# PARABOLIC BUNDLES AND REPRESENTATIONS OF THE FUNDAMENTAL GROUP

TOMÁS L. GÓMEZ AND T. R. RAMADAS

Abstract. Let $X$ be a smooth complex projective variety with Neron-Severi group isomorphic to $\mathbb{Z}$, and $D$ an irreducible divisor with normal crossing singularities. Assume $1 < r \leq 3$. We prove that if $\pi_1(X)$ doesn't have irreducible $PU(r)$ representations, then $\pi_1(X-D)$ doesn't have irreducible $U(r)$ representations. The proof uses the non-existence of certain stable parabolic bundles. We also obtain a similar result for $GL(2)$ when $D$ is smooth and $X$ is a complex surface.

## Introduction

Let $X$ be a smooth complex projective variety of dimension $n$. Let $D \subset X$ be an irreducible divisor with normal crossing singularities. One would like to relate the fundamental groups of $X$ and $X-D$. There is a short exact sequence

$$1 \to N \to \pi_1(X-D) \to \pi_1(X) \to 1.$$

Fix once and for all an element $\lambda$ of $\pi_1(X-D)$ going once around $D$. The kernel $N$ is generated by the set

$$\{a\lambda^\sigma a^{-1} : \sigma = \pm 1,\ a \in \pi_1(X-D)\}.$$

The most definitive results on the fundamental group of $X-D$ are due to Nori. If $X$ is a surface [No, prop. 3.27] implies that if $D^2 > 2r(D)$ (where $r(D)$ is number of nodes of $D$) then $N$ is a finitely generated abelian group, and its centralizer is a subgroup of finite index. In particular, his result implies Zariski's conjecture: if $X = \mathbb{P}^2$, then $\pi_1(\mathbb{P}^2-D)$ is abelian (since in this case $D^2 > 2r(D)$ is automatically satisfied).

We will make the following assumption on the Neron-Severi group $\mathrm{NS}(X)$ and the rank $r$ of the representations:

$$\mathrm{NS}(X) \cong \mathbb{Z}L \quad \text{and} \quad 1 < r \leq 3. \tag{1}$$

The main result of this paper is that if $\pi_1(X)$ has no irreducible $PU(r)$ representations, then $\pi_1(X-D)$ has no irreducible $U(r)$ representations.

The motivation for this kind of result is the following. If $\rho : \pi_1(X-D) \to U(r)$ is a representation such that $\rho(\lambda)$ is a multiple of the identity, then $\rho$ descends to give a representation $\overline{\rho} : \pi_1(X) \to PU(r)$. If $\overline{\rho}$ is reducible, then also $\rho$ is reducible. Of course this argument doesn't work if $\rho(\lambda)$ is not a

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multiple of the identity, but using parabolic bundles (and assuming (1)) we show that this cannot happen (corollary 1.4).

We also give a similar result for $GL(2)$ representations (theorem 2.2). Here we have to assume that $X$ is a surface and $D$ is smooth, since the correspondence between representations and parabolic Higgs bundles is only known under those conditions.

## 1. Unitary representations of fundamental groups

Let $(X, H)$ be a polarized smooth projective variety. All degrees and stability will be with respect to the polarization $H$, unless otherwise stated. Let $D \subset X$ be an irreducible divisor with normal crossing singularities. Let

$$\pi : \widetilde{D} \to D \subset X$$

be the composition of the normalization of $D$ with the inclusion in $X$. Let $\rho : \pi_1(X-D) \to U(r)$ be a representation of the fundamental group of the complement. This gives a local system on $X-D$. Let $(E, \nabla)$ be the Deligne extension ([D], [Ka]), i.e. $E$ is a holomorphic vector bundle on $X$ (with $\overline{\partial}_E = \nabla^{0,1}$ ) and $\nabla : E \to E \otimes \Omega_X\langle \log D\rangle$ is a holomorphic logarithmic connection. The restriction of the residue $\mathrm{Res}(\nabla) : E|_{\widetilde{D}} \to E|_{\widetilde{D}}$ to a point in the normalization $\widetilde{D}$ of $D$ is (up to conjugation) $\Gamma$, where $\exp(-2\pi i\Gamma)$ is the holonomy of $\rho$ on a small loop around $D$, and the eigenvalues of $\Gamma$ satisfy $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_r < 1$.

Recall that a parabolic structure on a vector bundle $E$ or rank $r$ is a filtration of $E|_{\widetilde{D}}$ by holomorphic subbundles

$$E|_{\widetilde{D}} = \pi^* E = F_1 \supsetneq F_2 \supsetneq \ldots \supsetneq F_p \supsetneq F_{p+1} = 0$$

with weights

$$0 \leq \mu_1 < \mu_2 < \ldots < \mu_p < 1$$

where $\mu_j$ are the different eigenvalues (without repetitions) of $\Gamma$. We say that the parabolic structure is trivial if $p = 1$ (i.e. all weights $\alpha_i$ are equal).

If $E$ is a torsion-free sheaf, then $E|_{\widetilde{D}}$ won't be locally free in general. In this case we only define the filtration in a Zariski open subset of $\widetilde{D}$, in which $E|_{\widetilde{D}}$ is locally free. This will be sufficient for our purposes (for a more general definition and for the notion of morphism between parabolic sheaves, see [M-Y]).

