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Frobenius Splitting and Ordinarity

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1 Introduction

Let k be a perfect field of characteristic $p > 0$. An abelian variety A over k is said to be ordinary if the p-rank of A is the maximum possible, namely, equal to the dimension of A. The notion of ordinarity was extended by Mazur [14] to a smooth projective variety X over k, using notions from crystalline cohomology. A more general definition was given by Bloch and Kato [4] and Illusie and Raynaud [11] using coherent cohomology. Most varieties are ordinary, in that the ordinary locus is open in any smooth family, and the crystalline cohomology of ordinary varieties is better understood than those of general varieties.

Frobenius split varieties were introduced by Mehta and Ramanathan [15]. These varieties enjoy a number of properties, for instance they satisfy Kodaira vanishing, and over the past decade and a half, Frobenius splitting has played an important role in the study of Schubert varieties and related matters. The motivations for this paper is to study the crystalline cohomology of Frobenius split varieties, and in particular to study the relationship between the Frobenius split and ordinary varieties. The question of whether Frobenius split varieties are ordinary was raised by Mehta (oral communication). It was shown by Mehta and Srinivas [16] that smooth, projective varieties with trivial cotangent bundle, in particular abelian varieties, are ordinary if and only if they are Frobenius split. It can be seen that smooth, projective F-split surfaces are ordinary, and in the converse direction that any smooth, projective, ordinary variety with trivial canonical bundle is F-split (see Proposition 3.1).

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Ordinary varieties are Hodge-Witt, in that the de Rham-Witt cohomology groups $H^i(X, W\Omega_X^j)$ are finitely generated over W. A natural weaker variant of Mehta's question that arises is whether Frobenius split varieties are Hodge-Witt. We show (see Theorem 4.2) that for any smooth projective F-split variety over an algebraically closed field, the cohomology groups $H^i(X, W(\mathcal{O}_X))$ are of finite type as W-modules. Using the work of Illusie and Raynaud [11], we also see that $d_1^{i,0}$, the first differential of the slope spectral sequence, is zero for all $i \geq 0$. During the course of writing this paper, the first author refined these methods to control the nature of crystalline torsion for F-split varieties (see [12]) and has also shown that any smooth, projective and Frobenius split threefold is Hodge-Witt.

The foregoing results, it would appear, lend credence to Mehta's expectation: that Frobenius split varieties be ordinary. However, this general expectation turns out to be false. One of the main results of this paper (see Section 5) is that we provide examples of Frobenius split varieties which are not ordinary (of dimension at least three) and are not even Hodge-Witt (dimension at least four).

2 Frobenius splitting

The notion of Frobenius splitting was introduced by Mehta and Ramanathan in [15] and a number of remarkable properties were also investigated in that paper. In this section, we recall a few basic facts about Frobenius split (or F-split) varieties.

Let X be a smooth proper variety over a perfect field of characteristic $p > 0$, and let $F : X \to X$ denote the absolute Frobenius of X. The natural map $\mathcal{O}_X \to F_*\mathcal{O}_X$ is given by $f \mapsto f^p$ for a section f of $\mathcal{O}_X$, and let C be the quotient sheaf.

Definition 2.1. The variety X is a *Frobenius split* if the exact sequence of sheaves (the sheaf B_X^1 being defined by the sequence)

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X \longrightarrow B_X^1 \longrightarrow 0 \tag{2.1}$$

is split; that is, there exists a section $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that the composite $\mathcal{O}_X \to F_*\mathcal{O}_X \to \mathcal{O}_X$ is the identity.

Note that when X is smooth, this is an exact sequence of locally free $\mathcal{O}_X$-modules, and the sheaf B_X^1 can be identified more explicitly (see Section 3). Considering the relative Frobenius morphism, it can be shown that the F-split condition is an open condition on the base for a smooth, proper family of varieties.

2.1 Splitting sections and duality for Frobenius

Let X be a smooth, projective variety of dimension n, with canonical bundle ω_X. We now recall a criteria for X to be Frobenius split [15, Proposition 7]. Since any global map $\mathcal{O}_X \to \mathcal{O}_X$ is a constant, in order to split X, it is enough to produce a section $F_*\mathcal{O}_X \to \mathcal{O}_X$ such that the composite $\mathcal{O}_X \to F_*\mathcal{O}_X \to \mathcal{O}_X$ is nonzero. By Serre duality, $H^0(X, F_*\mathcal{O}_X^*) = \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$ is isomorphic to $H^n(X, \omega_X \otimes F_*\mathcal{O}_X)^*$. By the projection formula and the fact that F is an affine morphism, we obtain a semilinear (for Frobenius of k) isomorphism,

$$H^n\left(X, F^*\omega_X\right)^* \simeq H^n\left(X, \omega_X^p\right)^* \simeq H^0\left(X, \omega_X^{1-p}\right), \tag{2.2}$$

where the last isomorphism follows once again from Serre duality.

