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Deformations of complex structures on $\Gamma\backslash SL_2(\mathbb{C})$

C S RAJAN

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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Abstract. Let G be a connected complex semisimple Lie group. Let Γ be a cocompact lattice in G. In this paper, we show that when G is $SL_2(\mathbb{C})$, nontrivial deformations of the canonical complex structure on X exist if and only if the first Betti number of the lattice Γ is non-zero. It may be remarked that for a wide class of arithmetic groups Γ, one can find a subgroup Γ' of finite index in Γ, such that $\Gamma'/[\Gamma', \Gamma']$ is finite (it is a conjecture of Thurston that this is true for all cocompact lattices in $SL(2, \mathbb{C})$).

We also show that G acts trivially on the coherent cohomology groups $H^i(\Gamma\backslash G, \mathcal{O})$ for any $i \geqslant 0$.

Keywords. Deformations; lattice; cohomology.

1. Introduction

Let M be a compact smooth manifold. We assume that M can be equipped with a hyperbolic structure. In ([3]), Johnson and Millson show that the space of deformations of 'marked conformal structures' on M has dimension at least r, where r is the largest number of disjoint, nonsingular, totally geodesic hypersurfaces in M. Such hypersurfaces are known to contribute to the first Betti number of M. Now, it is known that if the dimension of M is n, then M is diffeomorphic to $\Gamma\backslash SO(n, 1)/K$, where Γ is a torsion-free cocompact lattice in $SO(n, 1)$ and K is a maximal compact subgroup of $SO(n, 1)$. When $n = 3$, $SO(3, 1)$ is locally isomorphic to $SL(2, \mathbb{C})$ and thus carries a complex structure. One can raise the question, whether there exists nontrivial deformations of the complex structure on $\Gamma\backslash SL(2, \mathbb{C})$, and if so whether the deformations are related to the 'topology of Γ'.

In a different direction, Matsushima raised the question whether the canonical complex structure on $\Gamma\backslash G$ is infinitesimally rigid, where G is a connected complex semisimple Lie group and Γ is an irreducible torsion-free cocompact lattice in G. In [8], Raghunathan showed that whenever G has no 3-dimensional components, the canonical complex structure on $\Gamma\backslash G$ is infinitesimally rigid. It is easy to extend this result to all G, provided G is not three-dimensional. We remark that when G is not three dimensional, the first Betti number of Γ is zero.

From these results, we are thus led to relating the 'topology of Γ' to the deformations of the complex structure on $\Gamma\backslash G$, where $G = SL(2, \mathbb{C})$. Our main result states that nontrivial deformations of the canonical complex structure on $\Gamma\backslash SL(2, \mathbb{C})$ exist if and only if the first Betti number of the cocompact torsion-free lattice Γ in $SL(2, \mathbb{C})$ is nonzero.

It is a conjecture of Thurston that for any cocompact lattice Γ in $SL(2, \mathbb{C})$, one can find a sublattice Γ' of finite index in Γ, such that the first Betti number of Γ' is nonzero. Thurston's conjecture is known to be true for a wide class of arithmetic lattices Γ([5], [6], [7]).

We also show that the natural action of G on $H^*(\Gamma\backslash G, \mathcal{O})$ is the trivial action for G a complex semisimple Lie group and Γ any cocompact torsion-free lattice in G.

2. Cohomology computations

Let G be a connected, semisimple complex Lie group with Lie algebra $L(G)$ of left invariant vector fields on G. Let $L(G)^{\mathbb{C}}$ denote the complexification of $L(G)$. Let Γ be a discrete subgroup of G and let $X = \Gamma\backslash G$. Elements of $L(G)^{\mathbb{C}}$ can be regarded as complex vector fields on X. Then the space U_1 of holomorphic left invariant vector fields on G project to X, to give a trivialisation of the holomorphic tangent bundle of X. Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on X. There is a natural holomorphic action of G on $H^*(X, \mathcal{O})$. Our aim in this section is to show that G acts trivially on $H^*(\Gamma\backslash G, \mathcal{O})$.