The parabolic degree of $E$ is defined as

$$\text{par-deg}\, E = \deg E + \Big(\sum_{i=1}^{r} \alpha_i\Big) \deg D.$$

We say that a parabolic bundle $E$ is parabolic stable (resp. semistable) if for any saturated torsion-free parabolic subsheaf $E'$ or rank $r'$,

$$\frac{\text{par-deg}\, E'}{r'} < \frac{\text{par-deg}\, E}{r}, \quad \text{(resp. } \leq).$$

And in general, all notions related to Mumford stability (slope, polystability, etc...) have a corresponding parabolic notion, changing the usual degree with the parabolic degree.

**Proposition 1.1.** *Let $\rho : \pi_1(X - D) \to U(r)$ be a representation. Then there is an associated polystable parabolic bundle $E$ of rank $r$. If the representation $\rho$ is irreducible then $E$ is parabolic stable.*

*The Chern characters of $E$ are given by* [E-V, cor (B.3)]

$$\mathrm{ch}_1(E) = -\operatorname{tr}(\Gamma)[D], \qquad \mathrm{ch}_2(E) = \frac{1}{2}\operatorname{tr}(\Gamma^2)[D]^2. \tag{2}$$

*Equivalently, we have the Chern classes*

$$c_1(E) = -\operatorname{tr}(\Gamma)[D], \qquad c_2(E) = \frac{1}{2}\Big((\operatorname{tr}\ \Gamma)^2 - \operatorname{tr}\ (\Gamma^2)\Big)[D]^2. \tag{3}$$

*This, in turn, says that the parabolic Chern classes of $E$ are zero.*

If $D$ is smooth and $X$ is a surface, this follows from [B2]. The same proof works here with the only variation that the parabolic structure is defined on the normalization $\widetilde{D}$ of $D$. The details of the proof are given in section 3.

**Lemma 1.2.** *With the same notation as above, assume that $D^2 > 0$ and that not all weights $\alpha_i$ are equal. Then $E$ is Mumford unstable.*

*Proof.* Using (2) we calculate the discriminant of $E$

$$\Delta = \Big(-\mathrm{ch}_2(E) + \frac{1}{2r}(\mathrm{ch}_1(E))^2\Big)H^{n-2} =$$
$$-\frac{1}{2}\Big(\sum \alpha_i^2 - \frac{1}{r}(\sum \alpha_i)^2\Big)D^2H^{n-2} < 0,$$

then, by Bogomolov inequality ([H-L, thm 7.3.1]), $E$ is Mumford unstable. □

From now on we will assume that $\mathrm{NS}(X) = \mathbb{Z}L$.

**Proposition 1.3.** *Assume (1). Let $E$ be a parabolic vector bundle with vanishing parabolic Chern classes and rank $r$. If $E$ is parabolic stable, then the parabolic structure is trivial (all the weights are equal).*

*Proof.* Assume that not all weights are equal. By lemma 1.2, $E$ is Mumford unstable. We will prove that $E$ is not parabolic stable by showing that at least one of the subsheaves of the Harder-Narasimhan filtration contradicts the parabolic stability of $E$. Since the parabolic degree of $E$ is zero, we have to prove that the parabolic degree of one of the subbundles is non-negative.

First we assume that the Harder-Narasimhan filtration of $E$ has only one term, i.e. there is a short exact sequence

$$0 \to E' \to E \to E'' \to 0 \tag{4}$$

such that both $E'$ and $E''$ are Mumford semistable torsion-free sheaves. The objective is to show that the parabolic degree of $E'$ is non-negative.

Let $s = (1/r)\sum \alpha_i$, and $\alpha_i^0 = \alpha_i - s$. By the formula for the first Chern class of $E$ we have $c_1(E) \equiv -rsdL$, where "$\equiv$" means numerical equivalence, and $dL \equiv [D]$. Let $a'L \equiv c_1(E')$, $r' = \mathrm{rk}(E')$, and analogously for $a''$ and $r''$.

Both $E'$ and $E''$ are Mumford semistable, then by Bogomolov inequality

$$0 \leq \Big( c_2(E') - \frac{r'-1}{2r'} a'^2 L^2 + c_2(E'') - \frac{r''-1}{2r''} a''^2 L^2 \Big) H^{n-2}.$$

Using $c_2(E) = c_2(E') + c_2(E'') + a'a''L^2$ and the formula for $c_2(E)$ in terms of $\Gamma$ this inequality becomes

$$0 \leq \Big( (r^2 - r)s^2 d^2 - \sum (\alpha_i^0)^2 d^2 - 2a'a'' - \frac{r'-1}{r'} a'^2 - \frac{r''-1}{r''} a''^2 \Big) \frac{L^2}{2} H^{n-2}$$

Using the formula for $c_1(E)$ in terms of $\Gamma$ we have $a'' = -rsd - a'$. Substituting this into the inequality and simplifying we obtain

$$0 \leq \Big( \frac{rr'}{r''} (\frac{a'}{r'} + sd)^2 - \sum (\alpha_i^0)^2 d^2 \Big) \frac{L^2}{2} H^{n-2}$$

Note that because $E' \subset E$ is a Harder-Narasimhan filtration we have

$$\frac{a'}{r'} + sd = \frac{a'}{r'} - \frac{a}{r} > 0,$$

and then we have

$$\frac{a'}{r'} + sd - \sqrt{\frac{r''}{r'r} \sum (\alpha_i^0)^2 d^2} \geq 0. \tag{5}$$

There is an induced parabolic structure on $E'$. There is a subset $I' \subset I = \{1, \ldots, r\}$ with $r'$ elements such that the weights of the parabolic structure on $E'$ induced by $E$ are $\alpha_i$ for $i \in I'$.

