# Oded Schramm and the Schramm–Loewner evolution: In memoriam

Oded Schramm, the inventor of the stochastic Loewner evolution (SLE), died on 1 September 2008 in a tragic hiking accident in the Cascades Mountains. He was forty-six. He is survived by his wife and two children.

Schramm left behind an impressive legacy of research in probability and stochastic processes, statistical mechanics, and discrete and computational geometry. He brought highly original mathematical insights to the study of critical phenomena in lattice models in statistical physics. Schramm will be sorely missed by the mathematics and the statistical physics communities.

Oded Schramm was born in Jerusalem in December 1961. At the Hebrew University, he received a B Sc in mathematics and computer science, and an M Sc in mathematics under the direction of Gil Kalai. His Ph D (1990) was from Princeton – his supervisor: William Thurston. After a postdoctoral stint at the University of California, San Diego, Schramm moved to the Weizmann Institute in 1992. In 1999, he joined the Theory Group at Microsoft Research in Redmond.

Schramm was an avid hiker. He listed his interests – on his Microsoft webpage some years back – thus: 'Percolation, two-dimensional random systems, critical systems, SLE, conformal mappings, dynamical random systems, discrete and coarse geometry, mountains'.

## Conformal invariance of scaling limits and the SLE

The most celebrated of Schramm's contributions is his work on scaling limits of two-dimensional lattice models in statistical mechanics. The goal of statistical mechanics, roughly speaking, is to study the thermodynamic properties of large ensembles of interacting particles. The dimension of the 'configuration space' associated with this ensemble is the number of its degrees of freedom. In the large-number limit, the configuration space is infinite-dimensional. In mathematical terms, statistical mechanics is concerned with the construction of thermodynamically significant probability measures on these infinite dimensional spaces. These measures are constructed as limits of measures on suitably truncated finite-dimensional systems. Typically, there are only finitely many variables (dimensions) associated with a finite volume $V$ – and the limit involves taking $V \to \infty$ ('infinite volume') limit. The approximating measures depend on the interaction energy $E_V$ of the specific system, as well as on thermodynamic parameters (e.g. temperature $T$, magnetic field, etc.).

In two dimensions, the theory is particularly rich, and a variety of techniques – probabilistic, algebraic and heuristic – have been deployed to great effect, starting with Onsager's solution of the Ising model. *We shall use this familiar model as a guide to some of the technical constructs* that underpin the work of Schramm and his colleagues. To begin with, one considers a 'square lattice', namely the infinite grid $\Gamma_a$ in $\mathbb{R}^2$. (The parameter $a$ determines how fine a grid we get.) To each vertex $v$ of the grid, one associates a variable ('spin') $\sigma_v$ taking values 1 (denoting 'up' spin) or –1 (denoting 'down' spin). The 'configuration space' consists of all assignments of spin at every vertex. (One can view this space as an infinite cartesian product of $\{1, -1\}$ indexed by the vertices $v$.) Given a finite region $V$, the function $E_V$ is defined on the space $\prod_{v\in V}\{1, -1\}$ by the following formula:

$$E_V = \sum_{l\in V} J\sigma_{v_1(l)}\sigma_{v_2(l)},$$

where the sum runs over edges $l$ of the grid contained in $V$, and $v_i(l)$ are the vertices at either end of the edge $l$. Here $J$ is a real parameter that measures how nearby spins interact.

For instance, if $J < 0$, it is clear that configurations with nearby spins 'aligned' have lower energy, and hence higher probability. The relative weight of a configuration is

$$\exp(-E_V/T),$$

where $T$ is the temperature. The Ising model seeks to describe magnetic phenomena, e.g. when $J < 0$, the above is a model for ferromagnetism.