We recall now the duality theory for the finite, flat morphism given by the Frobenius F (see [17, Sections 1.14–1.16] and [15] for more details). Let $f : X \to Y$ be a finite, flat morphism. Define, for a coherent sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules on Y, the sheaf on X given by

$$f^!(\mathcal{G}) = \mathrm{Hom}_Y\left(f_*\mathcal{O}_X, \mathcal{G}\right), \tag{2.3}$$

thought of as $f_*\mathcal{O}_X$-modules. The duality theory of f gives that the natural morphism,

$$f_*\,\mathrm{Hom}_X\left(f_*\mathcal{F}, f^!\mathcal{G}\right) \to \mathrm{Hom}_Y\left(f_*\mathcal{F}, \mathcal{G}\right), \tag{2.4}$$

is an isomorphism.

Let f be the absolute Frobenius $F = F_X$ of X. We have the identification [17, Section 1.16],

$$F^!\mathcal{O}_X \simeq \omega_X^{1-p}. \tag{2.5}$$

The identification $F_*\omega_X^{1-p}$ with $F_*\mathcal{O}_X^*$ is given by the Cartier operator as follows: the Cartier operator gives a pairing

$$F_*\left(\mathcal{O}_X\right) \otimes F_*\left(\omega_X\right) \longrightarrow \omega_X, \tag{2.6}$$

given by $(f, \omega) \mapsto C(f\omega)$, where C is the Cartier operator $C : F_*(\omega_X) \to \omega_X$. This is equivalent to giving a pairing

$$\omega_X^{-1} \otimes F_*\left(\omega_X\right) \longrightarrow F_*\left(\mathcal{O}_X\right)^*. \tag{2.7}$$

By the projection formula, the left-hand side can be identified with $F_*(F^*\omega_X^{-1} \otimes \omega_X) = F_*\omega_X^{1-p}$. Thus by Cartier duality, there is a functorial morphism of sheaves [15, Proposition 5],

$$F_*\omega_X^{1-p} \simeq \mathrm{Hom}_{\mathcal{O}_X}\left(F_*\mathcal{O}_X, \mathcal{O}_X\right), \tag{2.8}$$

expressing the fact that the Cartier operator is essentially the trace map in Grothendieck duality theory for the Frobenius applied to the structure sheaf. The above morphism in (2.8) can be seen to be an isomorphism by the local nature of the Cartier operator and by reduction to the case when X is isomorphic to the affine line $\mathbf{A}^1$. By means of this, we obtain a natural identification for global sections

$$H^0\left(X, \omega_X^{1-p}\right) \simeq \mathrm{Hom}\left(F_*\mathcal{O}_X, \mathcal{O}_X\right). \tag{2.9}$$

In fact, using the Cartier operator as in [15, Proposition 5], or the relative duality for the finite, flat morphism $F : X \to X$ as in [17], it is possible to make the isomorphism (2.8) explicit in terms of local coordinates on X. Using this explicit description, the following useful criterion for Frobenius splitting is proved in [15].

Proposition 2.2 (see [15, Proposition 7])**.** Let X be a smooth, projective variety over a perfect field k of characteristic $p > 0$. Then X is Frobenius split if there is a global section s of ω_X^{1-p}, and a point P in X such that the divisor of zeros of s is defined at P by $x_1 \cdots x_n = 0$, where $x_1, \ldots, x_n$ is a regular system of parameters in the completion $\widehat{\mathcal{O}}_X$ of $\mathcal{O}_X$; that is, the divisor of s is of the form $Z_1 + \cdots + Z_n + D$, with $P \in Z_1, \ldots, Z_n$, $P \notin D$, and Z_i is defined at P by $x_i\widehat{\mathcal{O}}_X$, and D is effective. □

Definition 2.3. A section $\sigma \in H^0(X, \omega_X^{1-p})$ which, under the above isomorphism (2.8), provides a splitting of X, will be called a *splitting section*.