**Claim.** For any subset $I' \subset I$ of cardinality $r'$,

$$\frac{\sum_{i \in I'} \alpha_i^0}{r'} \geq -\sqrt{\frac{r''}{r'r} \sum_{i \in I} (\alpha_i^0)^2}. \tag{6}$$

Proving this is an easy calculus exercise. Use the method of Lagrange multipliers to minimize $\sum_{i \in I'} \alpha_i^0$ subject to the conditions $\sum_{i \in I} \alpha_i^0 = 0$ and $\sum_{i \in I} (\alpha_i^0)^2 = R$ where $R$ is some constant (note that we are minimizing a linear function restricted to a sphere).

Then combining inequalities (5) and (6) we get

$$0 \leq \frac{a' + \sum_{i \in I'} \alpha_i^0}{r'} + sd = \frac{a' + \sum_{i \in I'} \alpha_i}{r'},$$

but this is the parabolic degree of $E'$, so $E$ cannot be parabolic stable.

Now we assume that the Harder-Narasimhan filtration has length 2, i.e. we have

$$E_1 \subset E_2 \subset E$$

Then $\mathrm{rk}(E) = 3$ (recall that we assume $r \leq 3$), $\mathrm{rk}(E_i) = i$, and $E_2/E_1$ and $E/E_2$ are torsion free sheaves of rank one. Let $a_1 = c_1(E_1)$, $a_2 = c_1(E_2/E_1)$, $a_3 = c_1(E/E_1)$ (since $NS(X) \cong \mathbb{Z}L$, we can think of the first Chern class as an integer number). By the definition of the Harder-Narasimhan filtration we have

$$a_1 > a_2 > a_3 \tag{7}$$

Using the formula (3) for $c_1(E)$ we have $-\sum \alpha_i d = c_1(E) = \sum a_i$, where $d$ is the degree of $D$. Then there exist (rational) numbers $x$, $y$ such that

$$\alpha_1 d = -\frac{c_1(E)}{3} + x, \quad \alpha_2 d = -\frac{c_1(E)}{3} + y, \quad \alpha_3 d = -\frac{c_1(E)}{3} - x - y. \tag{8}$$

Using the formula for $\mathrm{ch}_2(E)$ we have

$$\sum \alpha_i^2 d^2 L^2 = 2\,\mathrm{ch}_2(E) = \Big( -l + \sum a_i^2 \Big) L^2, \tag{9}$$

where $lL^2 = 2c_2(E_2/E_1) + 2c_2(E/E_1)$, hence $l \geq 0$. Combining this with (8) we obtain

$$y = -\frac{x}{2} \pm \sqrt{\frac{3}{4}(x_m^2 - x^2)}, \text{ where } x_m^2 = \frac{2}{3}\Big( -l + \sum a_i^2 \Big) - \frac{2}{9}\Big( \sum a_i \Big)^2.$$

The number $y$ must be real, then $-x_m \leq x \leq x_m$.

Now we calculate the parabolic degrees of $E_1$ and $E_2$ as functions of $x$ (we take the polarization $(1/L^2)L$). Using $\text{par-deg}\, E_1 = a_1 + \alpha_1 d$ and $\text{par-deg}\, E_2 = a_1 + \alpha_1 d + a_2 + \alpha_2 d$ we obtain

$$\text{par-deg}\, E_1 = x + \frac{2a_1 - a_2 - a_3}{3} \tag{10}$$

$$\text{par-deg}\, E_2 = \frac{a_1 + a_2 - 2a_3}{3} + \frac{x}{2} \pm \sqrt{\frac{3}{4}(x_m^2 - x^2)} \tag{11}$$

Note that if we fix $x$, $\text{par-deg}\, E_1$ is fixed, but $\text{par-deg}\, E_2$ could take two values, hence the two signs in the formula. We want to show that for any value of $x$ (with $x^2 \leq x_m^2$), at least one of these is non-negative.

Equation (11) defines a conic with coordinates $(x, \text{par-deg}\, E_2)$ in the plane $\mathbb{R}^2$. This conic intersects the axis $\text{par-deg}\, E_2 = 0$ in the two points $x_-$ and $x_+$

$$x_\pm = \frac{-a_1 - a_2 + 2a_3}{6} \pm \frac{1}{2}\sqrt{(a_1 - a_2)^2 - 2l}$$

For $x = x_m$, $\text{par-deg}\, E_2 = \frac{1}{3}(a_1 + a_2 - 2a_3) + \frac{1}{2}x_m > 0$ (by equation (7)). Assume $\text{par-deg}\, E_2 < 0$. Then

$$x_- < x < x_+,$$

but if $x$ is in this interval, using $l \geq 0$ we have

$$x > x_- > \frac{-a_1 - a_2 + 2a_3}{6} - \frac{1}{2}\sqrt{(a_1 - a_2)^2} = \frac{-2a_1 + a_2 + a_3}{3}$$

(for the equality we used equation (7), to get the correct sign of the square root), and then equation (10) implies that $\text{par-deg}\, E_1 > 0$. Then either $\text{par-deg}\, E_1$ or $\text{par-deg}\, E_2$ is non-negative.

