, \ldots, e_N$. We note then that the denominators of the entries of $\gamma \in \Gamma$ when reduced to the minimal form remain bounded.

For $\alpha \in \Delta$, we denote by Φ_α the set

$$\{\beta \mid \beta \in \Phi^+, \beta = \sum_{\theta \in \Delta} m_\beta(\theta)\, \theta,\ m_\beta(\alpha) > 0\}.$$

Then

$$\mathfrak{u}_\alpha = \coprod_{\beta \in \Phi_\alpha} \mathfrak{g}^\beta$$

is a Lie subalgebra of G. Its normalizer $\mathfrak{p}_\alpha$ in $\mathfrak{g}$ is easily seen to be

$$\mathfrak{n} \oplus \mathfrak{z} \oplus \coprod_{\beta \in \Phi^+ - \Phi_\alpha} \mathfrak{g}^{-\beta}.$$

We denote the corresponding Lie subgroup by P_α. Then P_α is a parabolic subgroup of G defined over $\boldsymbol{Q}$ and is maximal with respect to this property.

With this notation, we have the following crucial

Lemma 2.1. *Let $\eta \subset P_{\boldsymbol{R}}^0$ be any relatively compact open subset, $\underline{t}$: $\Delta \to \boldsymbol{R}^+$ any map and p be any integer. We fix a root $\alpha \in \Delta$. Then there exists $s > 0$ such that the following holds: let $\underline{t}'$: $\Delta \to \boldsymbol{R}^+$ be the map $\underline{t}'(\beta) = t(\beta)$ for $\beta \neq \alpha$ $t'(\alpha) = s$; let $g \in G_{\boldsymbol{Q}}$ be any element all of whose entries as well as those of g^{-1} when reduced to the simplest form have denominators which divide p; t then*

$$K A_{\underline{t}'} \eta g \cap K A_{\underline{t}} \eta \neq \emptyset$$

only if $g \in P_\alpha$. Moreover if $\varepsilon > 0$ is any given number, such that $\underline{t}(\alpha) - \varepsilon > 0$, then we can choose s to satisfy further the following: if $x = k \cdot a \cdot \theta$, $k \in K$, $a \in A_{\underline{t}'}$, $\theta \in \eta$ and $x g = k' \cdot a' \cdot \theta'$, $k' \in K$, $a' \in A_{\underline{t}}$ $\theta' \in \eta$, then $\alpha(a') < t(\alpha) - \varepsilon$.

Remark. The first part of the lemma is due to BOREL [2]. The proof below however is different from that of BOREL and is included because the same technique yields both results.

Proof of Lemma 2.1. We first remark that

$$\eta' = \{a \theta a^{-1} \mid a \in A_{\underline{t}}, \theta \in \eta\}$$

is again a relatively compact subset of $P_{\boldsymbol{R}}$ (for a proof, see [2]). Clearly, we have

$$K A_{\underline{t}} \eta \subset K \eta' A_t \quad \text{and} \quad K A_{\underline{t}'} \eta \subset K \eta' A_{\underline{t}'}.$$

Now $K\eta'$ being a relatively compact subset of G, there exist constants m, $M > 0$ such that for any $v \in V$ and $Y \in K\eta'$, we have, denoting $\operatorname{Ad} g$ (for $g \in G$) simply g,

(I) $$m_1 \|v\|^2 \leqq \|Yv\|^2 \leqq M_1 \|v\|^2$$

where $\| \ \|$ denotes the norm on g defined by the hermitian scalar product with respect to which $\{e_i\}_{1 \leqq i \leqq N}$ is an orthonormal basis. Suppose now that $X \in K A_{\underline{t}'} \eta$ and $X' \in K A_{\underline{t}} \eta$. Consider now any $e_i \in \mathfrak{g}^\beta$, $\beta \in \Phi_\alpha$. Now X may be written as Ya where $a \in A_{\underline{t}'}$ and $Y \in K\eta'$. In view of (*I*), then we have

$$\|X e_i\|^2 \leqq M_1 \|a e_i\|^2 \leqq M_1 \beta(a)^2 \|e_i\|^2.$$

Now since $\beta \in \Phi_\alpha$ we have

$$\beta = \alpha + \sum_{\theta \in \Delta} m_{\beta'}(\theta) \cdot \theta$$

where $m'_\beta(\theta) \geqq 0$. Now, since $a \in A_{\underline{t}'}$

$$\beta(a)^2 = \alpha(a)^2 \prod_{\theta \in \Delta} \theta(a)^{2 m'_\beta(\theta)} \leqq t'(\alpha)^2 \prod_{\theta \in \Delta} t'(\theta)^{2 m'_\beta(\theta)}.$$

It follows that if we have $t'(\theta) \leqq t(\theta)$ for all $\theta \in \Delta$, then

$$\| X e_i \|^2 \leqq M_1 \, t'(\alpha)^2 \prod_{\theta \in \Delta} t(\theta)^{2 m'_{\beta'}(\theta)}$$

so that if

$$M_2 = \sup_{\beta \in \Phi_\alpha} \{ \prod_{\theta \in \Delta} t(\theta)^{2 m_{\beta'}(\theta)} \}$$

we have for any $e_i \in \mathfrak{u}_\alpha$,

(II) $$\| X e_i \|^2 \leqq M_1 \, M_2 \, t'(\alpha)^2.$$

On the other hand let v be any vector in the lattice $\mathscr{L}$ such that the component of v in $\mathfrak{z} \oplus \mathfrak{n}^-$ is non-zero. Then v has a non-zero component either in $\mathfrak{z}$ or in one at least of the $\mathfrak{g}^{-\beta}$ for some $\beta \in \Phi^+$. We fix one such non-zero component and denote it by v_1. Clearly $v_1 \in \mathscr{L}$. Now for any $X' \in K A_{\underline{t}} \eta$, we have $X' = Y' a'$ where $Y' \in K \eta'$ and $a' \in A_{\underline{t}}$ so that

