Math. Ann. 333, 797–809 (2005) **Mathematische Annalen**

DOI: 10.1007/s00208-005-0697-4

Congruences for rational points on varieties over finite fields

N. Fakhruddin and C. S. Rajan

Received: March 4, 2004 / Revised: April 7, 2005
Published online: October 28, 2005 –

Abstract. We prove the existence of rational points on singular varieties over finite fields arising as degenerations of smooth proper varieties with trivial Chow group of 0-cycles. We also obtain congruences for the number of rational points of singular varieties appearing as fibres of a proper family with smooth total and base space and such that the Chow group of 0-cycles of the generic fibre is trivial. In particular this leads to a vast generalization of the classical Chevalley-Warning theorem. The above results are obtained as special cases of our main theorem which can be viewed as a relative version of a theorem of H. Esnault on the number of rational points of smooth proper varieties over finite fields with trivial Chow group of 0-cycles.

1. Introduction

In this paper we prove the existence of rational points on singular varieties over finite fields arising as degenerations of smooth proper varieties with trivial Chow group of 0-cycles. We also obtain congruences for the number of rational points of singular varieties appearing as fibres of a proper family with smooth total and base space and such that the Chow group of 0-cycles of the generic fibre is trivial. In particular this leads to a vast generalization of the classical Chevalley-Warning theorem. The above results are obtained as special cases of our main theorem which can be viewed as a relative version of the following theorem of H. Esnault [10]: if X is a smooth proper variety over a finite field k with $CH_0(X_{\overline{k(X)}})_{\mathbb{Q}} = \mathbb{Q}$ then $|X(k)| \equiv 1 \mod |k|$.

One of the main points of the paper is that our results are valid for proper morphisms of smooth varieties rather than for morphisms of smooth proper varieties. It is possible to extend Esnault's theorem to a relative version for proper varieties using the classical theory of correspondences. To deal with non-proper varieties, we introduce the notion of proper correspondences which plays a key role in our proofs. This requires the use of the refined intersection theory developed by Fulton and Macpherson [11]. The use of Bloch's decomposition of the diagonal in Esnault's proof is replaced here by a relative version. The other ingredients of

N. Fakhruddin and C. S. Rajan
Tata Institue of Fundamental Research, School of Mathematics, Homi Bhabha Road, Mumbai 400005, India
(e-mail: naf@math.tifr.res.in, rajan@math.tifr.res.in)

our proofs are the Grothendieck-Lefschetz trace formula and Deligne's integrality theorem [6] for the eigenvalues of Frobenius on l-adic cohomology.

The methods of this paper can be applied in other contexts where the yoga of weights is applicable. For a Hodge theoretic illustration see Theorem 4.3.

1.1.

The main result of this paper is the following:

Theorem 1.1. *Let $f_i : X_i \to Y$, $i = 1, 2$ be proper dominant morphisms of smooth irreducible varieties over a finite field k and let $g : X_1 \to X_2$ be a dominant morphism over Y. Let Z_i be the generic fibre of f_i, let $Z_{i\overline{k(X_1)}} = Z_i \times_{k(Y)} \overline{k(X_1)}$, and assume that $g_* : CH_0(Z_{1\overline{k(X_1)}})_{\mathbb{Q}} \to CH_0(Z_{2\overline{k(X_1)}})_{\mathbb{Q}}$ is an isomorphism. Then for all $y \in Y(k)$,*

$$|f_1^{-1}(y)(k)| \equiv |f_2^{-1}(y)(k)| \mod |k|.$$

We remark that there are no flatness assumptions in the above theorem. Specialising to the case $X_2 = Y$ and $f_2 = Id_Y$, we obtain:

Corollary 1.2. *Let $f : X \to Y$ be a proper dominant morphism of smooth irreducible varieties over a finite field k. Let Z be the generic fibre of f and assume that $CH_0(Z_{\overline{k(X)}})_{\mathbb{Q}} = \mathbb{Q}$. Then for all $y \in Y(k)$, $|f^{-1}(y)(k)| \equiv 1 \mod |k|$.*

When $Y = \mathrm{Spec}(k)$, the corollary reduces to the theorem of H. Esnault cited above. Since we do not assume that f is smooth, we are able to obtain congruences even for singular varieties.