□

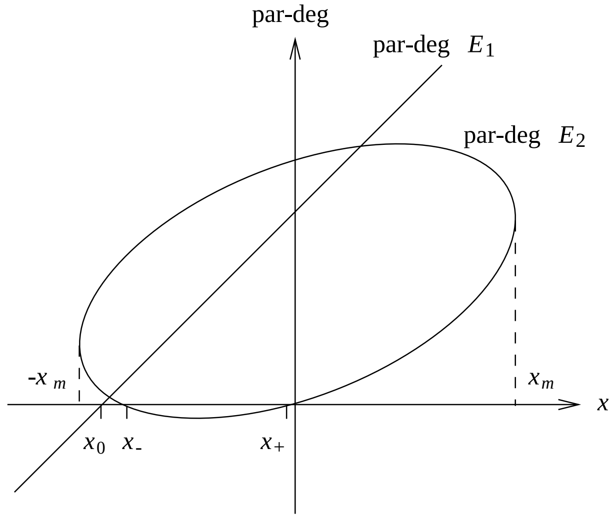


FIGURE 1. $x_0 = \frac{-2a_1+a_2+a_3}{3}$, $x_\mp = \frac{-a_1-a_2+2a_3}{6} \mp \frac{1}{2}\sqrt{(a_1-a_2)^2-2l}$

**Corollary 1.4.** *Assume (1). If $\rho : \pi_1(X{-}D) \to U(r)$ is an irreducible representation then $\rho(\lambda)$ is a multiple of the identity.*

*Proof.* Let $E$ be the corresponding stable parabolic bundle given by proposition 1.1. By proposition 1.3 the parabolic structure is trivial (i.e. all weights are equal), and then $\rho(\lambda)$ is a multiple of identity. $\square$

**Theorem 1.5.** *Assume (1). If $\pi_1(X)$ has no irreducible $PU(r)$ representations then $\pi_1(X{-}D)$ has no irreducible $U(r)$ representations.*

*Proof.* Let $\rho : \pi_1(X{-}D) \to U(r)$ be a representation, and $E$ the associated parabolic bundle. If $\rho$ is irreducible then, by corollary 1.4, $\rho(\lambda)$ is a multiple of identity. Then the representation $\rho : \pi_1(X{-}D) \to U(r)$ induces a representation $\overline{\rho} : \pi_1(X) \to PU(r)$. If $\rho$ is irreducible then $\overline{\rho}$ is also irreducible, and we get a contradiction. $\square$

## 2. Non-unitary representations of fundamental groups

One can also ask about non-unitary representations. In this section we will assume that $X$ is a surface and $D$ is smooth. We only deal with the case of $GL(2)$ representations.

Recall ([B3, déf 1.1]) that a parabolic Higgs bundle is a parabolic bundle $E$ together with a section $\phi \in H^0(\Omega^1\langle\log D\rangle \otimes ParEndE)$ with $\phi_\wedge\phi = 0$, where $ParEndE$ is the sheaf of parabolic endomorphisms of $E$. The residue $\mathrm{Res}_D\,\phi$ respects the filtration $F_\bullet$ of $E|_D$, and we require that its conjugation class is constant on each quotient $F_j/F_{j+1}$.

The following result follows from Biquard's theorem [B3, thm 11.4].

**Proposition 2.1.** *Let $D$ be a smooth divisor with $D^2 \neq 0$. Given an irreducible local system of rank $r$ on $X - D$, there is a rank $r$ stable parabolic Higgs bundle $(E, \phi)$ with vanishing parabolic Chern classes.*

*The eigenvalues of the residue of $\phi$ are zero. The slopes of the bundles $F_j/F_{j+1}$ are given by*

$$\frac{\deg(F_j/F_{j+1})}{\mathrm{rk}(F_j/F_{j+1})} = -\mu_j D^2, \tag{12}$$

*where $\mu_j$ are the parabolic weights of the parabolic Higgs bundle.*

*Proof.* Given an irreducible local system on $X - D$, consider its Deligne extension $(E, \nabla)$ ([Ka]). By construction, the real part of the eigenvalues of the residue of $\nabla$ are non-negative and less than 1. Define a parabolic structure on $E$, setting the parabolic weights equal to the real parts of the eigenvalues of the residue of $\nabla$. This integrable logarithmic connection satisfies the hypothesis of [B3, thm 11.4], and then we obtain a parabolic Higgs bundle. It is stable because of the irreducibility hypothesis and because it has a Hermite-Einstein metric.