3 Ordinary varieties

Let X be a smooth, projective variety over a perfect field k of positive characteristic. Following Bloch and Kato [4] and Illusie and Raynaud [11], we say that X is ordinary if $H^i(X, B_X^j) = 0$ for all $i \geq 0$, $j > 0$, where

$$B_X^j = \mathrm{image}\left(d : \Omega_X^{j-1} \longrightarrow \Omega_X^j\right). \tag{3.1}$$

If X is an abelian variety, then it is known that this definition coincides with the usual definition [4]. By [10, Proposition 1.2], ordinarity is an open condition in the following sense: if $X \to S$ is a smooth, proper family of varieties parameterized by S, then the set of points s in S, such that the fiber X_s is ordinary is a Zariski-open subset of S.

3.1 Cartier operator

Let X be a smooth, proper variety over a perfect field of characteristic $p > 0$, and let F_X (or F) denote the absolute Frobenius of X. We recall a few basic facts about Cartier operators from [8]. The first fact we need is that we have a fundamental exact sequence of locally free sheaves

$$0 \longrightarrow B^i_X \longrightarrow Z^i_X \xrightarrow{C} \Omega^i_X \longrightarrow 0, \tag{3.2}$$

where Z^i_X is the sheaf of closed i-forms and C is the Cartier operator. The existence of this sequence is the fundamental theorem of Cartier (see [8]). In particular, on applying $\mathrm{Hom}(-, \Omega^n_X)$ to the exact sequence

$$0 \longrightarrow B^n_X \longrightarrow Z^n_X \longrightarrow \Omega^n_X \longrightarrow 0, \tag{3.3}$$

we obtain the exact sequence (2.1).

3.2 Frobenius splitting and ordinarity

We compare now the notions of ordinarity and Frobenius splitting in some simple situations.

Proposition 3.1. Let X be any smooth, projective variety over a perfect field.

(a) If X is a Frobenius split surface, then X is ordinary.

(b) If X is ordinary and the canonical bundle of X is trivial, then X is Frobenius split. □

Proof. (a) It follows from the proof of Theorem 4.2 that, for any F-split variety X,

$$H^i(X, B^1_X) = 0 \tag{3.4}$$

for all i. So when X is a surface, we need to check that the same vanishing is also valid for B^2_X. But this is immediate from Serre duality and the following fact: the perfect pairing of (2.6) under the assumption that X is a surface,

$$F_*(\mathcal{O}_X) \otimes F_*(\Omega^2_X) \longrightarrow \Omega^2_X, \tag{3.5}$$

given by $(f, w) \mapsto C(fw)$ and this induces a perfect pairing $B^1_X \otimes B^2_X \to \Omega^2_X = \omega_X$ (see [16]).

(b) The obstruction to the splitting of the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_*(\mathcal{O}_X) \longrightarrow B^1_X \longrightarrow 0 \tag{3.6}$$

is an element of $\mathrm{Ext}^1(B^1_X, \mathcal{O}_X) \simeq H^1(X, (B^1_X)^*)$. The duality pairing induced by the Cartier operator implies that

$$(B^1_X)^* \simeq B^n_X \otimes \omega_X, \tag{3.7}$$

where ω_X denotes the canonical bundle. Since we have assumed that ω_X is trivial, it follows that

$$\mathrm{Ext}^1(B^1_X, \mathcal{O}_X) \simeq H^1(X, (B^1_X)^*) \simeq H^1(X, B^n_X) = 0, \tag{3.8}$$

where the vanishing follows from the ordinarity assumption. Hence X is Frobenius split. ∎

4 de Rham-Witt cohomology of F-split varieties

4.1 de Rham-Witt cohomology

The standard reference for de Rham-Witt cohomology is [8]. Throughout this section, the following notations will be in force. Let k be an algebraically closed field of characteristic $p > 0$, and X a smooth, projective variety over k. Let $W = W(k)$ be the ring of Witt vectors of k. Let $K = W[1/p]$ be the quotient field of W. Note that as k is perfect, W is a Noetherian local ring with a discrete valuation and with residue field k. For any $n \geq 1$, let $W_n = W(k)/p^n$. The Witt vectors W comes equipped with a lift $\sigma : W \to W$, of the Frobenius morphism of k, which will be called the Frobenius of W. Let $\{W_n\Omega^*_X\}_{n\geq 1}$ be the de Rham-Witt pro-complex constructed in [8]. It is standard that for each $n \geq 1$, $i, j \geq 0$, $H^i(X, W_n\Omega^j_X)$ are of finite type over W_n. We define

$$H^i(X, W\Omega^j_X) = \varprojlim_n H^i(X, W_n\Omega^j_X), \tag{4.1}$$

which are W-modules of finite type up to torsion. These cohomology groups are called Hodge-Witt cohomology groups of X.