$$\| X' v \|^2 = \| Y' a' v \|^2 \geqq m \, \| a' v \|^2$$

in view of (*I*). Also since the $\{\mathfrak{g}^\alpha\}_{\alpha \in \Phi^-}$ and $\mathfrak{z}$ are mutually orthogonal subspaces each stable under A, it follows that

$$\| X' v \|^2 \geqq m_1 \, \| a' v_1 \|^2.$$

Now if $v_1 \in \mathfrak{g}^{-\beta}$ for some $\beta \in \Phi^+$, we have

$$\| a' v_1 \|^2 = \beta(a')^{-2} \, \| v_1 \|^2.$$

Now $a' \in A_{\underline{t}}$; on the other hand,

$$\beta = \sum_{\theta \in \Delta} m_\beta(\theta) \, \theta$$

with $m_\beta \geqq 0$ so that

$$\beta(a') = \prod_{\theta \in \Delta} \theta(a')^{m_\beta(\theta)} \leqq \prod_{\theta \in \Delta^-} t(\theta)^{m_\beta(\theta)}.$$

It follows that

$$\| X' v_1 \|^2 \geqq m_1 \prod_{\theta \in \Delta^-} t(\theta)^{-2 m_\beta(\theta)} \cdot \| v_1 \|^2.$$

On the other hand if $v_1 \in \mathfrak{z}$ then

$$\| a' v_1 \|^2 = \| v_1 \|^2$$

so that $\|X' v_1\|^2 \geqq m_1 \|v_1\|^2$. It follows that if we set

$$m_2 = \mathrm{Inf}\big(1, \operatorname*{Inf}_{\beta \in \Phi^+} \prod_{\theta \in \Delta^-} t(\theta)^{-2 m_\beta(\theta)}\big)$$

we have for any v *in* $\mathscr{L}$ *which has a non-zero component in* $\mathfrak{z} \oplus \mathfrak{n}^-$ and $X' \in KA_{\underline{t}}\eta$,

$$\text{(III)} \qquad \|X' v\|^2 \geqq m_1 m_2 \|v_1\|^2 \geqq m_1 m_2 .$$

(Since $v_1 \in \mathscr{L}$, $\|v_1\|^2 \geqq 1$.) Note that m_2, M_2 depend only on t. Also m_1 and M_1 are determined by η and $\underline{t}$ since η' is determined by them. Now let $c>0$ be any positive constant such that

$$c^2 < \frac{m_1 m_2}{M_1 M_2 p^2}$$

(p as in the statement of the lemma). We then claim then for any choice of $s<c$, the first assertion of the lemma holds. Suppose then that $X \in KA_{t'}\eta$ and $X' \in KA_t \eta$ and that $Xg = X'$ for some $g \in G_Q$ satisfying the conditions stated in the lemma. Now for $e_i \in \mathfrak{u}_\alpha$, we have in view of (II),

$$\|X e_i\|^2 \leqq M_1 M_2 t_1(\alpha)^2 < M_1 M_2 c^2.$$

On the other hand if $g^{-1} e_i$ is not contained in $\mathfrak{n}$, it has a non-zero component in $\mathfrak{z} \oplus \mathfrak{n}^-$ and since $p \cdot g^{-1} e_i \in \mathscr{L}$, we have in view of (III)

$$\|X e_i\|^2 = \|X' g^{-1} e_i\|^2 = \frac{1}{p^2} \|X' p g^{-1} e_i\|^2$$
$$\geqq \frac{m_1 m_2}{p^2} > M_1 M_2 c^2.$$

Thus we see that for $e_i \in \mathfrak{u}_\alpha$, $g^{-1} e_i \in \mathfrak{n}$. In other words $g^{-1}(\mathfrak{u}_\alpha) \subset \mathfrak{n}$. Taking orthogonal complements with respect to the *Killing form*, this means

$$g^{-1}(\mathfrak{p}_\alpha) \supset \mathfrak{n} \oplus \mathfrak{z}$$

i.e. $\mathfrak{p}_\alpha \supset g(\mathfrak{n} \oplus \mathfrak{z})$. On the other hand $\mathfrak{p}_\alpha \supset \mathfrak{n} \oplus \mathfrak{z}$ or going over to the corresponding groups, P_α contains both P and gPg^{-1}. Now gPg^{-1} is a minimal parabolic subgroup of G defined over Q so that $G/(gPg^{-1})$ is compact. It follows that $P_\alpha/(gPg^{-1})$ is compact and hence that gPg^{-1} is a parabolic subgroup of P_α defined over Q as well. But now P is a minimal parabolic subgroup defined over Q of P_α as well so that there exists $u \in P_\alpha$ such that $uPu^{-1} = gPg^{-1}$. But then $u^{-1}g$ normalizes P; but P is its own normalizer. Hence $u^{-1}g \in P \subset P_\alpha$. It follows that $g \in P_\alpha$. Thus the first assertion of the lemma is proved. We note further that since $g \in P_\alpha$, $g^{-1}(\mathfrak{u}_\alpha) = \mathfrak{u}_\alpha$.

To prove the second part of the lemma, we first observe that if $v \in \mathscr{L}$ is any vector such that it has a non-zero component v_1 in g^α, we have for $X' = k'a'\theta'$, $k' \in K$, $a' \in A_{\underline{t}}$, $\theta' \in \eta$, with $\alpha(a') \geqq \underline{t}(\alpha) - \varepsilon$

$$\text{(IV)} \qquad \|X' v\|^2 \geqq m_1 \|a' v\|^2 = \alpha(a')^2 \|v\|^2 \geqq (t(\alpha) - \varepsilon)^2 m_1 .$$

(Since $v_1 \in \mathscr{L}$, $\|v_1\| \geqq 1$.) Choose $c_1 > 0$ such that

$$c_1^2 = \operatorname{Inf}\left(\frac{m_1 m_2}{M_1 M_2 p^2}, \frac{(t(\alpha)-\varepsilon)^2 m}{M_1 M_2 p^2}\right).$$