The vanishing of the eigenvalues of the residue of $\phi$ is given in [B3, lemme 7.1]. By [B3, lemme 11.2], the eigenvalues of the residue of $\nabla$ are real, and by [B3, prop 10.1] they are equal to the parabolic weights of the parabolic Higgs bundle, and applying again [B3, lemme 11.2] we obtain the formula for the slopes.

□

We use this result to prove the following theorem.

**Theorem 2.2.** *Assume that $X$ is a projective surface, $D$ is a smooth divisor and $\mathrm{NS}(X) \cong \mathbb{Z}L$. If $\pi_1(X)$ has no irreducible $PGL(2)$ representations, then $\pi_1(X-D)$ has no irreducible $GL(2)$ representations.*

*Proof.* Assume that there is an irreducible representation $\rho : \pi_1(X-D) \to GL(2)$. Let $(E, \phi)$ be the stable parabolic Higgs bundle associated by proposition 2.1.

We will use the following fact:

$$H^0(\Omega^1\langle\log D\rangle) = H^0(\Omega^1). \tag{13}$$

The proof was shown to us by I. Biswas. Consider the sequence

$$0 \to \Omega^1 \to \Omega^1\langle\log D\rangle \to \mathcal{O}_D \to 0, \tag{14}$$

where the second map is the residue, and note that the image of the constant function 1 under the co-boundary map is represented by the Čech cocycle $dz_i/z_i - dz_j/z_j = d\log(z_i/z_j)$, where $z_i$ is a local equation of $D$ on an open set $U_i$, and this is (up to a non-zero constant) the Chern class of the line bundle $\mathcal{O}(D)$ (cf. [A, prop 12]).

If $M$ is a nontrivial line bundle on $X$ with $\deg M \leq 0$, we still have

$$H^0(M \otimes \Omega^1\langle\log D\rangle) = H^0(M \otimes \Omega^1). \tag{15}$$

To see this, note that $H^1(M(-D)) = 0$ by Kodaira vanishing theorem, hence $H^0(M|_D) = H^0(M) = 0$, and then use the sequence (14) tensored by $M$.

Consider first the case when the parabolic structure of $E$ is non-trivial (weights are different). We already know that the eigenvalues of $\operatorname{Res}_D \phi$ are zero. Since the residue of the Higgs field also preserves the parabolic filtration, it yields a map of line bundles $F_1/F_2 \to F_2$. But by (12) we have

$$\deg(F_1/F_2) = -\mu_1 D^2 > -\mu_2 D^2 = \deg(F_2)$$

so this map has to be zero. In other words, $\operatorname{Res}_D \phi = 0$, and then $(E, \phi)$ defines a Higgs pair on $X$. Using Bogomolov inequality for Higgs bundles (cf. [Si, prop 3.4 and thm 1]) we see that $(E, \phi)$ is not stable as a Higgs pair. Then there is a short exact sequence

$$0 \to M \to E \to M' \otimes I_Z \to 0$$

where $I_Z$ is the ideal sheaf of a zero-dimensional subscheme and $M$ and $M'$ are line bundles with $\deg M \geq \deg M'$. A calculation similar to the proof of proposition 1.3 (for filtrations of length 1), using formulae (3) and the fact that $c_2(I_Z) = l(Z) \geq 0$, shows that $E$ is not parabolic Higgs stable (since $M$ is $\phi$-invariant), contradicting the hypothesis and then finishing the proof in this case.

The case when the parabolic structure is trivial is more subtle (it is to cover this case that we need the assumption on the tangent bundle). Since all the weights are equal, $E$ is Mumford stable iff it is parabolic stable. But if $E$ is parabolic stable then there is a corresponding $U(2)$ irreducible representation $\rho_{U(2)}$ of $\pi_1(X-D)$, and since the weights are equal, $\rho_{U(2)}(\lambda)$ is a multiple of the identity. Then there is an induced irreducible $PU(2)$ representation of $\pi_1(X)$, in contradiction with the hypothesis.

Then $E$ is not Mumford stable, and thus there is a sequence as before

$$0 \to M \to E \to M' \otimes I_Z \to 0$$

where $M$ and $M'$ are line bundles, $I_Z$ is the ideal sheaf of a zero-dimensional subscheme $Z$, and $\deg M \geq \deg M'$, but now $M$ might not be $\phi$-invariant, so $M$ doesn't contradict parabolic Higgs stability. The Higgs field $\phi$ defines a map $\phi' : M \to M' \otimes \Omega^1\langle \log D\rangle$. Regard this as a section of $H^0(M^{-1} \otimes M' \otimes \Omega^1\langle \log D\rangle)$. By (15), this is a section of $H^0(M^{-1} \otimes M' \otimes \Omega^1)$. In other words, $\operatorname{Res} \phi' = 0$. On the other hand, $\phi'$ vanishes on $Z$, since it factors through $M' \otimes I_Z \otimes \Omega^1\langle \log D\rangle$. Putting both facts together, $\phi'$ can be seen as a section of $H^0(M^{-1} \otimes M' \otimes I_Z \otimes \Omega^1)$. If the tangent bundle is globally generated we see that this group is zero (and thus $M$, being invariant under $\phi$, contradicts stability of the parabolic Higgs pair) unless $M = M'$ and $Z$ is empty.

