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On strong multiplicity one for automorphic representations

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Abstract

We extend the strong multiplicity one theorem of Jacquet, Piatetski-Shapiro and Shalika. Let π be a unitary, cuspidal, automorphic representation of $GL_n(\mathbf{A}_K)$. Let S be a set of finite places of K, such that the sum $\sum_{v\in S} Nv^{-2/(n^2+1)}$ is convergent. Then π is uniquely determined by the collection of the local components $\{\pi_v | v \notin S,\ v \text{ finite}\}$ of π. Combining this theorem with base change, it is possible to consider sets S of positive density, having appropriate splitting behavior with respect to a solvable extension L of K, and where π is determined up to twisting by a character of the Galois group of L over K.

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1. Introduction

Let K be a number field with ring of adeles $\mathbf{A}_K$. Let $\mathscr{A}(n, K)$ denote the set of isomorphism classes of irreducible, unitary, cuspidal automorphic representations of $GL_n(\mathbf{A}_K)$. Given $\pi \in \mathscr{A}(n, K)$, π can be expressed as a restricted tensor product '$\otimes'_{v\in\Sigma_K}\pi_v$,' where v runs over all the places Σ_K of K, and π_v is an irreducible, unitary representation of $GL_n(K_v)$. At almost all finite places v, π_v is unramified, and let

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$A_v(\pi)$ denote the conjugacy class in $GL(n, \mathbf{C})$ determined by the Langlands–Satake parameter associated to π_v. Let S be a finite set of places of K. The strong multiplicity one theorem of Jacquet, Piatetski-Shapiro, and Shalika [JS], asserts that any $\pi \in \mathscr{A}(n, K)$ is uniquely determined by the collection of the local components $\{\pi_v | v \in \Sigma_K - S\}$ of π. In view of the possible applications to the functoriality principle of Langlands, it is desirable to have refined versions of the strong multiplicity one theorem. In this note, we give an analytical proof of a stronger version of the strong multiplicity one theorem, based on the analytical properties of the Rankin–Selberg L-functions.

Let $\mathcal{O}_K$ denote the ring of integers of K. If v is a finite place of K, then we denote the corresponding prime ideal by P_v, and by Nv the norm of v to be the number of elements in the residue field $\mathcal{O}_K/P_v$. Let S be a set of finite places of K satisfying the following convergence condition:

$$\sum_{v \in S} Nv^{-\frac{2}{n^2+1}} < \infty. \tag{1}$$

Our main theorem is as follows:

Theorem 1. *Let π_1 and π_2 be irreducible, unitary, cuspidal automorphic representations of $GL_n(\mathbf{A}_K)$. Let S be a set of finite places of K satisfying the convergence criterion given in* (1). *Suppose that at almost all but finitely many places $v \notin S$, the local components of π_1 and π_2 are isomorphic*:

$$\pi_{1,v} \simeq \pi_{2,v}.$$

Then $\pi_1 \simeq \pi_2$.

An advantage with this formulation, is that the ignorant set of places S can be infinite, and in fact the convergence criterion allows us to handle sets S of positive density of special type, as provided by the following example:

Exmple. Let $K/\mathbf{Q}$ be a cyclic extension of prime degree greater than $(n^2+1)/2$ over $\mathbf{Q}$, and take S to be the collection of unramified finite places in K, which are not of degree one over $\mathbf{Q}$. More generally, let $K/\mathbf{Q}$ be a Galois extension with Galois group G. Let σ_v denote the Frobenius element at an unramified finite place v of K over $\mathbf{Q}$. Then the collection S of finite unramified places v of K, such that the order of σ_v is greater than $(n^2+1)/2$, satisfies the convergence condition (1).