An immediate consequence of Corollary 1.2 is that for $f : X \to Y$ as above, $|X(k)| \equiv |Y(k)| \mod |k|$. By taking Y to be a point in Theorem 1.1 we obtain:

Corollary 1.3. *Let $g : X_1 \to X_2$ be a dominant morphism of smooth proper varieties over a finite field k. If $g_* : CH_0(X_{1\overline{k(X_1)}})_{\mathbb{Q}} \to CH_0(X_{2\overline{k(X_1)}})_{\mathbb{Q}}$ is an isomorphism, then $|X_1(k)| \equiv |X_2(k)| \mod |k|$.*[1]

The above corollary as well as the Hodge theoretic analogue (see Section 4) applies to arbitrary proper birational morphisms of smooth varieties (see also Corollary 3.1 for a slightly stronger version in positive characteristics). This gives new restrictions on the cohomology of varieties which can occur as fibres of such morphisms.

In contrast to Corollary 1.2, the assumption that X_1 and X_2 be proper cannot be omitted here. For example, consider a pencil $h : X \to \mathbb{P}^1$ of high genus

[1] This can also be proved using results B. Kahn; see Remark 1 of [13].

curves in $\mathbb{P}^2$ with smooth generic fibre. Let X_2 be any affine open subset of $\mathbb{P}^1$, $X_1 = h^{-1}(X_2)$, and let $g : X_1 \to X_2$ be the induced map. In this case $CH_0(X_{1\overline{k(X_1)}})_{\mathbb{Q}} = CH_0(X_{2\overline{k(X_1)}})_{\mathbb{Q}} = 0$, but the fibres need not have rational points.

The triviality of the Chow group of zero cycles of degree 0, or even rational chain connectedness is not sufficient to guarantee the existence of a rational point for proper varieties over finite fields which are not smooth (see Remark 3.4). However, using alterations, we give a criterion for the existence of rational points which can be applied to all degenerations of smooth rationally chain connected varieties.

Corollary 1.4. *Let $f : X \to Y$ be a proper dominant morphism of irreducible varieties over a finite field k with Y smooth. Let Z be the generic fibre of f and assume that Z is smooth and $CH_0(Z_{\overline{k(X)}})_{\mathbb{Q}} = \mathbb{Q}$. Then for any $y \in Y(k)$, $f^{-1}(y)(k) \neq \emptyset$.*

The corollary below generalises the Chevalley–Warning theorem [12], which is the special case $P = \mathbb{P}^n$, $r = 1$ and $L_1 = \mathcal{O}(d)$ with $d \leq n$.

Corollary 1.5. *Let P be a smooth projective geometrically irreducible variety over a finite field k. Let $L_1, \cdots, L_r$ be very ample line bundles on P such that $(K_P \otimes L_1 \otimes \cdots \otimes L_r)^{-1}$ is ample, where K_P is the canonical bundle of P. For $i = 1, 2, \ldots, r$, let $s_i \in H^0(P, L_i)$. Then*

$$\left|\left\{x \in P(k) \middle| s_i(x) = 0\,,\ i = 1, 2, \ldots, r\right\}\right| \equiv 1 \mod |k|\,.$$

Note that the congruence formula of Katz [6], when it applies, only gives congruences modulo $p = \mathrm{char}(k)$. Examples of varieties to which Corollary 1.5 can be applied include toric Fano varieties and homogenous spaces G/P (with G a semisimple algebraic group and P a reduced parabolic subgroup) of index > 1, since in these cases ample line bundles are always very ample.

In [1], Bloch, Esnault and Levine formulate and prove a motivic version of the Chevalley-Warning theorem. In their work, they use the embedding of the hypersurface in the smooth ambient variety $\mathbb{P}^n$ and their theory of motivic cohomology with modulus. In contrast, we work with the family of all hypersurfaces, the total space of which is smooth, and use ordinary Chow groups.

It would be interesting to know whether an analogue of the Ax–Katz theorem [14] holds in the above situation or whether all low degree intersections as above over C_1 fields always have rational points.

1.2.