Then for $s < c_1$, we see from the preceding that

$$K A_{\underline{t}'} \eta g \cap K A_{\underline{t}} \eta \neq \emptyset$$

only if $g \in P_\alpha$. Suppose now that $X \in K A_{\underline{t}'} \eta$ and $X' = Xg = k' \cdot a' \cdot \theta$ with $k' \in K$, $a' \in A_{\underline{t}}$, $\theta \in \eta$ and $\alpha(a') \geqq t(\alpha) - \varepsilon$. Then if for some $e_i \in \mathfrak{u}_\alpha$, $g^{-1} e_i$ has a non-zero component in $\mathfrak{g}^\alpha$, we have in view of (II) and (IV),

$$\frac{(t(\alpha)-\varepsilon)^2 m_1}{p^2} \leqq \|X' g^{-1} e_i\|^2 = \|X e_i\|^2 \leqq M_1 M_2 t'(\alpha)^2 \leqq M_1 M_2 c_1^2$$

since $s < c_1$ (once again note that $p g^{-1} e_i$ belongs to the lattice). It follows that $g^{-1}(\mathfrak{u}_\alpha)$ is orthogonal to $\mathfrak{g}^\alpha$. But we have seen that $g^{-1}(\mathfrak{u}_\alpha) = u_\alpha$ and $\mathfrak{g}^\alpha \subset \mathfrak{u}_\alpha$, a contradiction. It follows that for any $X \in K A_{\underline{t}'} \eta$ if $Xg \in K A_{\underline{t}} \eta$, $Xg = k' \cdot a' \theta'$, $k' \in K$, $a' \in A_{\underline{t}}$, $\theta' \in \eta$ then $\alpha(a') < t(\alpha) - \varepsilon$.

Remark. The denominator of the entries of $\gamma \in \Gamma$ when reduced to the simplest form remain bounded and so there is a common integer p divisible by all of them. The same remark applies to the set of matrices

$$\bigcup_{j=1}^{m} \bigcup_{i=1}^{m} q_i \Gamma q_j^{-1} \qquad \text{where } q_1, \dots, q_m \in G_{\boldsymbol{Q}}.$$

§3. Construction of the Function

An element $g \in G_{\boldsymbol{R}}$ can be written in the form $g = k_g a_g \theta_g$, $k \in K$, $a \in A$, $\theta \in P^0$; here a_g is unique and the map $g \rightsquigarrow a_g$ is a smooth function on $G_{\boldsymbol{R}}$ which we denote by H. We let $\alpha \in \Delta$ also stand for the smooth function $\alpha \circ H$ on $G_{\boldsymbol{R}}$ with values in $\boldsymbol{R}^+$. If

$$\lambda = \sum_{\alpha \in \Delta} m_\alpha \cdot \alpha$$

is any *real* linear combination of simple roots we let λ also stand for the (smooth) function

$$\prod_{\alpha \in \Delta} \alpha(g)^{m_\alpha}.$$

We fix a set of representatives $q_1, \dots, q_m \in G_{\boldsymbol{Q}}$ for the set of double coset classes with $1 = q_j$ for some j. Then we can find real constants r, t, ε with $r > t + \varepsilon > t > 0$ and a relatively compact open subset η of $P_{\boldsymbol{R}}$ such that the following conditions are satisfied (see Borel's Theorem and Lemma 2.1 (§2)): Let $\underline{r}: \Delta \to \boldsymbol{R}^+$, $\underline{r}': \Delta \to \boldsymbol{R}^+$ and for $\alpha \in \Delta$, ${}_\alpha\underline{r}': \Delta \to \boldsymbol{R}^+$

be the functions defined as follows: $\underline{r}(\theta)=r$ for all $\theta\in\Delta$, $\underline{r}'(\theta)=r+2\varepsilon=r'$ for $\theta\in\Delta$ and ${}_\alpha\underline{r}'(\theta)=r+2\varepsilon=r'$ for $\theta\neq\alpha$ while ${}_\alpha\underline{r}'(\alpha)=t+\varepsilon$; then we have

(i) $\bigcup_{i=1}^{m} K A_{\underline{r}}\,\eta\, q_i$ $\left(\text{hence } \bigcup_{i=1}^{m} K A_{\underline{r}'}\,\eta\, q_i\right)$ is a fundamental domain for Γ.

(ii) $KA_{{}_\alpha\underline{r}'}\,\eta\, q_i\gamma\cap KA_{\underline{r}'}\,\eta\, q_j\neq\emptyset$ for $\gamma\in\Gamma$ only if $q_i\gamma\, q_j^{-1}\in P_\alpha$.

(iii) If k, $k'\in K$, θ, $\theta'\in\eta$, $a\in A_{{}_\alpha\underline{r}'}$, $a'\in A_{\underline{r}'}$ and $k\,a\,\theta\, q_i\gamma=k'a'\theta' q_j$ for some $\gamma\in\Gamma$ then $\alpha(a')<r$.

(iv) For $i=1,\ldots,m$, $(q_i^{-1}\eta\, q_i)\,(\Gamma\cap q_i^{-1}P_{\boldsymbol{R}}^0\, q_i)=q_i^{-1}P_{\boldsymbol{R}}^0\, q_i$.

In view of (iv), (ii) is equivalent to

(ii′) $KA_{{}_\alpha\underline{r}'}\,P_{\boldsymbol{R}}^0\, q_i\gamma\cap KA_{\underline{r}'}\,P_{\boldsymbol{R}}^0\, q_j\neq\emptyset$ for $\gamma\in\Gamma$ only if $q_i\gamma\, q_j^{-1}\in P_\alpha$.

Remark 3.1. The choice of t and ε in the above is very wide. We could replace them once chosen by anything smaller. Thus, we might at any stage demand that they be smaller than any positive constant depending upon r.