The above theorem, combined with the results on solvable base change of automorphic representations established by Langlands, Arthur, and Clozel [AC], and the characterization of the fibers of solvable base change [R3], allow us to handle special sets S of positive density, consisting of places of K having suitable splitting properties with respect to solvable extensions of K:

Corollary 1. *Suppose π_1 and π_2 are irreducible, unitary, cuspidal automorphic representations of $GL_n(\mathbf{A}_K)$. Let L/K be a solvable extension, and S be a set of finite places in K, unramified in L, such that for any place w of L dividing a place v in S of residue characteristic p, the norm Nw is greater than $p^{(n^2+1)/2}$. Suppose that at almost all but finitely many places v of K not belonging to S, the local components of π_1 and π_2 are isomorphic*:

$$\pi_{1,v} = \pi_{2,v}.$$

Assume further that the base change of π_1 and π_2 to $GL_n(\mathbf{A}_L)$ is cuspidal. Then there is an idele class character χ of K, corresponding via class field theory to a character of the Galois group of the extension L/K, such that

$$\pi_1 \simeq \pi_2 \otimes \chi.$$

Proof. Let $BC_{L/K}$ be the base change map defined by Arthur and Clozel [AC], from $\mathscr{A}(n,K)$ to $\mathscr{A}(n,L)$. By Theorem 1, it follows from the hypothesis of the corollary, that $BC_{L/K}(\pi_1) \simeq BC_{L/K}(\pi_2)$. Since the base change representations are cuspidal, the corollary follows from the description of the fibers of solvable base change given by Theorem 2 of [R3]. □

A conjecture of Ramakrishnan [DR2], asserts that if the Dirichlet density of the set of places S of K is less than $1/2n^2$, then the set of local components π_v for v not in S, determines π uniquely. Ramakrishnan in [DR1], proved this conjecture in the case $n = 2$, using the automorphicity of the symmetric square lifting for $GL(2)$. In [R2], Ramakrishnan's conjecture is verified for a particular class of automorphic representations of $GL_n(\mathbf{A}_K)$, and also that Ramanujan's conjecture (and not a weak Ramanujan conjecture, as wrongly stated there) implies Ramakrishnan's conjecture.

In the l-adic context, apart from Ramakrishnan's conjecture, much more is known about the nature of strong multiplicity one results [R1]. For example, suppose $\rho_1, \rho_2 : G_K \to GL_n(\mathbf{Q}_l)$ are absolutely irreducible l-adic representations of the absolute Galois group G_K of a global field K, unramified outside a finite set of places. Suppose that the algebraic envelope of the image of one of the representations is connected, and that the traces of the Frobenius elements agree at a set of unramified finite places of positive density of K. Then there is a character $\chi : G_K \to \mathbf{Q}_l^*$ of finite order, such that $\rho_2 \simeq \rho_1 \otimes \chi$.

Analogously, similar results can be expected in the automorphic context [R2]. For instance, suppose $n = 2$ and π_1 and π_2 be cuspidal, automorphic representations of $GL_2(\mathbf{A}_K)$ which are not automorphically induced from a Hecke character of a quadratic extension of K. If $a_v(\pi_1) = a_v(\pi_2)$ at a set of unramified finite places of positive density, then there should exist an idele-class character χ of K, such that $\pi_1 \simeq \pi_2 \otimes \chi$. Here for an unramified finite place v of an automorphic representation π, the Dirichlet coefficient $a_v(\pi)$ of π at v, is the trace of $A_v(\pi)$. The above results represent a small step towards this general direction.

In a different direction, the strong multiplicity one theorem was refined and proved by analtic methods by Moreno [M], who showed that given two cuspidal automorphic representations π_1 and π_2 of $GL_n(\mathbf{A}_K)$, then there is a constant $C(\pi_1, \pi_2)$ depending essentially on the conductors of π_1 and π_2, such that if the local components of π_1 and π_2 are isomorphic at the set of all finite places v of K with $Nv < C(\pi_1, \pi_2)$, then $\pi_1 \simeq \pi_2$. In [MR], results analogous to Lang–Trotter type conjectures were obtained: if the Rankin–Selberg L-functions $L(s, \mathrm{Sym}^m(\pi_1) \times \mathrm{Sym}^n(\pi_2))$ of the symmetric powers of π_1 and π_2 $(n = 2)$ have 'nice' analytic properties and satisfy the generalized Riemann hypothesis, then for any $\varepsilon > 0$,

$$\#\{Nv \leqslant x | \pi_{1,v} \simeq \pi_{2,v}\} = O(x^{\frac{5}{6}+\varepsilon}),$$

where the implied constant depends only on ε.