Suppose therefore that we have a non-trivial extension

$$0 \to M \to E \to M \to 0 \tag{16}$$

We now produce a "model" of the extension (16), which is adapted to the divisor $D$. Since the tangent bundle is generated by global sections, the anticanonical line bundle $K^{-1}$ has non-negative degree. Then, since the Neron-Severi group has rank one, $K^{-1} \otimes \mathcal{O}(D)$ is ample, and by Kodaira vanishing theorem $H^1(\mathcal{O}(D)) = 0$.

Let $\mathcal{N}_D = \mathcal{O}(D)|_D$ denote the normal bundle of $D$. From the sequence

$$0 \to \mathcal{O} \to \mathcal{O}(D) \to \mathcal{N}_D \to 0 \tag{17}$$

it follows that the map $H^0(\mathcal{N}_D) \to H^1(\mathcal{O})$ is onto. Note that this map is obtained from the natural multiplication map $\mathrm{Hom}(\mathcal{O}, \mathcal{N}_D) \times \mathrm{Ext}^1(\mathcal{N}_D, \mathcal{O}) \to \mathrm{Ext}^1(\mathcal{O}, \mathcal{O})$, using the extension class of (17). Then, given a section $\sigma$ of $\mathcal{N}_D$, the corresponding extension $E_\sigma$ is given as the kernel in the sequence

$$0 \to E_\sigma \to \mathcal{O}(D) \oplus \mathcal{O} \to \mathcal{N}_D \to 0 \tag{18}$$

the map on the right being $(u, v) \mapsto u|_D - \sigma v|_D$. From (16) we have

$$0 \to \mathcal{O} \to E \otimes M^{-1} \to \mathcal{O} \to 0,$$

and since $H^0(\mathcal{N}_D) \to H^1(\mathcal{O})$ is surjective, there is a section $\sigma$ such that $E = E_\sigma \otimes M$. From now on we will use this “model” of $E$.

Note the obvious inclusion $M \oplus M(-D) \hookrightarrow E$. Consider the composition

$$M \oplus M(-D) \hookrightarrow E \xrightarrow{\phi} E \otimes \Omega^1\langle \log D\rangle \hookrightarrow \{M(D) \oplus M\} \otimes \Omega^1\langle \log D\rangle,$$

where the last map comes from the left map on (18). We denote this composition by $\hat{\phi}$. One can now represent this map as a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{19}$$

where $a, d$ are sections of $\mathcal{O}(D) \otimes \Omega^1\langle \log D\rangle$, $c$ is a section of $\Omega^1\langle \log D\rangle$, and $b$ of $\mathcal{O}(2D) \otimes \Omega^1\langle \log D\rangle$. We will use this representation to show that the residue of the Higgs field is zero (note that $\operatorname{Res}\phi = \operatorname{Res}\hat{\phi}$).

By (13), $c \in H^0(\Omega^1)$. Consider a local holomorphic section $(f, 0)$ of $M \oplus M(-D)$. The residue of $\hat{\phi}$ sends $(f|_D, 0)$ to

$$(\operatorname{Res}(a) f|_D, \operatorname{Res}(c) f|_D) = (\operatorname{Res}(a) f|_D, 0),$$

hence $\operatorname{Res}(a)$ is an eigenvalue of $\operatorname{Res}(\hat{\phi})$, and hence of $\operatorname{Res}(\phi)$, but those eigenvalues are zero, then $\operatorname{Res}(a) = 0$ and $a \in H^0(\mathcal{O}(D) \otimes \Omega^1)$.

Since the eigenvalues of $\hat{\phi}$ are zero, $0 = \operatorname{tr}\operatorname{Res}(\hat{\phi}) = \operatorname{Res}(a) + \operatorname{Res}(d) = \operatorname{Res}(d)$, and then $d \in H^0(\mathcal{O}(D) \otimes \Omega^1)$.

Finally we will impose the condition that the map $\hat{\phi}$ comes from the Higgs field on $E$, i.e. $\hat{\phi}$ extends to $E$. The exact sequence (18) shows (after tensoring with $M$) that a local section of $E$ is the same thing as a local section $(f', g')$ of $M(D) \oplus M$, with the condition that $f'|_D - \sigma g'|_D = 0$. We will write local sections of $M(D) \oplus M$ as $((1/z)f, (1/z)g)$, where $f$ and $g$ are respectively holomorphic local sections of $M$ and $M(-D)$ and $z$ is a local equation of $D$. Take a local holomorphic section $g$ of $M(-D)$ that doesn't vanish identically on $D$, and let $f$ be a local holomorphic section of $M$ such that $f|_D - \sigma g|_D = 0$, so that $((1/z)f, (1/z)g)$ is in fact a local section of $E$. The matrix (19) acting on this gives

$$(\frac{1}{z}af + \frac{1}{z}bg, \frac{1}{z}cf + \frac{1}{z}dg).$$

This should be a local section of $\{M(D) \oplus M\} \otimes \Omega^1\langle \log D\rangle$. We have seen that $a$ is in fact a section of $\mathcal{O}(D) \otimes \Omega^1$, then $(1/z)af$ is a local section of $M(D) \otimes \Omega^1\langle \log D\rangle$. We need then $(1/z)bg$ to be also a local section of $M(D) \otimes \Omega^1\langle \log D\rangle$. Recall that $b$ is a section of $\mathcal{O}(2D) \otimes \Omega^1\langle \log D\rangle$ and $g$ a local section of $M(-D)$ (that doesn't vanish identically on $D$). Then for $(1/z)bg$ to be a local section of $M(D) \otimes \Omega^1\langle \log D\rangle$ we need the residue of $b$ to be zero, i.e. $b \in H^0(\mathcal{O}(2D) \otimes \Omega^1)$. Then we obtain that the residue of (19) (and hence of $\phi$) is zero.