That we can choose a η relatively compact in $P_{\boldsymbol{R}}$ and satisfying (iv) follows from the fact that

$$q_i^{-1}\,P_{\boldsymbol{R}}^0\, q_i/(q_i^{-1}\,P_{\boldsymbol{R}}^0\, q_i\cap\Gamma)$$

is compact. By enlarging it if necessary we can choose an $r>0$ such that (i) holds (this is the theorem of BOREL). Then we can take $\varepsilon>0$ any constant such that $r-\varepsilon>0$ and by appealing to Lemma 2.1 select a t so that (ii) and (iii) are satisfied. (Note that the entries of all the elements in

$$\bigcup_{\gamma\in\Gamma} q_i\,\gamma\, q_j^{-1}$$

when reduced to the simplest form have denominators which remain bounded and so we can find an integer p which is divisible by all of them.)

In the sequel we fix $q_1,\ldots,q_m$, η, r, t, ε chosen as above. Let $\varphi\colon \boldsymbol{R}^+\to I$ (the unit interval) be a smooth function such that

i) $\varphi(x)=1$ for $x\leqq t$,

ii) $\varphi(x)=0$ for $x\geqq t+\varepsilon$ and

iii) $\varphi'(x)\leqq 0$.

Also let ψ be the C^∞ function on $\boldsymbol{R}^+$ into the unit interval defined by

$$\psi(x)=1-\varphi(x) \qquad \text{for } x\leqq r,$$
$$\psi(x)=\varphi(x-r+t) \qquad \text{for } x\geqq r.$$

For a subset $I\subset\Delta$, we define $\Phi_I\colon G_{\boldsymbol{R}}\to I$ by

$$\Phi_I(g)=\prod_{\alpha\in I}\varphi\big(\alpha(g)\big)\prod_{\alpha\in\Delta-I}\psi\big(\alpha(g)\big).$$

Then the function $\Phi_I(g)$ is invariant under the action $P_{\mathbf{R}}^0$ on the right. Let

$$\Lambda = \sum_{\alpha \in \Delta} m_\alpha \lambda_\alpha$$

be any real linear combination of the fundamental weights (λ_α is defined by $\langle \lambda_\alpha, \beta \rangle = \delta_{\alpha\beta}$ for $\beta \in \Delta$), such that $m_\alpha > 0$ for all $\alpha \in \Delta$. Then for any subset $I \subset \Delta$, we have

$$\Lambda = \sum_{\alpha \in I} m_{I\alpha} \lambda_\alpha + \sum_{\alpha \in \Delta - I} n_{I\alpha} \alpha$$

where $m_{I\alpha} > 0$ (in fact $m_{I\alpha} \geqq m_\alpha$) and $n_{I\alpha} \geqq 0$. (See Lemma 1.1.) We set

$$\Lambda_I = \sum_{\alpha \in I} m_{I\alpha} \lambda_\alpha .$$

We denote again by Λ_I as before the function it defines on $G_{\mathbf{R}}$. For later use we state the above facts as

Lemma 3.1.
$$\Lambda_I = \sum_{\alpha \in I} m_{I\alpha} \lambda_\alpha \qquad \textit{with } m_{I\alpha} > 0 .$$

Also, for $\beta \in \Delta$ $\Lambda_{I \cup \beta} - \Lambda_I$ is a non-negative linear combination of the simple roots; moreover the coefficient $C_{I\beta\beta}$ of β in this expression is non-zero if $\beta \notin I$.

Proof. The first assertion is already proved. To prove the second assertion we need only consider the case $\beta \notin I$. Let $I' = I \cup \beta$ and

$$\Lambda = \sum_{\alpha \in I'} m_{I'\alpha} \lambda_\alpha + \sum_{\alpha \in \Delta - I'} n_{I'\alpha} \cdot \alpha$$

where $m_{I'\alpha}$, $n_{I'\alpha} \geqq 0$ (Lemma 1.1). On the other hand by Lemma 1.1, we have,

$$\lambda_\beta = \sum_{\alpha \in I} a_{\beta\alpha} \lambda_\alpha + \sum_{\alpha \in \Delta - I} b_{\beta\alpha} \alpha$$

where $a_{\beta\alpha}$, $b_{\beta\alpha} \geqq 0$. It follows that

$$\Lambda = \sum_{\alpha \in I} (m_{I'\alpha} + m_{I'\beta} \cdot a_{\beta\alpha}) \lambda_\alpha + \sum_{\alpha \in \Delta - I'} (n_{I'\alpha} + m_{I'\beta} b_{\beta\alpha}) \alpha + m_{I'\beta} b_{\beta\beta} \cdot \beta .$$

It follows that

$$\Lambda_{I'} - \Lambda_I = m_{I'\beta} \sum_{\alpha \in \Delta - I} b_{\beta\alpha} \cdot \alpha$$

and since $m_{I'\beta} > 0$, and $b_{\beta\alpha} \geqq 0$ for all $\alpha \in \Delta - I$, to conclude the proof of the lemma we need only show that $b_{\beta\beta} > 0$. To see this, we have, since $\beta \notin I$,

$$1 = \langle \lambda_\beta, \beta \rangle = \sum_{\alpha \in \Delta - I} b_{\beta\alpha} \langle \alpha, \beta \rangle ;$$

since $b_{\beta\alpha} \geqq 0$ and $\langle \alpha, \beta \rangle \leqq 0$ for $\alpha \neq \beta$, we must necessarily have $b_{\beta\beta} > 0$. Hence the lemma.