2. Proof

The method of proof is analogous to the proof of Ramakrishnan's conjecture for Artin (finite) representations of the absolute Galois group, and can be considered as a relative analogue of the method of [DR3]: let G be a finite group, and let ρ_1, ρ_2 be inequivalent representations of G to $GL(n, \mathbf{C})$. Then

$$\#\{g \in G | Tr(\rho_1(g)) = Tr(\rho_1(g))\} \leqslant (1 - 1/2n^2)|G|.$$

The proof of this result follows from considering the equality,

$$\frac{1}{|G|} \sum_{g \in G} |Tr(\rho_1(g)) - Tr(\rho_2(g))|^2 \geqslant 2,$$

and using the fact that the character value at an element g in G, is a sum of n roots of unity. In analogy, we consider the L-function associated to an automorphic representation, as the 'character' of the representation. Then the convolution product of characters becomes the Rankin–Selberg L-function $L(s, \pi_1 \times \pi_2)$ associated to a pair of unitary, cuspidal automorphic representations π_1 and π_2 of $GL_n(\mathbf{A}_K)$. The analytic properties of the convolution L-function have been studied extensively by Jacquet, Piatetskii-Shapiro, Shalika, Shahidi and Waldspurger in a series of papers [JS,JPSh,Sh,MoW].

The local factors of the Rankin–Selberg L-function $L(s, \pi_1 \times \pi_2)$ at an unramified finite place v of π_1 and π_2 is defined as

$$L(s, \pi_{1,v} \times \pi_{2,v}) = \det(1 - A_v(\pi_1) \otimes A_v(\pi_2) Nv^{-s})^{-1}. \tag{2}$$

Let T be a set of places of K containing the archimedean places of K. Define the Dirichlet series

$$L_T(s, \pi_1 \times \pi_2) = \prod_{v \notin T} L(s, \pi_{1,v} \times \pi_{2,v}).$$

If T is finite, then $L_T(s, \pi_1 \times \pi_2)$ satisfies the following properties [JS,JPSh,MoW,Sh]:

- The Dirichlet series $L_T(s, \pi_1 \times \pi_2)$ is absolutely convergent in the half-plane $\mathrm{Re}(s) > 1$.
- The function $L_T(s, \pi_1 \times \pi_2)$ admits a meromorphic continuation to 1. Further, if $L_T(s, \pi_1 \times \pi_2)$ is holomorphic at $s = 1$, then it is non-vanishing at $s = 1$.
- Let $\tilde{\pi}$ denote the contragredient representation of an automorphic representation π. The function $L_T(s, \pi_1 \times \pi_2)$ has a simple pole at $s = 1$ if and only if $\pi_1 \simeq \tilde{\pi}_2$.

Consider the following Dirichlet series:

$$L_T(s) = \frac{L_T(s, \pi_1 \times \tilde{\pi}_1) L_T(s, \pi_2 \times \tilde{\pi}_2)}{L_T(s, \pi_1 \times \tilde{\pi}_2) L_T(s, \pi_2 \times \tilde{\pi}_1)}.$$

It follows from the properties of Rankin–Selberg convolutions listed above, that if T is finite, then $L_T(s)$ has a pole of order 2, provided $\pi_1 \not\simeq \pi_2$.