We now give a brief indication of the methods of our paper. For simplicity we begin by describing the proof specialised to Corollary 1.3: let $g : X_1 \to X_2$ be as in Corollary 1.3, and let W be a multisection of g, i.e. an irreducible subvariety of X_1 mapping dominantly and generically finitely onto X_2 with degree d. Let Γ_g be the graph of g, and let Γ_W be the transpose of the graph of $g|_W$ embedded in $X_2 \times X_1$. Let $\Gamma_1 = \Gamma_W \circ \Gamma_g$ be the correspondence in $X_1 \times_k X_1$ given by the composition of the correspondences Γ_W and Γ_g. Consider the two classes η_1 and η_2 in $CH_0(\overline{k(X_1)} \times_k X_1)$ given respectively by the pullback of the diagonal Δ_{X_1} and Γ_1/d. By construction both these classes project to the same class in $CH_0(\overline{k(X_1)} \times_k X_2)$ given by the pullback of the graph of g. The injectivity hypothesis on the Chow groups implies that $\eta_1 = \eta_2$. Hence there exists a correspondence Γ_2 in $X_1 \times_k X_1$ whose support maps to a proper subvariety of X_1 via the first projection such that

$$\Delta_{X_1} = \Gamma_1/d + \Gamma_2.$$

Deligne's integrality theorem then implies that the eigenvalues of the (geometric) Frobenius acting on the cokernel of the map $g^* : H^*_{et}((X_2)_{\bar{k}}, \mathbb{Q}_l) \to H^*_{et}((X_1)_{\bar{k}}, \mathbb{Q}_l)$ are algebraic integers divsible by $|k|$. The proof of Corollary 1.3 is completed by appealing to the Grothendieck-Lefschetz trace formula.

For the proof of Theorem 1.1, we apply the diagonal decomposition described above to the induced morphism on the generic fibres. The induced decomposition is then spread out to obtain a relative decomposition of the diagonal Δ_{X_1} in $X_1 \times_Y X_1$. Since the varieties need not be proper, we introduce the notion of proper correspondences. This can be applied in our context since the morphisms involved are all proper. This gives rise to a relative diagonal decomposition consisting of proper correspondences (see the following section for the definition and properties of proper correspondences). The rest of the proof proceeds as indicated above.

2. Proper Correspondences

2.1.

In this section we introduce the concept of proper correspondences and prove some of its properties.[2] Most of the proofs are simple modifications of those in Fulton's book [11] and we prove only what we need for later use; several other results in [11, Chapter 16] for (usual) correspondences have analogues for proper correspondences.

Definition 2.1. *Let X and Y be smooth irreducible varieties over a field k. The group of* proper correspondences *from X to Y, $PCorr(X, Y)$, is the free abelian*

[2] A. Nair has informed us that a variant of this definition has been considered, in a different context, by E. Urban in the preprint: Sur les représentations p-adiques associées aux représentations cuspidales de $GSp_4/\mathbb{Q}$, 2001.

group generated by irreducible subvarieties $\Gamma \subset X \times Y$ *which are proper over* Y *modulo the subgroup generated by elements of the form*

$$\{\mathrm{div}(f) | f \in k(Z)^*,\ Z \subset X \times Y \textit{ irreducible and proper over } Y\}\ .$$

This is clearly a graded abelian group; we shall grade it by dimension (lower indices) or codimension (upper indices) as is convenient. For a cycle Γ (not necessarily irreducible) in the free abelian group as above we shall denote its class in $PCorr(X, Y)$ by $[\Gamma]$.

If $f : X \to Y$ is a proper morphism, then the graph of f gives an element $[\Gamma_f]$ of $PCorr(X, Y)$. We shall show below that proper correspondences induce maps on cohomology generalising the maps induced by proper morphisms.

2.2.

We recall some properties of étale (co)homology and cycle class maps that we shall need. The main reference is [7, Exposés VI-IX]; we note that the quasi-projectivity hypotheses there can be removed using the methods of [11]. The compatibility of refined intersection products and refined cycle class maps stated below can be deduced using the methods of [11, Chapter 19]; see also [3].

Let K be an algebraically closed field and fix a prime number $l \neq char(K)$. For a variety X over K let $H^i(X) := H^i_{et}(X, \mathbb{Q}_l)$, $H^i_c(X) := H^i_{c,et}(X, \mathbb{Q}_l)$ and let $H^i_Z(X) := H^i_{Z,et}(X, \mathbb{Q}_l)$ for Z a closed subvariety of X. We also let $H_i(X)$ denote the locally finite (or Borel-Moore) homology $H^{et}_i(X, \mathbb{Q}_l)$. For any of the above groups H, we denote by $H(n)$, for an integer n, the corresponding Tate twisted group.