Consider now for each k, $1 \leqq k \leqq m$, the function f_k on G defined by the following series:

$$f_k(g) = \sum_{\gamma \in (\Gamma / q_k^{-1} P^0 q_k \cap \Gamma)} \sum_{I \subset \Delta} \Phi_I(g \gamma q_k^{-1}) \log \Lambda_I(g \gamma q_k^{-1}).$$

This requires some justification. Firstly the functions $g \mapsto \Phi_I(g)$ and $g \mapsto \Lambda_I(g)$ are invariant under the right action of $P^0_{\boldsymbol{R}}$. It follows that if γ', $\gamma'' \in \Gamma$ are two elements such that $\gamma'' = \gamma' \cdot \gamma$ with $\gamma \in q_k^{-1} P^0 q_k \cap \Gamma$ we have

$$\Phi_I(g \gamma'' q_k^{-1}) = \Phi_I(g \gamma'' q_k^{-1} q_k \gamma^{-1} q_k^{-1}) = \Phi_I(g \gamma' q_k^{-1});$$

similarly for Λ_I. Hence formally the series makes sense. Next we assert for fixed $g \in G_{\boldsymbol{R}}$, all but a finite number of terms of the series vanish identically in a neighborhood of g. To see this first notice that the support of Φ_I is contained in the interior of the domain $K A_{\underline{r}'} P^0_{\boldsymbol{R}}$; on the other hand we may assume that g is contained in some Siegel domain $S_{\underline{t}_1 \eta_1}$. Now the set

$$\{\gamma \mid \gamma \in \Gamma, S_{\underline{t}_1 \eta_1} \gamma q_k^{-1} \cap K A_{\underline{r}'} P^0_{\boldsymbol{R}} \neq \emptyset\}$$

is finite mod $P^0_{\boldsymbol{R}} \cap \Gamma$: in fact in view of (iv) in the choice of η, we see that each element of the above set is equivalent mod $P^0 \cap \Gamma$ to one of the elements in

$$\{\gamma \in \Gamma \mid S_{\underline{t}_1 \eta_1} \gamma \cap K A_{\underline{h}'} \eta q_k \neq \emptyset\}$$

(note that $q_j = 1$ for some j) which is finite according to the theorem of BOREL (§2). Hence $f_k(g)$ is a smooth C^∞ function on G; it is clearly invariant under Γ. The main result in more precise form then is

Theorem. *The function $\tilde{f} = -\sum f_k$ is a smooth function on $G_{\boldsymbol{R}}$ invariant under Γ. The function f on the quotient $G_{\boldsymbol{R}}/\Gamma$ defined by $\tilde{f}$ maps $G_{\boldsymbol{R}}/\Gamma$ properly into $[c, +\infty)$ for some $c \in \boldsymbol{R}$ and has no critical points outside a compact set.*

Proof. To prove that f is proper into $[c, +\infty)$ it is sufficient to show the following: (i) $\tilde{f}$ is bounded below and (ii) let $x_n \in K A_{\underline{r}} \cdot \eta q_k$ be any sequence such that if we write $x_n = k_n a_n \theta_n q_k$, $k_n \in K$, $a_n \in A_{\underline{r}}$, $\theta_n \in \eta$, $\alpha(a_n) \to 0$ for some $\alpha \in \Delta$, then $\tilde{f}(x_n) \to \infty$. We will show in fact that the f_j remain less than a fixed constant M, while $f_k(x_n)$ tends to $-\infty$

$$f_j(g) = \sum_{\gamma \in (\Gamma / q_j^{-1} P^0 q_j \cap \Gamma)} \sum_{I \subset \Delta} \Phi_I(g \gamma q_j^{-1}) \log \Lambda_I(g \gamma q_j^{-1}).$$

Now when $g \in K A_{\underline{r}} \eta q_k$, the above sum reduces to a finite sum

$$\sum_{\gamma \in S_{jk}} \sum_{I \subset \Delta} \Phi_I(g \gamma q_j^{-1}) \log \Lambda_I(g \gamma q_j^{-1})$$

where S_{jk} is the image in $\Gamma/(q_j^{-1} P^0 q_j \cap \Gamma)$ of the set

$$S_{jk} = \{\gamma \mid \gamma \in \Gamma, K A_{\underline{r}} \eta\, q_k \gamma\, q_j^{-1} \cap K A_{\underline{r}} P_{\mathbf{R}}^0 \neq \emptyset\}$$

that is of the set

$$\{\gamma \mid \gamma \in \Gamma, K A_{\underline{r}} \eta\, q_k \gamma \cap K A_{\underline{r}} P_{\mathbf{R}}^0 q_j = \emptyset\}.$$

Now since we have assumed η so chosen that

$$(q_j^{-1} \eta\, q_j)(\Gamma \cap q_j^{-1} P_{\mathbf{R}}^0 q_j) = q_j^{-1} P_{\mathbf{R}}^0 q_j,$$

we see that for any $\gamma \in S'_{jk}$ there exists $\gamma' \in \Gamma \cap q_j^{-1} P_{\mathbf{R}}^0 q_j$ such that

$$K A_{\underline{r}} \eta\, q_k \gamma \gamma' \cap K A_{\underline{r}} \eta\, q_j \neq \emptyset.$$

It follows from the theorem of Borel that S_{jk} is a finite set. Thus to show that $f(x_n)$ tends to infinity as n tends to infinity it suffices to show that each of the terms in the right hand side of (I) are bounded above and that when $j=k$, at least one of them tends to $-\infty$ (as n tends to ∞).

Now whenever $\Phi_I(g\,\gamma\, q_j^{-1}) \neq 0$, $g\,\gamma\, q_j^{-1} \in K A_{\underline{r'}} P_{\mathbf{R}}^0$ so that $\theta(g\,\gamma\, q_j^{-1}) < r'$ for all $\theta \in \Delta$. On the other hand Λ_I is a non-negative linear combination

$$\sum_{\theta \in \Delta} m_{I\theta}\, \theta$$

of the simple roots. It follows that

$$\Lambda_I(g\,\gamma\, q_j^{-1}) = \prod_{\theta \in \Delta} (g\,\gamma\, q_j^{-1})^{m_{I\theta}} \leqq \prod r'^{\,m_{I\theta}} = r'^{\,\sum m_{I\theta}}.$$