Let K be a maximal compact subgroup of G. Let $Y = \Gamma\backslash G/K$ and let π denote the natural projection $X \to Y$. Let $L(K)$ be the Lie algebra of K.

PROPOSITION 1

Let ρ be a nontrivial holomorphic representation of G on F. Then for any $i \geqslant 0$,

$$H^i(\Gamma, \rho) = (0)$$

Proof. By Matsushima's formula it is enough to show the following:

$$H^i(L(G), L(K): W \otimes F) = (0) \quad \forall i \geqslant 0,$$

where W is an irreducible unitary $(L(G), L(K))$-module (see [2, page 224]).

Let $L(H)^+ \subset L(K)$ be a maximal abelian subalgebra of $L(K)$. Let $L(H)$ be the centralizer in $L(G)$ of $L(H)^+$. Then $L(H)$ is a Cartan subalgebra of $L(G)$. Let Φ denote the root system of $(L(G)^{\mathbb{C}}, L(H)^{\mathbb{C}})$. Let θ denote the Cartan involution of the pair $(L(G), L(K))$. Fix a positive system of roots Φ^+ of Φ as in ([2, page 65]). In particular this implies that if α is a positive root, so is $\theta\alpha$. Let δ denote half the sum of the positive roots. Then $\theta\delta = \delta$.

Let J denote the complex structure on $L(G)$. We can choose θ such that $\theta J \theta = -J$. The space of holomorphic (resp. antiholomorphic) vectors of $L(G)^{\mathbb{C}}$ can be taken as the kernel of $(J - i)$ (resp. $(J + i)$). By the compatibility relation between θ and J given above, θ interchanges the space of holomorphic and antiholomorphic vectors of $L(G)^{\mathbb{C}}$.

Let λ be the highest weight of F^* with respect to the ordering defined by Φ^+. Since the representation ρ is assumed to be holomorphic, λ vanishes on the space of antiholomorphic vectors of $L(H)^{\mathbb{C}}$. But then $\theta\lambda$ vanishes on the space of holomorphic vectors of $L(H)^{\mathbb{C}}$. Since λ is assumed to be nontrivial we have $\theta\lambda \neq \lambda$. Since $\theta\delta = \delta$, we have $\theta(\lambda + \delta) \neq \lambda + \delta$. But then by Proposition 6.12 1), page 69 of ([2]), $H^*(L(G), L(K), W \otimes F) = (0)$. Hence the proposition.

Remark. For the purposes of studying the deformations of $\Gamma\backslash SL_2(\mathbf{C})$, it is enough to have this proposition when $G = SL_2(\mathbf{C})$. In this case, the proposition has been proved by Raghunathan. See [2, page 225].

Let L_ρ be the local system on X associated to the representation ρ.

In ([8]), it is shown that the E_2 term of the Hodge–de Rham spectral sequence, which converges to $H^*(X, L_\rho)$ is given by

$$E_2^{pq} = H^p(L(G), H^q(X, \mathcal{O}) \otimes_{\mathbf{C}} F),$$

where the G-module structure on $H^q(X, \mathcal{O}) \otimes_{\mathbf{C}} F$ is the tensor product of the representations.

There is also the Leray spectral sequence associated to the K-principal fibration $\pi: X \to Y$, converging to $H^*(X, L_\rho)$, and whose E_2 term is given by

$$'E_2^{pq} = H^p(Y, R^q\pi_* L_\rho).$$

Now for a fibration $\pi: X \to Y$ with fiber Z, and given a local system L_ρ on X, $R^q\pi_* L_\rho$ is the local system associated to the representation of $\pi_1(Y)$ on $H^q(Z, L_\rho|_Z)$.