Now let T be a finite set of places of K containing the archimedean places of K, the ramified places of π_1 and π_2, and the finite number of places w not in S, as in the hypothesis of the theorem, where it is not known that the local $\pi_{1,w}$ and $\pi_{2,w}$ are isomorphic. Let S', T' denote, respectively, the complement of S and T in the set of all places of K. We have the hypothesis of the theorem that the local components $\pi_{1,v}$ and $\pi_{2,v}$ are isomorphic at all places of K outside T and S. Hence we obtain

$$L_T(s) = L_{T \cup S'}(s) = \prod_{v \in S \cap T'} \frac{L(s, \pi_{1,v} \times \tilde{\pi}_{1,v}) L(s, \pi_{2,v} \times \tilde{\pi}_{2,v})}{L(s, \pi_{1,v} \times \tilde{\pi}_{2,v}) L(s, \pi_{2,v} \times \tilde{\pi}_{1,v})}. \tag{3}$$

At an unramified finite place v of an automorphic representation π, with associated Langlands–Satake parameter $A_v(\pi)$, define for any natural number k,

$$a_{v,k}(\pi) = \sum_{i=1}^{n} \alpha_{v,i}(\pi)^k,$$

where $\alpha_{v,1}, \dots, \alpha_{v,n}$ are the eigenvalues of matrix representative of $A_v(\pi)$. Expanding $\log L_T(s)$ in terms of the Euler product expansion given by (3), and by the definition of the local L-factors as given by (2), we obtain in the region $\sigma = \mathrm{Re}(s) > 1$

$$\log L_T(s) = \sum_{v \in S \cap T'} \sum_{k=1}^{\infty} |a_{v,k}(\pi_1) - a_{v,k}(\pi_2)|^2 N v^{-ks} k^{-1}. \tag{4}$$

We recall the estimate for an eigenvalue α_v of $A_v(\pi)$ proved by Luo, Rudnick, and Sarnak [LRS] for an unramified finite place v of a unitary, automorphic representation π of $GL_n(\mathbf{A}_K)$:

$$|\alpha_v| \leqslant Nv^{\frac{1}{2}-\frac{1}{n^2+1}}. \tag{5}$$

It follows from this estimate, that for $\sigma > 1$ we have a majorization

$$|\log L_T(s)| \leqslant 4n^2 \sum_{v \in S \cap T'} \sum_{k=1}^{\infty} k^{-1} Nv^{-k\sigma+k-2k/(n^2+1)}.$$

The convergence condition satisfied by S as given in (1), ensures that the right-hand expression has a finite limit as $s \to 1^+$. But this implies that $\pi_1 \simeq \pi_2$, and that proves the theorem.

Remark. The theorem can be extended to pairs of isobaric, unitary automorphic representations π_1 and π_2 of $GL_n(\mathbf{A}_K)$.

Remark. It is possible to use the method to compare two isobaric, unitary automorphic representations π_1 and π_2 on $Gl_n(\mathbf{A}_K)$ and $GL_m(\mathbf{A}_K)$, respectively $(n \geqslant m)$. Instead of requiring the equality of local components, we impose the hypothesis

$$a_{v,k}(\pi_1) = a_{v,k}(\pi_2), \quad k = 1, \ldots, [(n^2+1)/2].$$

The Luo–Rudnick–Sarnak estimates (5), ensure that the contribution of the sum over the places $v \in S'$ to $\log L_T(s)$ is $O(1)$ as $s \to 1^+$, and the rest of the proof goes through. If we assume further the Ramanujan conjectures, then the method of proof allows us to show that if the Dirichlet density of S is less than $1/2n^2$, then the collection of Dirichlet coefficients (rather than the local components) $\{a_v(\pi) | v \notin S\}$ uniquely determines π.