Finally, if the residue of $\phi$ is zero, since we have assumed that the weights of the parabolic structure of $E$ are equal, we obtain that $\rho(\lambda)$ is a multiple of the identity, then there is an induced $PGL(2)$ representation, and since this is reducible by hypothesis, we conclude that $\rho$ is also reducible.

□

## 3. Proof of proposition 1.1

We will follow closely [Ko, (V.8.5)] and [B2, (4.1)]. This proof works for $X$ a Kähler manifold, not only for projective varieties. We will denote by $\omega$ the Kähler form.

The representation $\rho$ gives a local system on $X - D$. Let $(E, \nabla)$ be the Deligne extension. Let $p$ be a point of $D$. There are local coordinates $z_1, \dots, z_n$ on $X$ and a local holomorphic trivialization of $E$, where $D$ is defined by $t = z_1 \cdots z_l$, and the Deligne extension is ([Ka])

$$\nabla = d + \sum_{j=1}^{l} \frac{dz_j}{z_j} \Gamma_j$$

where $\Gamma_j = \mathrm{diag}(\alpha_1^{(j)}, \dots, \alpha_r^{(j)})$, and $\alpha_1^{(j)}, \dots, \alpha_r^{(j)}$ is a permutation of $\alpha_1, \dots, \alpha_r$.

Fix once and for all a point $x_0 \in X$ and a Hermitian metric on the fiber $E|_{x_0}$. Since the connection is flat and the holonomy is unitary, using this fixed metric and parallel transport we can define a Hermitian metric on $E$ (degenerate on $D$). In the previous local coordinates

$$h = \begin{pmatrix} |z_1|^{\alpha_1^{(1)}} \dots |z_l|^{\alpha_1^{(l)}} & & 0 \\ & \ddots & \\ 0 & & |z_1|^{\alpha_r^{(1)}} \dots |z_l|^{\alpha_r^{(l)}} \end{pmatrix} h_0$$

where $h_0$ is a fixed constant matrix (depending on the metric chosen on $E|_{x_0}$). This is an "adapted metric", in the sense of [B1, déf 2.3]. By direct computation we can check that $\nabla$ is the Chern connection associated to the metric $h$.

Now we define the parabolic structure ([B2, thm 2.1]). Since the local equation for $D$ is $t = z_1 \cdots z_l = 0$, where $l$ can be different from 1 in general, we need to go to the normalization $\widetilde{D}$ of $D$ to separate the different branches. Let $\tilde{p} \in \widetilde{D}$ be a point mapping to $p \in D$, such that a small neighborhood $\widetilde{V}$ of $\widetilde{D}$ maps to the subset of $D$ defined by the equation $z_k = 0$, with $1 \leq k \leq l$. We define a filtration of $E|_{\widetilde{V}} = \pi^*_{\widetilde{V}} E$

$$E|_{\widetilde{V}} = F_1|_{\widetilde{V}} \supset F_2|_{\widetilde{V}} \supset \ldots \supset F_r|_{\widetilde{V}} \supset 0$$

by the property that if $s(z)$ is a local section of $E$, then

$$s|_{\widetilde{V}} \in F_i|_{\widetilde{V}} - F_{i+1}|_{\widetilde{V}} \quad \Leftrightarrow \quad \|s(z)\|_h \sim |z_k|^{\alpha_i}$$

i.e. if $\|s(z)\|_h = |z_k|^{\alpha_i} f$, with $f$ a continuous function that doesn't vanish identically on the branch $z_k = 0$. This local construction defines a filtration on $E|_{\widetilde{D}}$ and hence a parabolic structure on $E$.

Now we will prove that if the representation $\rho$ is irreducible, this parabolic structure is stable (it will then follow that if $\rho$ is not irreducible, then the parabolic structure will still be polystable: decompose $\rho$ in irreducible representations, and then the parabolic bundle will be a direct sum of stable parabolic bundles, i.e. a polystable parabolic bundle).

Let $E'$ be a saturated coherent torsion-free subsheaf of $E$ of rank $r'$. There is a naturally induced parabolic structure on $E'$ whose weights are a subset of the weights of $E$ (to check stability it is enough to look at parabolic subsheaves with this parabolic structure, since they have maximal parabolic degree). It is defined by considering the Hermitian metric $h'$ on $E'$ obtained by restriction of the metric $h$ on $E$. The metric $h'$ is degenerate on the points where $E'$ is not locally free or where $i : E' \to E$ is not an injection of vector bundles. Let $S$ be the union of all such points. We define a parabolic structure as before, but only on $X - S$ (i.e. the filtration of $\pi^* E' = E'|_{\widetilde{D}}$ is only defined on $\widetilde{D} - \pi^{-1}(S)$. This is enough for our purpose: to calculate the parabolic degree). We denote by $\sum_{i \in I'} \alpha_i$ the sum of the parabolic weights of $E'$ (with repetitions), where $I' \subset \{1, \ldots, r\}$.