Definition 4.1. The variety X is Hodge-Witt if for $i, j \geq 0$, the Hodge-Witt cohomology groups $h^i(X, W\Omega^j_X)$ are finite type W-modules.

4.2 A finiteness result

For a general variety X, the de Rham-Witt cohomology groups are not of finite type over W, and the structure of these groups reflects the arithmetical properties of X. For instance, in [4, 11] it is shown that for ordinary varieties, $H^i(X, W\Omega^j)$ are of finite type over W.

Theorem 4.2. Let X/k be any smooth, projective, F-split variety over an algebraically closed field k of characteristic $p > 0$. Then for each $i \geq 0$, $H^i(X, W(\mathcal{O}_X))$ is a finite type $W(k)$-module. □

Proof. We use the theory of higher Cartier operators as outlined in [8]. The higher Cartier sheaves, $B_n\Omega^i_X \subset B_n\Omega^i_X \subset \Omega^i_X$, are defined inductively (see [8, page 519]),

$$\begin{aligned} B_1\Omega^n_X &= \text{image}\left(d : \Omega^{n-1}_X \longrightarrow \Omega^n_X\right), \\ Z_1\Omega^n_X &= \ker\left(d : \Omega^n_X \longrightarrow \Omega^{n+1}_X\right). \end{aligned} \tag{4.2}$$

In particular, we have $B_1\Omega^1_X = B^1_X = \text{image}(d : \mathcal{O}_X \to \Omega^1_X)$. For $n \geq 2$, we have by induction

$$\begin{aligned} B_n\Omega^1_X &\xrightarrow{C^{-1}} B_{n+1}\Omega^1_X/B_1\Omega^1_X, \\ Z_n\Omega^1_X &\xrightarrow{C^{-1}} Z_{n+1}\Omega^1_X/B_1\Omega^1_X, \end{aligned} \tag{4.3}$$

where C denotes the Cartier operator.

To prove the theorem, it suffices by [8, page 613, Proposition 2.16] to prove that for all $j \geq 0$ and for all $n \geq 0$, $H^j(X, Z_n\mathcal{O}_X)$ and $H^j(X, B_n\Omega^1_X)$ have bounded dimension. But $Z_1\mathcal{O}_X = \ker(d : \mathcal{O}_X \to \Omega^1_X)$ and by definition, $Z_n\mathcal{O}_X \to Z_{n-1}\mathcal{O}_X$ is an isomorphism for all $n \geq 2$ (the arrow in this isomorphism is the Cartier operator, (see [8, equation (2.5.1.2), page 531])). Thus the required cohomology groups have dimension independent of n. Next, we need to check that $H^j(X, B_n\Omega^1_X)$ have bounded dimension for all n. We show now that these cohomology groups vanish by induction on n. Since X is F-split, it follows that $F_*(\mathcal{O}_X) = \mathcal{O}_X \oplus B^1_X$. Hence

$$H^i(X, F_*(\mathcal{O}_X)) = H^i(\mathcal{O}_X) \oplus H^i(X, B^1_X). \tag{4.4}$$

But by the Leray spectral sequence applied to the Frobenius morphism and the projection formula, we see that

$$H^i(X, F_*(\mathcal{O}_X)) \simeq H^i(X, \mathcal{O}_X) \tag{4.5}$$

and hence we see that

$$\dim H^i(\mathcal{O}_X) = \dim H^i(\mathcal{O}_X) + \dim H^i(B^1_X) \tag{4.6}$$

and so the cohomology group $H^i(B^n_X)$ vanishes when $n = 1$. The result for general n follows from the result for $n = 1$, and the exact sequence given by the definition

$$0 \longrightarrow B_1\Omega^1_X \longrightarrow B_{n+1}\Omega^1_X \xrightarrow{C^{-1}} B_n\Omega^1_X \longrightarrow 0. \tag{4.7}$$

■

Remark 4.3. The theorem is of interest when the formal Brauer group $\widehat{Br^i}(X)$ of X (associated by Artin and Mazur [1]) is representable. It turns out that $H^i(X, W(\mathcal{O}_X))$ is the Cartier module of this formal group. When this module is free of finite type over W, the formal Brauer group is a p-divisible group of height equal to the dimension of $H^i(X, W(\mathcal{O}_X)) \otimes_W K$.