In the above situation, since Γ is torsion-free, $L_\rho|_K$ is a trivial local system and hence $R^q\pi_* L_\rho = H^q(K, \mathbf{C}) \otimes_{\mathbf{C}} F$ as $\pi_1(Y) = \Gamma$-modules, where the action of Γ on $H^q(K, \mathbf{C})$ is trivial and on F it acts as ρ. Thus,

$$'E_2^{pq} = H^p(\Gamma, \rho) \otimes_{\mathbf{C}} H^q(K, \mathbf{C})$$

With notation as above, we prove the following:

Theorem 1. *G acts trivially on $H^i(\Gamma\backslash G, \mathcal{O})$ for all $i \geq 0$.*

Proof. Let ρ denote an irreducible, holomorphic representation of G on F, occurring in $\mathrm{Hom}(H^i(X, \mathcal{O}), \mathbf{C})$. Suppose ρ is nontrivial. Then by the proposition proved above, $H^p(\Gamma, \rho) = (0) \forall p \geq 0$. Hence $'E_2^{pq} = 0$ and so we have that $H^p(X, L_\rho) = (0) \forall p \geq 0$.

We have by the Hodge–de Rham spectral sequence computed above

$$E_2^{pq} = H^p(L(G), H^q(X, \mathcal{O}) \otimes_{\mathbf{C}} F)$$

and this converges to $H^*(X, L_\rho)$.

We claim that $E_2^{pq} = 0 \,\forall\, p, q \geq 0$. If not, there is a maximal p_0 and q_0 such that $E_2^{pq} \neq 0$. i.e., $E_2^{pq} = (0)$ if either $p > p_0$ or $q > q_0$ and $E_2^{p_0q_0} \neq (0)$. Since the differentials $d_r (r \geq 2)$ of the spectral sequence increases the indices p or q by at least 1, we see that $E_2^{p_0q_0}$ survives to E_∞ and is nonvanishing. But this contradicts the vanishing of $H^*(X, L_\rho)$ shown above. Hence $E_2^{pq} = (0)$ $\forall p, q \geq 0$.

But then $E_2^{i0} = H^0(L(G), H^i(X, \mathcal{O}) \otimes F)$ is nonzero since by assumption on $F, H^i(X, \mathcal{O}) \otimes F$ contains a copy of the trivial representation and hence the invariants $H^0(L(G), H^i(X, \mathcal{O}) \otimes F)$ can never be zero. Hence ρ has to be trivial and this proves the theorem.

PROPOSITION 2

Let G be a connected, simply connected, complex Lie group and Γ a cocompact lattice in G. Then

$$H^1(\Gamma, \mathbf{C}) = H^1(\Gamma\backslash G, \mathcal{O})$$

Proof. Since $X = \Gamma\backslash G$ is compact, $H^*(X, \mathcal{O})$ are finite dimensional. By Whitehead lemma for semisimple Lie algebras, $H^1(L(G), \mathbb{C}) = H^2(L(G), \mathbb{C}) = (0)$ (see [2]). Hence in the Hodge–de Rham spectral sequence given above (for $F = \mathbb{C}$), we have $E_2^{10} = E_2^{20} = 0$.

Therefore $E_\infty^{10} = 0$ and $E_\infty^{01} = E_2^{01}$. Hence $H^1(X, L_\rho) = H^0(L(G), H^1(X, \mathcal{O}))$. But from the Leray spectral sequence given above, we have $'E_2^{01} = 0$ and $'E_\infty^{10} = {'E_2^{10}} = H^1(\Gamma, \mathbb{C})$. Hence $H^1(\Gamma, \mathbb{C}) = H^1(X, L_\rho)$. Since $L(G)$ acts trivially on $H^1(X, \mathcal{O})$ by the theorem proved above, we have

$$H^1(\Gamma, \mathbb{C}) = H^1(X, L_\rho) = H^0(L(G), H^1(X, \mathcal{O})) = H^1(X, \mathcal{O}).$$

This proves the proposition.

Remark. When G has more than one three dimensional component and Γ is an irreducible lattice in G, then the first Betti number of Γ vanishes by a theorem of Bernstein–Kazhdan, see ([1].) Let Θ denote the sheaf of germs of holomorphic vector fields on X. Since the holomorphic tangent bundle of X is trivial,

$$H^1(X, \Theta) = H^1(X, \mathcal{O}) \otimes U_1.$$

By the above proposition $H^1(X, \Theta)$ vanishes. This extends the following rigidity theorem of Raghunathan for groups G with no three dimensional components, (see [8]). (note that it is enough to prove the results for simply connected G).