**Lemma 3.1.**

$$\frac{\sqrt{-1}}{2\pi} \int_{X-(D\cup W)} \operatorname{tr} \overline{\partial}({h'}^{-1} \partial h')_{\wedge} \omega^{n-1} = \text{par-deg}\, E' \tag{20}$$

*Proof.* Since $h'$ (and also the associated Chern connection) is singular on $D$, we cannot directly apply the Chern-Weil formula. We will modify the metric to make it smooth, and this will produce the second summand.

First note that

$$\operatorname{tr} \overline{\partial}({h'}^{-1} \partial h') = \overline{\partial} \partial \log(\det h'). \tag{21}$$

This can be proved by first showing that both sides are invariant under change of holomorphic trivialization, and then computing in a holomorphic trivialization where $h = \mathrm{I} + O(|z|^2)$ ([W, III lemma 2.3]). Note that

$\det h'$ is a Hermitian metric on $\det E'$, singular on $D$ (in fact $\det h' \sim |z_1 \cdots z_l|^{2(\sum_{i\in I'}\alpha_i)}$).

Following [B2, (4.1)], choose a section $t \in H^0(\mathcal{O}_X(D))$ with a zero of order one along $D$. Take any Hermitian metric on the line bundle $\mathcal{O}_X(D)$, and then $\|t\|$ is a smooth function on $X$, vanishing on $D$. Near $D$, $t = z_1 \cdots z_l$ (using an appropriate trivialization of $\mathcal{O}_X(D)$), and then

$$\frac{\det h'}{\|t\|^{2(\sum_{i\in I'}\alpha_i)}}$$

defines a smooth metric on $(\det E')|_{X-S}$, and we can apply the Chern-Weil formula as in [Ko, (V.8.5) formula (**)]. The integral (20) can be written, using (21)

$$\begin{aligned}&\frac{\sqrt{-1}}{2\pi}\int_{X-(D\cup S)}\Big(\overline{\partial}\partial \log \frac{\det h'}{\|t\|^{2(\sum_{i\in I'}\alpha_i)}} + (\sum_{i\in I'}\alpha_i)\overline{\partial}\partial\log\|t\|^2\Big)_\wedge\omega^{n-1}\\ &= \deg E' + (\sum_{i\in I'}\alpha_i)\deg D\end{aligned}$$

where the second integral is given by the Poincaré-Lelong formula. This finishes the proof of the lemma.

□

If we take $E' = E$, then we obtain par-deg $E = 0$, since the connection is flat on $X-D$. Using this lemma, to prove that $E$ is stable we have to show that for any proper saturated torsion-free subsheaf $E'$ of $E$,

$$\frac{\sqrt{-1}}{2\pi}\int_{X-(D\cup S)} \operatorname{tr}\overline{\partial}({h'}^{-1}\partial h')_\wedge\omega^{n-1} < 0. \tag{22}$$

First note that the Chern connection associated to $h'$ is equal to $\nabla' = \pi_{E'}\circ\nabla$, where $\pi_{E'}$ is the orthogonal projection (with respect to $h$) on the subbundle $E'$. Then

$$\overline{\partial}({h'}^{-1}\partial h) = \Theta_{\nabla'} = \Theta_\nabla|_{E'} - A_\wedge A^*,$$

where $\Theta_{\nabla'}$ is the curvature of $\nabla'$, $A$ is the second fundamental form and the second equality if given by the Gauss-Codazzi formula (see [Ko]). But $\Theta_\nabla = 0$ on $X-D$, since $\nabla$ is flat, and then the integral (22) can be written as

$$\frac{\sqrt{-1}}{2\pi}\int_{X-(D\cup S)} -\operatorname{tr} A_\wedge A^*{}_\wedge\omega^{n-1}$$

For any $A$, this integral is non-positive. It is zero only if the second fundamental form $A$ is identically zero, but this would imply that there is a holomorphic splitting $E|_{X-(D\cup S)} = E'|_{X-(D\cup S)} \oplus {E'}^\perp|_{X-(D\cup S)}$. The same argument in [Ko, (V.8.5) p. 183] shows that this extends to a holomorphic splitting on $X-D$: since we have a splitting on $X-(D\cup S)$, the holonomy group $G$ of $(E,h)$ on $X-(D\cup S)$ is in $U(r')\times U(r-r')$, and since $S$ is a closed subvariety (of $\operatorname{codim} S \geq 2$), the holonomy group on $X-D$ is contained in the closure of $G$, and this is still in $U(r')\times U(r-r')$.

This contradicts the irreducibility of the representation $\rho$. Then this integral has to be negative, and (22) is proved. This finishes the proof of proposition 1.1.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha road, Mumbai 400 005, India

*E-mail address*: `tomas@math.tifr.res.in, ramadas@math.tifr.res.in`