*The infinite-volume limit need not be unique, and it can vary non-analytically with the thermodynamic parameters*. Points of non-uniqueness or non-analyticity in the parameter space are called *phase transitions*. In the case of the Ising model, one wishes to study how the magnetization of the infinite-volume limit (the model for a real magnet) varies with temperature $T$. As $T$ is raised, magnetization weakens until, at a certain critical temperature $T_c$, it becomes zero. This is in fact a phase transition. One can say even more: in the Ising model, as well as in a large menagerie of models, correlations – which we shall not define rigorously (in the Ising model, 'two-point correlation' is the moment $\langle\sigma_v\sigma_{v'}\rangle$) – fall off with a power law (i.e. polynomially as opposed to, say, exponentially) in terms of distance between the vertices $v$ and $v'$. A consequence of this *slow decay of correlations at criticality* is the following. By taking a suitable limit (in the case of the square lattice $\Gamma_a$, for example, letting $a \to 0^+$, while simultaneously letting $T \to T_c$), one is able to:

- define thermodynamically meaningful measures that live on a *space of paths* in $\mathbb{R}^2$ (paths of a certain description that is dictated by the specific model), and
- rigorously deduce information about the infinite-volume limit.

This limit is called a *scaling limit*. Consider again the Ising model: given a configuration of spins, one can define a set of 'contours' – closed loops in the graph $\Gamma_a$ – that separate regions of 'up' spins from regions of 'down' spins. The aforementioned 'space of paths in $\mathbb{R}^2$ of a certain description' in this case is the set of limits of all these contours. Indeed, this sort of construction is possible for a large zoo of other lattice models at criticality.

An important postulate in the study of lattice models is the following.

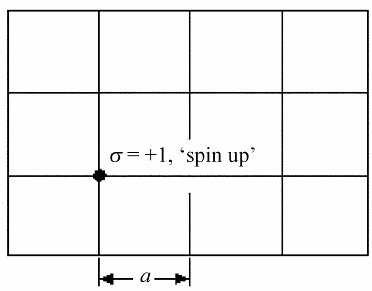


**Figure 1.** The square lattice $\Gamma_a$, lattice spacing $a$.

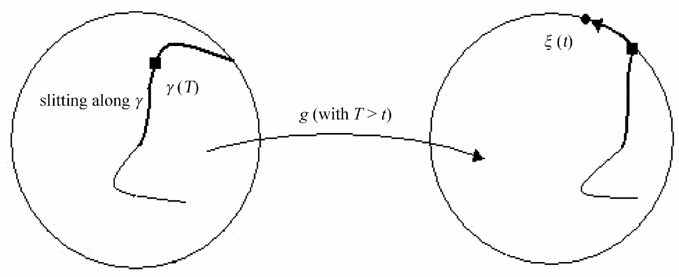


**Figure 2.** Loewner evolution showing a conformal transformation $g := g_T$, $T > t$.

**Postulate/conjecture.** *Scaling limits of two-dimensional models at criticality are conformally invariant.*

Given a lattice model and a region $\mathscr{D}$ in $\mathbb{R}^2$, let $\mu_{\mathscr{D}}$ denote the aforementioned measure on the relevant 'space of paths' in $\mathscr{D}$ associated to the scaling limit. The above conjecture says that if $\mathscr{D}$ and $\mathscr{D}'$ are two regions related by a conformal map $\Phi : \mathscr{D} \to \mathscr{D}'$, then the associated measures $\mu_{\mathscr{D}}$ and $\mu_{\mathscr{D}'}$ are related in terms of $\Phi$. *Much of the work by physicists in the past three decades assumes the existence of suitable scaling limits, as well as the conformal invariance of this limit.* Under this assumption, techniques from representation theory are powerful enough to enable detailed computations to be made.

Schramm's key contribution to all this originates from a study of the scaling limits of a lattice model known as the *loop-erased random walk* (LERW). The technology sketched out in the previous paragraphs applies to the LERW – any scaling limit of a LERW is supported on *loop-free curves* in the plane. The crux of Schramm's analysis of the conformal-invariance question (which forms a part of the influential article in *Israel J. Math.*, 2000, **118**, 221–288) lies in his discovery of a new family of manifestly conformally invariant stochastic processes which subsumes within it the scaling limit of (among other two-dimensional lattice models) the LERW. This stochastic process is the SLE. In the last couple of years, the SLE has come to be identified with Schramm by name; it is now called the *Schramm–Loewner evolution*.