Since $0 \leqq \Phi_I(g\,\gamma\, q_j^{-1}) \leqq 1$, we see that

$$\Phi_I(g\,\gamma\, q_j^{-1}) \log \Lambda_I(g\,\gamma\, q_j^{-1}) \leqq \sum m_{I\theta} \log r',$$

a constant independent of $g \in K A_{\underline{r}}\, \eta\, q_k$. Thus we have only to show that for a suitable choice of $\gamma \in S_{kk}$ and $I \subset \Delta$,

$$\Phi_I(x_n \gamma\, q_k^{-1}) \log \Lambda_I(x_n \gamma\, q_k^{-1})$$

tends to $-\infty$ as n, tends to infinity. We take γ to be the identity coset in S_{kk}. Now $x_n = k_n a_n x_n q_k$ where $k_n \in K$, $a_n \in A_{\underline{r}}$, $u_n \in \eta$ and $\alpha(a_n)$ tends to zero as n tends to infinity. We choose for I the following subset:

$$\{\theta \mid \varphi(\theta(a_n)) > \tfrac{1}{2} \text{ for all large } n\}.$$

This subset is non-empty as it clearly contains $\alpha(\alpha(a_n) \to 0)$. With this choice of I and γ consider $\Phi_I(x_n \gamma\, q_k^{-1})$. We have in fact, (because of our chice of γ)

$$\begin{aligned}\Phi_I(x_n \gamma q_k^{-1}) &= \Phi_I(x_n q_k^{-1}) = \Phi_I(k_n a_n u_n)\\ &= \Phi_I(a_n) = \prod_{\theta\in I} \varphi((\theta)(a_n)) \prod_{\theta\in\Delta-I} \psi((\theta)(a_n)).\end{aligned}$$

Now, for $\theta\in I$ and n large $\varphi((\theta)(a_n))>\frac{1}{2}$ and for $\theta\notin I$, $\varphi(\theta(a_n))\leqq\frac{1}{2}$ so that $\psi(\theta(a_n))=1-\varphi(\theta(a_n))\geqq\frac{1}{2}$ (note that $\theta(a_n)\leqq r$ and in the range $x\leqq r$, $\varphi(x)+\psi(x)=1$). It follows that for n large,

$$\Phi_I(a_n)\geqq\tfrac{1}{2}\, l$$

where l is the number of simple roots. Thus $\Phi_I(x_n\gamma q_k^{-1})\geqq\frac{1}{2} l$ for all large n. Hence we have only to show that $\log \Lambda_I(x_n\gamma q_k^{-1})$ tends to $-\infty$ as n tends to infinity.

Now by Lemmas 1.1 and 3.1, we have

$$\Lambda_I = b_I^\alpha \lambda_\alpha + \sum_{\theta\in\Delta-\alpha} a_{I\theta}^\alpha \theta$$

where $a_{I\theta}^\alpha\geqq 0$ and $b_I^\alpha>0$ since $a\in I$ by our choice of I. Hence

$$\begin{aligned}\Lambda_I(x_n\gamma q_k^{-1}) &= \Lambda_I(x_n q_k^{-1}) = \Lambda_I(k_n a_n u_n)\\ &= \Lambda_I(a_n) = \lambda_\alpha(a_n)^{b_I^\alpha}\cdot \prod_{\theta\in\Delta-\alpha} \theta(a_n)^{a_{I\theta}^\alpha}\\ &\leqq \lambda_\alpha(a_n)^{b_I^\alpha}\cdot r'^{\,\Sigma a_{I\theta}^\alpha}\\ &= \lambda_\alpha(a_n)^{b_I^\alpha} C\end{aligned}$$

(note that $a_n\in A_{\underline{r}'}$) where C is a positive constant. Once again, by Lemma 1.1,

$$\lambda_\alpha = \sum_{\theta\in\Delta} C_{\alpha\theta}\,\theta$$

where $C_{\alpha\theta}\geqq 0$ and $C_{\alpha\alpha}>0$ so that

$$\lambda_\alpha(a_n)^{b_I^\alpha} = \prod_{\theta\in\Delta} \theta^{C_{\alpha\theta}}(a_n) \leqq \alpha^{C_{\alpha\alpha}}(a_n)\, r'^{\,\sum_{\theta\neq\alpha} C_{\alpha\theta}}.$$

thus

$$\Lambda_I(x_n\gamma q_k^{-1}) \leqq \alpha^{b_I^\alpha C_{\alpha\alpha}}(a_n)\cdot M$$

where $M>0$ is a constant independent of n.

Hence

$$\log \Lambda_I(x_n\gamma q_k^{-1}) \leqq \log M + b_I^\alpha C_{\alpha\alpha} \log \alpha(a_n)$$

where $b_I^\alpha\cdot C_{\alpha\alpha}>0$; but $\alpha(a_n)\to 0$ as $n\to\infty$ so that $\log \Lambda_I(x_n\gamma q_k^{-1})\to-\infty$ as $n\to\infty$.

Thus f is proper into the real line and bounded below.

§ 4. The Critical Points of f

In this section we complete the proof of the main theorem. Continuing with the notation of § 3 we need only show the following. Let $\mathfrak{a}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to A. Let $H_{\lambda_\alpha} \in \mathfrak{a}$ be the unique element of $\mathfrak{a}$ defined by

$$\beta(\exp t H_{\lambda_\alpha}) = e^{t\,\delta_{\alpha\beta}} \qquad \text{(exp is the exponential map)}$$

for $\beta \in \Delta$. Then we have fixing an $\alpha \in \Delta$, the following

Assertion. *For $x \in K A_{\alpha \underline{r}} \eta\, q_j$, we have for $1 \leqq k \leqq m$*

$$\left\{\frac{d}{ds} f_k(x\, q_j^{-1} \exp s H_{\lambda_\alpha} q_j)\right\}_{s=0} \geqq 0$$

and

$$\left\{\frac{d}{ds} f_j(x\, q_j^{-1} \exp s H_{\lambda_\alpha} q_j)\right\}_{s=0} > 0 .$$

(We recall that $A_{\alpha \underline{r}} = \{a \mid a \in A,\ \beta(a) \leqq r \text{ for } \beta \in \Delta \text{ and } \alpha(a) \leqq t\}$ where r, ε, t, η etc. are chosen as described in the beginning of § 3.)