Corollary 4.4. Let X be as in Theorem 4.2. Then for all $i \geq 0$, the differential

$$d_1^{i,0} : H^i(X, W(\mathcal{O}_X)) \longrightarrow H^i(X, W\Omega^1_X) \tag{4.8}$$

is zero. □

Proof. This is immediate from Theorem 4.2 and [11]. ■

Remark 4.5. Let X be any smooth, projective variety. It has been shown in [11], that for all j, $H^1(X, W\Omega^j_X)$ are finite type W-modules.

Remark 4.6. By combining the theory of dominoes with Theorem 4.2 and Remark 4.5, it can be shown that Frobenius split, smooth, projective threefolds are Hodge-Witt [12].

5 Examples

We begin by recalling Mehta's question.

Question 5.1. Is any smooth projective, Frobenius split variety over a perfect field of characteristic p ordinary or of Hodge-Witt type?

We know by Proposition 3.1 that a smooth, projective Frobenius split surface is ordinary. In [12], it is shown that any Frobenius split smooth, projective threefold is Hodge-Witt. Further it is known that for abelian varieties, the notions of Frobenius splitting and ordinarity coincide [16]. However, in contrast to the expectation created by these results, we show in this section that the first question is false in dimensions greater than two, and the second question is false in dimensions greater than three.

We also give examples of varieties which are Hodge-Witt, but are not ordinary. These examples also provide examples of smooth, projective varieties $f : X \to Y$, such that both Y and the (smooth) fibers of f are ordinary, but X is not ordinary.

5.1 Compatible splittings and blowups

The key result we need is a criteria on the relative embedding of a smooth subvariety in a smooth, Frobenius split variety, such that the blowup along the subvariety remains Frobenius split. We recall now some of the concepts and results regarding compatible Frobenius splitting of subvarieties (see [15, 17]). Let X be a Frobenius split variety, and let

$$\sigma : F_*(\mathcal{O}_X) \longrightarrow \mathcal{O}_X \tag{5.1}$$

be a splitting of the Frobenius morphism. Suppose that Y is a subvariety of X, defined by a sheaf of ideals $\mathcal{I}_Y \subset \mathcal{O}_X$. In this case, we have a notion of Y being compatibly Frobenius split in X as follows.

Definition 5.2. The subvariety Y is said to be compatibly split by σ in X, if

$$\sigma(F_*(\mathcal{I}_Y)) \subset \mathcal{I}_Y. \tag{5.2}$$

Let X be a nonsingular variety and Y a nonsingular subvariety of codimension $d \geq 2$. Denote by $B_Y(X)$ the blowup of X along Y, and by E the exceptional divisor. For a smooth, projective variety X, let K_X denote the canonical divisor. The following result follows quite easily from [13, Proposition 2.1].

Proposition 5.3. Let $s \in H^0(X, \omega_X^{-1})$. Suppose that s^{p-1} is a splitting section of X, and that it vanishes to order at least $(d-1)$ generically along Y. Then s^{p-1} extends to a splitting of $B_Y(X)$. □

Proof. Let $\pi : B_Y(X) \to X$ be the blowing-up morphism. The canonical divisors of $B_Y(X)$ and X are related by

$$K_{B_Y(X)} = \pi^*(K_X) + (d-1)E. \tag{5.3}$$

Dually, multiplying both sides by $(1-p)$, we obtain

$$(1-p)K_{B_Y(X)} = \pi^*\big((1-p)K_X\big) - (p-1)(d-1)E. \tag{5.4}$$

Hence a section of ω_X^{1-p} remains regular as a section of $\omega_{B_Y(X)}^{1-p}$ if and only if it vanishes to order at least $(d-1)(p-1)$ generically along Y. ■

On the other hand, we have the following criterion for a blowup to be ordinary or Hodge-Witt [6, 10].

Proposition 5.4. The blowup $B_Y(X)$ is ordinary (or Hodge-Witt) if and only if both the variety X along the subvariety Y is ordinary (resp., Hodge-Witt). □

The proof of this proposition follows from the decomposition of W-modules, compatible with the action of the Frobenius [6, IV 1.1.9],

$$H^j(X, W\Omega_X^i) \oplus \left(\bigoplus_{0<l<d} H^{j-l}(Y, W\Omega_Y^{i-l})\right) \xrightarrow{\sim} H^j(B_Y(X), W\Omega^i_{B_Y(X)}), \tag{5.5}$$

and the fact that a smooth, proper variety Z is ordinary if and only if

$$F : H^j(Z, W\Omega_Z^i) \longrightarrow H^j(Z, W\Omega_Z^i) \tag{5.6}$$

is an isomorphism for all i and j.