Remark. We give a brief indication of the proof of a slight strenthening of Ramakrishnan's theorem [DR1], that if the Dirichlet density

$$d \coloneqq \lim_{s \to 1^+} - \sum_{v \in S} Nv^{-s} / \log(s-1)$$

of S is at most $1/8$, then the collection of Dirichlet coefficients $\{a_v(\pi) | v \notin S\}$ of π (rather than local components) uniquely determines π, where π is a cuspidal automorphic of $GL_2(\mathbf{A}_K)$. Using the estimate $|\alpha_v(\pi)| \leqslant Nv^{1/5}$ proved in [LRS] (even the weaker estimate $Nv^{1/4-\varepsilon}$ for some positive ε will suffice), we see that the hypothesis implies

$$\begin{aligned}
\log L_T(s) &= \sum_{v \in S \cap T'} |a_v(\pi_1) - a_v(\pi_2)|^2 Nv^{-s} + O(1) \quad \text{as } s \to 1^+ \\
&\leqslant 2 \sum_{v \in S \cap T'} (|a_v(\pi_1)|^2 + |a_v(\pi_2)|^2) Nv^{-s} + O(1) \quad \text{as } s \to 1^+.
\end{aligned} \tag{6}$$

For $\pi \in \mathcal{A}(2, K)$, let $\mathrm{Ad}(\pi)$ denote the adjoint lift of π, an automorphic representation of $GL(3, \mathbf{A}_K)$ constructed by Gelbart and Jacquet [GJ]. If $\mathrm{Ad}(\pi)$ is not cuspidal, then it is automorphically induced and we have the estimate $|a_v(\pi)|^2 \leqslant 4$. Hence we obtain

$$-\lim_{s\to 1} \frac{\sum_{v\in S\cap T'} |a_v(\pi)|^2 Nv^{-s}}{\log(s-1)} = 4d. \tag{7}$$

If $\mathrm{Ad}(\pi)$ is cuspidal, then consider

$$L^4(s) := L_T(s, \pi\times\tilde{\pi}\times\pi\times\tilde{\pi}) = L_T(s, \mathrm{Ad}(\pi)\times\mathrm{Ad}(\pi)) L_T(s, \mathrm{Ad}(\pi))^2 \zeta_{K,T}(s).$$

From the properties of the Rankin–Selberg L-functions we obtain

$$\log L^4(s) = -2\log(s-1) + f(s) = \sum_{v\notin T} \frac{|a_v(\pi)|^4}{Nv^s} + O(1) \quad \text{as } s\to 1^+,$$

where $f(s)$ is holomorphic at $s=1$. By Cauchy–Schwarz, we get

$$-\lim_{s\to 1} \frac{\sum_{v\in S\cap T'} |a_v(\pi)|^2 Nv^{-s}}{\log(s-1)} = \sqrt{2d}. \tag{8}$$

From (6)–(8), we obtain

$$2 = \lim_{s\to 1} -\frac{\log L_T(s)}{\log(s-1)} \leqslant \begin{cases} 4\sqrt{2d} & \text{if } \mathrm{Ad}(\pi_1) \text{ and } \mathrm{Ad}(\pi_2) \text{ are cuspidal,} \\ 2\sqrt{2d} + 8d & \text{if only one of } \mathrm{Ad}(\pi_1) \text{ or } \mathrm{Ad}(\pi_2) \text{ is cuspidal,} \\ 16d & \text{if } \mathrm{Ad}(\pi_1) \text{ and } \mathrm{Ad}(\pi_2) \text{ are not cuspidal.} \end{cases}$$

In all the three cases, this leads to a contradiction if $d < 1/8$. This proves Ramakrishnan's theorem.

It is not clear whether the methods contained in this paper, will be sufficient to prove that S can be taken to be a set of places of density at most $1/2$, provided that both $\mathrm{Ad}(\pi_1)$ and $\mathrm{Ad}(\pi_2)$ are cuspidal. Such a result can be expected in analogy with the results in the l-adic context (and hence valid also for holomorphic modular forms) obtained in [R1].

Our original approach to the results of this paper was to use the Cauchy–Schwarz inequality, and to obtain a region of convergence to the left of 2 for the sum $\sum_v |a_v(\pi)|^4 Nv^{-s}$. Using the methods contained in [DI], it is possible to obtain a region of convergence to the left of 2, provided $K = \mathbf{Q}$. But over number fields, it is not clear how to obtain such a result. However, even over $\mathbf{Q}$, these results do not yield stronger results than our main theorem towards strong multiplicity one.

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