Theorem 2. *Let G be a connected complex semisimple Lie group and Γ an irreducible cocompact lattice in G. Assume that G is not locally isomorphic to $SL(2, \mathbb{C})$. Then the canonical complex structure on X is rigid.*

Essentially the only interesting case left is when $G = SL_2(\mathbb{C})$. In this case if the canonical complex structure on X is not locally rigid, then $H^1(X, \Theta) \neq 0$. Hence we obtain that the first Betti number of Γ is nonzero.

3. Deformations of $\Gamma\backslash SL_2(\mathbb{C})$

From now onwards $G = SL_2(\mathbb{C})$.

In this section we show the existence of non-trivial deformations of the canonical complex structure on X, whenever the first Betti number of Γ is nonzero.

Let T denote the holomorphic tangent bundle of X. By means of the projection $G \to X$, U_1 defines a trivialisation of T. Let A^p denote the space of $(0, p)$ forms of X with values in T. We have $\bar{\partial}: A^p \to A^{p+1}$. It is well known that on $A = \Sigma_{p \geqslant 0} A^p$, one can define a bilinear operation,

$$[,]: A^p \otimes A^q \to A^{p+q}$$

which turns A into a graded Lie algebra complex. The graded Lie algebra structure descends down to a graded Lie algebra structure on $H = \Sigma_{p \geqslant 0} H^{(0,p)}(T)$. The group G acts as graded Lie algebra automorphisms on A and this action descends to an action of G on H, compatible with the graded Lie algebra structure on H.

Let K be a maximal compact subgroup of G. Fix a hermitian metric on T, on which K acts as isometries.

With respect to this metric one can define the adjoint $\bar{\partial}^*$ of $\bar{\partial}$, the Laplacian $\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, the Green's operator $\mathbf{G}$ and the harmonic projection operator $\mathbf{H}$. Since K acts as isometries the action of K commutes with that of Δ, $\mathbf{G}$ and $\mathbf{H}$.

By means of the Dolbeault isomorphism $H^p(X, \Theta) = H^{0,p}(T)$ and by the theory of harmonic forms, we can think of $H^p(X, \Theta)$ as a subspace of A^p consisting of harmonic forms and these spaces are isomorphic as K-modules.

Theorem 3. *Let $G = SL_2(\mathrm{C})$ and Γ a cocompact lattice on G. Let $X = \Gamma\backslash G$. Assume that the first Betti number of Γ is nonzero. Then there exists nontrivial deformations of the canonical complex structure on X.*

Proof. Let Θ denote the sheaf of germs of holomorphic vector fields on X. Since T is trivialised by the projection of U_1 to X, we have as G-modules,

$$H^1(X, \Theta) = H^1(X, \mathcal{O}) \otimes U_1$$

and

$$H^2(X, \Theta) = H^2(X, \mathcal{O}) \otimes U_1$$

where the G-module structure is the tensor product of the G-modules. Since the duality relationship is compatible with the G-action and the canonical bundle is trivial $H^1(X, \Theta) \simeq H^2(X, \Theta)$ as G-modules.

Fix a maximal torus S of K and a system of positive roots of G with respect to S and let λ be a highest weight for the representation of K on $H^1(X, \Theta)$. Let V denote the corresponding highest weight subspace of $H^1(X, \Theta)$. V is nonzero as $H^1(X, \Theta)$ is a nontrivial $L(G)$ module. We also think of $H^1(X, \Theta)$ as the subspace of harmonic forms in A^1.