(This assertion completes the proof of the theorem in view of the fact that it implies that $\bar{f}$ has no critical points in

$$\Omega_1 = \bigcup_{\alpha \in \Delta} \bigcup_{k=1}^{m} K A_{\alpha \underline{r}} \eta\, q_k$$

and that the complement of Ω_1 in the fundamental domain

$$\Omega = \bigcup_{k=1}^{m} K A_{\underline{r}} \eta\, q_k$$

is relatively compact in $G_{\boldsymbol{R}}$.)

Proof of the Assertion. We have, writing $x = k \cdot a \cdot \theta\, q_j$ where $k \in K$, $a \in A$, $\theta \in \eta$,

$$\begin{aligned} &\frac{d}{ds} \Phi_I(x\, q_j^{-1} \exp s H_{\lambda_\alpha} q_j \gamma\, q_k^{-1}) \\ &= \frac{d}{ds} \left\{ \prod_{\beta \in I} \varphi(\beta(k\, a\, \theta \exp s H_{\lambda_\alpha} q_j \gamma\, q_k^{-1}) \prod_{\beta \in \Delta - I} \psi(\beta(k\, a\, \theta \exp s H_{\lambda_\alpha} q_j \gamma\, q_k^{-1}) \right\}. \end{aligned}$$

Clearly the right hand side is non-zero at $s=0$ only if

$$k\, a\, \theta \exp s H_{\lambda_\alpha} q_j \gamma\, q_k^{-1} \in \bigcap_{\beta \in I} K A_{\beta \underline{r}} P_{\boldsymbol{R}}^0$$

for all small values of s; in particular only if

$$k\,a\,\theta\cdot q_j\gamma q_k^{-1}\in\bigcap_{\beta\in I} K A_{\beta r} P_{\mathbf{R}}^0.$$

On the other hand, $k\,a\,\theta\in KA_{\alpha r}\eta$. It follows from (ii′) that

$$q_j\gamma q_k^{-1}\in\bigcap_{\beta\in I\cup\alpha} P_\beta.$$

Writing then $q_j\gamma\, q_k^{-1}$ in the form $k_1' a_1' \theta_1'$, where $k_1'\in K$, $a_1'\in A$, $\theta_1'\in P_{\mathbf{R}}^0$, it is easily seen that k_1 and a_1 commute with $\exp s\, H_{\lambda_\alpha}$. It follows that

$$\begin{aligned} k\,a\,\theta\exp s\,H_{\lambda_\alpha} q_j\gamma q_k^{-1} &= k\,a\,\theta\, k_1'\, a_1' \exp s\, H_{\lambda_\alpha}\theta_1' \\ &= k_1\, a_1 \exp s\, H_{\lambda_\alpha}\,\theta_s' \end{aligned}$$

where $k_1\in K$, $a_1\in A$ are independent of s and $\theta_s'\in P_{\mathbf{R}}^0$. Now if

$$\frac{d}{ds}\,\Phi_I(x\,q_j^{-1}\exp s\,H_{\lambda_\alpha} q_j\gamma\, q_k^{-1})$$

is to be non-zero at $s=0$ we have necessarily, for all $\beta\in I$, for small s,

$$\beta(k_1\,a_1\exp s\,H_{\lambda_\alpha}\theta_s')=\beta(a_1)\cdot\beta(\exp s\,H_{\lambda_\alpha})\leqq t+\varepsilon$$

and for all $\beta\notin I$,

$$\beta(a_1)\,\beta(\exp s\,H_{\lambda_\alpha})\leqq r+\varepsilon.$$

Also in view of (iii) since we have assumed that $x\in KA_{\alpha r}\eta\, q_j$, we have necessarily $\alpha(a_1)<r$. Now for $\beta\neq\alpha$, we have,

$$\frac{d}{ds}\,\beta(k_1\,a_1\exp s\,H_{\lambda_\alpha}\theta_s')=\frac{d}{ds}\,\beta(a_1)=0.$$

Also,

$$\frac{d}{ds}\,\alpha(k_1\,a_1\exp s\,H_{\lambda_\alpha}\theta_{s'})=\frac{d}{ds}\,\alpha(a_1)\,e^s=\alpha(a_1)\,e^s.$$

It follows that we have if $\alpha\in I$

$$\begin{aligned} &\left\{\frac{d}{ds}\,\Phi_I(x\,q_j^{-1}\exp s\,H_{\lambda_\alpha} q_j\gamma\, q_k^{-1})\right\}_{s=0} \\ &\quad=\prod_{\beta\in I-\alpha}\varphi\big(\beta(x\gamma\,q_k^{-1})\big)\prod_{\beta\in\Delta-I}\psi\big(\beta(x\gamma\,q_k^{-1})\big)\,\varphi'\big(\alpha(x\gamma\,q_k^{-1})\big)\cdot\alpha(a_1) \end{aligned}$$

where a_1 is defined by

$$x\gamma q_k^{-1} = k_1 a_1 \theta, \qquad k_1 \in K,\ a_1 \in A,\ \theta_1 \in P_{\boldsymbol{R}}^0.$$

Similarly if $\alpha \notin I$, with the same notation

$$\left\{\frac{d}{ds} \Phi_I(x q_j^{-1} \exp s H_{\lambda_\alpha} q_j \gamma q_k^{-1})\right\}_{s=0}$$
$$= \prod_{\beta\in I} \varphi(\beta(x\gamma q_k^{-1})) \prod_{\beta\in\Delta-I-\alpha} \psi(\beta(x\gamma q_k^{-1}))\psi'(\alpha(x\gamma q_k^{-1}))\cdot\alpha(a_1).$$