Here is a brief but lucid explanation of this beautiful and important idea, excerpted from the Laudatio delivered by Michael Aizenman on the occasion of the presentation (in 2003) of the Henri Poincaré Prize to Schramm.

"Let $\gamma$: $[0, \infty) \to D$ be a curve forming a slit of the open unit disk $D$, growing from a boundary point into the interior. For each $t \geq 0$, let $g_t : D\backslash\gamma[0, t] \to D$, be the Riemann map of the partly slit disk onto $D$, which is made unique by imposing the conditions: $g_t(0) = 0$, and $g_t'(0) > 0$ is real and positive. As the slit is 'unzipped' the point of growth $\xi(t) = g_t(\gamma(t))$ is moving along the boundary $\partial D$. C. Loewner presented a procedure for solving the inverse problem: of recovering the curve $\gamma(t)$ from $\xi(t)$ and the rate information given by $g_t'(0)$ . . . Oded Schramm . . . applied Loewner's prescription in considering random curves with the property that the conditional distribution of a future segment of $\gamma$, conditioned on the past, depends in a conformally invariant way on the past trajectory. He noticed that such a 'conformal Markov property' requires $\xi(t)$ to evolve through independent increments, and thus to form a Brownian motion on the unit circle. . . This . . . led him to formulate the one-parameter family of random paths $SLE_\kappa$ . . ."

To recapitulate: Schramm unveiled a new family of stochastic processes described by the stochastic analogue of C. Loewner's differential equation (cf. *Math. Ann.*, 1923, **89**, 103–121) and whose solutions must satisfy a Markov-type condition that leads to conformal invariance. Schramm noticed that this 'conformal Markov property' requires that the trajectory of $\xi(t)$ describe a Brownian motion with some fixed diffusion rate $\kappa$. For a given choice of $\kappa$, the stochastic process driven by the Brownian motion $\{\xi(t)|t \geq 0\}$ is called $SLE_\kappa$. To complete the story of the LERW, the conformal invariance conjecture for the LERW was settled by viewing the trajectory of a LERW in the open unit disc $D$ as a slit that 'unzips' $D$, and proving that the scaling limit of the LERW is the stochastic process $SLE_2$.

There is more to this story. Briefly: the stochastic processes $SLE_\kappa$ are natural candidates for the scaling limits of many two-dimensional lattice models at criticality. In a series of deep papers, done mostly in collaboration with Greg Lawler and Wendelin Werner, Schramm established the conformal invariance of the scaling limits of several two-dimensional lattice models, and was able to rigorously establish the conjectured values of certain critical exponents for these models. A sampling:

1. They studied the scaling limit of percolation on a hexagonal lattice (conformal invariance had been proved earlier by Stanislav Smirnov) – specifically: a complete derivation of the critical exponents.
2. In 1982, Mandelbrot conjectured that the fractal dimension of the boundary of the trajectory of a Brownian path is 4/3. They proved this by establishing a connection with percolation.
3. They rigorously derived the values of the 'intersection exponents' for Brownian motion that had been conjectured by physicists Duplantier and Kwon (cf. *Phys. Rev. Lett.*, 1988, **61**, 2514–2517).

## A phenomenon and an inspiration

Oded Schramm was widely considered to be the most influential probabilist of this decade. The awards that he received include the Anna and Lajos Erdös Prize in Mathematics in 1996, the Salem Prize in 2001, the Clay Research Award in 2002, the Poincaré Prize in 2003 and the Pólya Prize in 2006. In 2008, he was elected as Member of the Royal Swedish Academy of Sciences. Wendelin Werner, whose Fields Medal of 2006 was the first awarded to a probabilist, recognized his debt to his collaborators Lawler and Schramm – and quoted, "the prize is also theirs, even if I am the only one under the age limit."

Amidst all this international attention, Schramm remained a calm and humble individual. Everyone who knew him speaks of the loss of not just a fine mathematician, but of a very kind and understanding person – always patient, and generous in sharing his insights.

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