Even though $H^1(X, \Theta) \simeq H^2(X, \Theta)(\neq 0)$, we will show below that the method of Kodaira–Spencer as outlined in ([4] p. 316), can be adapted to the subspace V of $H^1(X, \Theta)$, to obtain the existence of nontrivial deformations of the complex structure on X. Let $\{\beta_1, \ldots \beta_m\}$ be a basis of V and put

$$\phi_1(t) = \beta_1 t_1 + \ldots + \beta_m t_m.$$

Define inductively a sequence $\{\phi_k(t)\}_{k \geq 2}$ of A^1 valued homogeneous polynomials of degree k in the variables $t_1, \ldots t_m$ as follows:

$$\phi_k(t) = \tfrac{1}{2} \sum_{l=1}^{k-1} \bar{\partial}^* \mathbf{G}[\phi_l(t), \phi_{k-l}(t)]$$

Let $\phi(t) = \Sigma_{k=1}^{\infty} \phi_k(t)$. The inductive definition of the ϕ_k is made so as to secure the condition

$$\phi(t) = \phi_1(t) + \tfrac{1}{2}\bar{\partial}^* \mathbf{G}[\phi(t), \phi(t)]$$

$\phi(t)$ defines an almost complex structure on the real submanifold underlying the space X. One can proceed as in ([4], p. 316), to show that $\phi(t)$ converges with respect to the Holder norm for $|t| < \varepsilon$, provided $\varepsilon > 0$ is sufficiently small and that $\phi(t)$ is C^∞ on $X \times \Delta_\varepsilon$, where $\Delta_\varepsilon = \{t \in \mathrm{C}^m : |t| < \varepsilon\}$.

The almost complex structure given by $\phi(t)$ is integrable if

$$\mathbf{H}[\phi(t), \phi(t)] = 0$$

(see Lemma 6·3 [4], p. 316).

Now $[,]: A^1 \otimes A^1 \to A^2$ is a morphism of G-modules and if $\omega, \eta \in A^1$ are of weights $k\lambda$ and $l\lambda$ respectively for the compact torus S, then $[\omega, \eta]$ is of weight $(k+l)\lambda$. $\phi_1(t)$ is of weight λ with respect to the maximal torus S in K. By induction one can check that $[\phi_l(t), \phi_{k-l}(t)]$ is of weight $k\lambda$ for the torus $S (0 < l < k)$. Since the action of *K commutes with* $\bar{\partial}^*$ *and* **G**, we see that $\phi_k(t)$ is of weight $k\lambda$ for the torus $S \subset K$. Hence $[\phi(t), \phi(t)]$ is a sum of elements of weight $k\lambda$, where $k \geqslant 2$. Since Ker Δ has highest weight λ, and the action of K commutes with Δ and **H** we see that

$$\mathbf{H}[\phi(t), \phi(t)] = 0.$$

As in ([4]), one can also check that this forms a complex analytic family on Δ_ε, such that the Kodaira–Spencer deformation map from the tangent space of the base space at 0 to $H^1(X, \Theta)$ maps $T(\Delta_\varepsilon)_0$ isomorphically to V. Hence one gets a nontrivial family of deformations parametrized by V of the complex manifold X.

Remark. Let C be the cone of all highest weight vectors in $H^1(X, \Theta)$. By what we have shown above, we see that C is a complex analytic subvariety of the base space parametrizing the Kuranishi family. Hence we get a complex analytic family in the sense of Kuranishi of deformations of the complex structure on X over C, which is a subfamily of the universal family Kuranishi has constructed. It is not known whether this family is complete.

Remark. Thurston's conjecture states that given a uniform lattice Γ in $SL(2, \mathbb{C})$, there is a sublattice whose first Betti number is nonzero. We have shown that this conjecture is equivalent to showing existence of nontrivial deformations of the complex structure of some suitable finite cover of X. For a wide class of arithmetic lattices the conjecture has been shown to be true ([5], [6], [7]).

Jump phenomenon. G acts on C and the highest weight vectors in the same G orbit give raise to identical complex structures on the smooth manifold underlying X.

If we take a highest weight vector v, then the Borel subgroup in G corresponding to v, acts by scaling. Hence we have that the complex manifolds X_{tv} are all isomorphic for $t \neq 0$ and not isomorphic to the complex manifold $X = X_0$.

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