Combining the above propositions, we obtain the following theorem.

Theorem 5.5. Let X be a smooth, projective Frobenius split variety, with a splitting section s^{p-1} as above. Suppose that s vanishes to order at least $(d-1)$ generically along a smooth subvariety Y of codimension d in X. Further, assume that Y is not ordinary (or not Hodge-Witt). Then $B_Y(X)$ is Frobenius split but not ordinary (resp., not Hodge-Witt). □

Remark 5.6. It can be seen from local computations that if a splitting section s^{p-1} of ω_X^{1-p} vanishes to order at least $(p-1)(d-1)$ along Y, then s vanishes to order either d or $(d-1)$ generically along Y. Further, if s vanishes to order d generically along Y, then E is compatibly split in $B_Y(X)$, and it follows that Y is also compatibly split in X. Thus in the theorem above, it follows that s vanishes precisely to order $(d-1)$ along Y.

5.2 Examples

We can now give the examples of Frobenius split varieties which are not ordinary nor Hodge-Witt.

Example 5.7. Let E be a supersingular elliptic curve in the projective space $\mathbf{P}^3$. It is known that E is contained in the zero locus of nondegenerate quadric q. Let t_1 and t_2 be linear polynomials such that the ideals generated by choosing any combination of q, t_1, t_2 define complete intersection subvarieties in $\mathbf{P}^3$. Then it can be checked, using Proposition 2.2, that the section $s = qt_1t_2 \in \mathcal{O}(4)$ gives rise to a splitting of $\mathbf{P}^3$, vanishing to order 1 along E. Hence the blowup of $\mathbf{P}^3$ is Frobenius split (and is Hodge-Witt) but is not ordinary.

Example 5.8. A natural question that arises in the study of the geometry of ordinary varieties is whether a variety is ordinary, if it is fibered over an ordinary variety, such that the smooth fibres are ordinary. The above example also provides an example of a variety fibered over $\mathbf{P}^1$, such that the (smooth) fibers are ordinary but the variety itself is not ordinary. The elliptic curve is defined as the complete intersection of two nondegenerate quadrics, which generates a pencil of quadrics. The strict transform of these quadrics in the blowup gives a fibration of $B_E(\mathbf{P}^3)$ over $\mathbf{P}^1$. It is well known that smooth quadrics are Frobenius split varieties. By Proposition 3.1, we know that Frobenius split (smooth) surfaces are ordinary (this can also be checked more directly in the present case as all the fibres are quadrics), and so this gives us the desired example.

Example 5.9. The above example can be generalized. Let Y be a smooth hypersurface in $\mathbf{P}^{n+1} \subset \mathbf{P}^{n+2}$, for example, a Fermat hypersurface of degree m. Choose a system of coordinates $x_0, \ldots, x_{n+2}$ on $\mathbf{P}^{n+2}$, where $\mathbf{P}^{n+1}$ is given by $x_0 = 0$. By Proposition 2.2, it follows that $s^{p-1} = (x_0 \cdots x_{n+2})^{p-1}$ is a splitting section vanishing precisely to order $(p-1)$ generically along Y. The blowup of $\mathbf{P}^{n+2}$ along Y is then Frobenius split. Recall that results of [18] give explicit conditions on (n, p, m) under which the Fermat hypersurface Y is not Hodge-Witt. For instance assume that $p \not\equiv 1 \bmod m$, p does not divide m and $n \geq 5$, $m \geq 5$. Then this hypersurface is not Hodge-Witt and so the blowup is not Hodge-Witt. But the blowup is Frobenius split but neither Hodge-Witt nor ordinary. The results of [18] can also be used to give examples in dimensions four and five as well.

Example 5.10. Let $E/\mathbb{Q}$ be an elliptic curve. We embed E in $\mathbf{P}^3$ using the embedding given by the linear system $4(\infty)$. The blowup of $\mathbf{P}^3$ along E has F-split and Hodge-Witt reduction at all but finite number of primes. In fact, by the blowup formula for Hodge-Witt cohomology, the blowup of $\mathbf{P}^3$ along any smooth projective embedded curve is Hodge-Witt. By [5], we know that the reduction of E is supersingular at infinitely many primes, and if E has CM, then it has supersingular reduction at a set of primes of density $1/2$. Hence there are infinitely many primes where the blowup has F-split (and Hodge-Witt) but nonordinary reduction.

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