Also as already noted, for any γ such that the above derivative is non-zero, we have necessarily

$$\alpha(x\gamma q_k^{-1}) < r$$

so that $\varphi'(\alpha(x\gamma q_k^{-1})) = -\psi'(\alpha(x\gamma q_k^{-1})$; since in the domain $\{y\,|\,y\in\boldsymbol{R}, y\leqq r\}$, we have

$$\varphi(y)+\psi(y)=1).$$

Thus we see that we have

$$\left\{\frac{d}{ds} \Phi_I(x q_j^{-1} \exp s H_{\lambda_\alpha} q_j \gamma q_k^{-1})\right\}_{s=0}$$
$$= \pm\alpha(a_1) \prod_{\beta\in I-\alpha} \varphi(\beta(x\gamma q_k^{-1})) \prod_{\beta\in\Delta-I-\alpha} \psi(\beta(x\gamma q_k^{-1}))\varphi'(\alpha(x\gamma q_k^{-1}))$$

according as $\alpha\in I$ or $\alpha\in\Delta-I$. Once again,

$$\varphi'(\alpha(x\gamma q_k^{-1}))=0$$

if $\alpha(x\gamma q_k^{-1})\notin[t, t+\varepsilon]$. Consider now, when $t\leqq\alpha(x\gamma q_j^{-1})\leqq t+\varepsilon$, the sum

$$\sum_I \left\{\frac{d}{ds} \Phi_I(x q_j^{-1} \exp s H_{\lambda_\alpha} q_j \gamma q_k^{-1})\right\}_{s=0} \log \Lambda_I(x\gamma q_k^{-1}).$$

We divide the set of subsets of Δ into two parts: those which do not contain α denoted $\mathscr{E}$ and the rest; then the sum can be written as

$$\alpha(a_1)\varphi'(\alpha(x\gamma q_k^{-1}))\sum_{I\in\mathscr{E}}\prod_{\beta\in I}\varphi(\beta(x\gamma q_k^{-1}))\prod_{\beta\in\Delta-I-\alpha}\psi(\beta(x\gamma q_k^{-1}))$$
$$\cdot\{\log\Lambda_{I\cup\alpha}(x\gamma q_k^{-1})-\log\Lambda_I(x\gamma q_k^{-1})\}.$$

Now by Lemma 3.1, we see that

$$\log\Lambda_{I\cup\alpha}(x\gamma q_k^{-1})-\log\Lambda_I(x\gamma q_k^{-1})=\log\Big(\prod_{\beta\in\Delta}\beta^{c_{I\alpha\beta}}(x\gamma q_k^{-1})\Big)$$

where $c_{I\alpha\beta} \geqq 0$ and $c_{I\alpha\alpha} > 0$. Hence

$$\begin{aligned}\log \Lambda_{I\cup\alpha}(x\gamma q_k^{-1}) - \log \Lambda_I(x\gamma q_k^{-1}) \\ &= \sum_{\beta\in\Delta} c_{I\alpha\beta} \log \beta(x\gamma q_k^{-1}) \\ &\leqq \sum_{\beta\in\Delta-\alpha} c_{I\alpha\beta} \log(r+2\varepsilon) + c_{I\alpha\alpha} \log(t+\varepsilon).\end{aligned}$$

Now, as remarked in the beginning of § 3 (Remark 3.1) we could have assumed t so small that

$$\sum_{\beta\in\Delta-\alpha} c_{I\alpha\beta} \log(r+2\varepsilon) + c_{I\alpha\alpha} \log(t+\varepsilon) < 0.$$

We assume that t and ε were chosen to satisfy this inequality for all $\alpha \in \Delta$ (in addition to our earlier assumptions). We then see that

$$\sum_I \sum_{\gamma\in\Gamma/(\Gamma\cap q_k^{-1} P^0 q_k)} \left\{\frac{d}{ds} \Phi_I(x q_j^{-1} \exp s H_{\lambda_\alpha} q_j \gamma q_k^{-1})\right\}_{s=0} \log \Lambda_I(x\gamma q_k^{-1})$$

is greater than or equal to zero for any $x \in K A_{\alpha_{\underline{r}}} \eta\, q_j$: in fact φ and ψ are non-negative functions while φ' is non-positive in the domain $\{y \mid y \in \boldsymbol{R},\ y < r\}$; also $\log \Lambda_{I\cup\alpha} - \log \Lambda_I$ is non-positive whenever,

$$t \leqq \alpha(x\gamma q_k^{-1}) \leqq t+\varepsilon.$$

Finally writing as before $x\gamma q_k^{-1} = k_1 a_1 \theta_1$, we see that

$$\begin{aligned}\frac{d}{ds} \log \Lambda_I(x q_j^{-1} \exp s H_{\lambda_\alpha} q_j \gamma q_k^{-1}) \\ &= \frac{d}{ds} \log\{\Lambda_I(a_1) \cdot \Lambda_I(\exp s H_{\lambda_\alpha})\} \\ &= \Lambda_I(H_{\lambda_\alpha})\end{aligned}$$

where $\Lambda_I(H_{\lambda_\alpha})$ denotes the evaluation of Λ_I considered as a linear form on $\mathfrak{a}$ on H_{λ_α}. Now by Lemmas 1.1 and 3.1 $\Lambda_I(H_{\lambda_\alpha}) \geqq 0$ and is >0 if $\alpha \in I$. We thus see that

$$\begin{aligned}&\left\{\frac{d}{ds} f_k(x q_j^{-1} \exp s H_{\lambda_\alpha} q_j)\right\}_{s=0} \\ &= \left\{\frac{d}{ds} \sum_{I,\gamma} \Phi_I(x q_j^{-1} \exp s H_{\lambda_\alpha} q_j \gamma q_k^{-1}) \log \Lambda_I(x q_j^{-1} \exp s H_{\lambda_\alpha} q_j \gamma q_k^{-1})\right\}_{s=0}\end{aligned}$$

is greater than or equal to zero. Moreover, since $\Phi_I(x \cdot q_j^{-1}) \neq 0$ for some $I \subset \Delta$ with $\alpha \in I$ (note that $\alpha(x q_j^{-1}) \leqq t$) we see that for $k=j$, the above is greater than zero. Thus the proof of the theorem is complete.

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