For a cycle $\alpha = \sum_i a_i Z_i$, on a variety X, we denote by $|\alpha|$ the support $\cup_i Z_i$ of α. The following properties are proved in the references cited above:

(1) (Projection formula) For any variety X there are cap product maps $H^i(X) \otimes H_j(X) \to H_{j-i}(X)$ such that if $f : X \to Y$ is a proper morphism, $u \in H^i(Y)$, $v \in H_j(X)$ then $f_*(f^*(u) \cap v) = u \cap f_*(v)$ in $H_{j-i}(Y)$.
(2) For Z an irreducible variety of dimension n, $H_{2n}(Z)(-n)$ is one dimensional.
(3) For α a cycle of dimension k on a variety X the fundamental class defines a canonical element $cl(\alpha) \in H_{2k}(|\alpha|)(-k)$. This maps to an element, also denoted by $cl(\alpha)$, in $H_{2k}(X)(-k)$ which is zero if $\alpha = 0 \in CH_k(X)$.
(4) For X smooth of pure dimension n there is a canonical isomorphism $H_{2k}(|\alpha|)\ (-k) \cong H^{2n-2k}_{|\alpha|}(X)(n-k)$, so we also get an element $cl(\alpha) \in H^{2n-2k}_{|\alpha|}(X)(n-k)$.
(5) For $f : X \to Y$ a morphism and α a cycle on X of dimension n whose support is proper over Y, $f_*(cl(\alpha)) = cl(f_*(\alpha))$ in $H^*_{f(|\alpha|)}(Y)(-n)$.

(6) For α and β cycles of pure dimension k and l respectively in a smooth irreducible variety X of dimension n

$$cl(\alpha \cdot \beta) = cl(\alpha) \cup cl(\beta) \in H^{4n-2(k+l)}_{|\alpha| \cap |\beta|}(X)(n-k-l) ,$$

where the product on the left is the refined intersection product.

2.3.

Let $\dim(X) = n$, $\dim(Y) = m$ and let Γ be a representative for an element of $PCorr_r(X, Y)$. Then Γ induces a linear map $\Gamma_* : H^i(X) \to H^{2m-2r+i}(Y)(m-r)$ as the composite of the sequence of linear maps:

$$H^i(X) \xrightarrow{p_X^*} H^i(X \times Y) \xrightarrow{\cdot cl(\Gamma)} H^{i+2(n+m-r)}_{|\Gamma|}(X \times Y) \cong$$
$$H_{2r-i}(|\Gamma|)(-r) \xrightarrow{p_{Y*}} H_{2r-i}(Y)(-r) \cong H^{2m-2r+i}(Y)(m-r) .$$

Since X and Y are smooth, by duality we also get maps $\Gamma^* : H^i_c(Y) \to H_c^{2n-2r+i}(X)(n-r)$.

Now suppose that k is a finite field and K an algebraic closure of k. All the cohomology groups discussed above, for varieties over K which are base changed from varieties over k, have a continuous action of $Gal(K/k)$. The maps Γ^* and Γ_* are then compatible with the Galois action.

The following lemma implies that the maps Γ_* and Γ^* only depends on $[\Gamma] \in PCorr(X, Y)$.

Lemma 2.2. *Let Γ be as above. Suppose there exists a closed subvariety $Z \subset X \times Y$ such that $|\Gamma| \subset Z$, $[\Gamma] = 0$ in $CH_*(Z)$ and $p_Y|_Z : Z \to Y$ is proper. Then Γ^* and Γ_* are the zero maps.*

Proof. Suppose Z is a closed subvariety of $X \times Y$ such that the projection $p_Y|_Z : Z \to Y$ is proper. It follows from the projection formula that Γ_* can also be defined as,

$$H^i(X) \xrightarrow{p_X|_Z^*} H^i(Z) \xrightarrow{\cdot cl(\Gamma)} H_{2r-i}(Z)(-r)$$
$$\xrightarrow{p_{Y*}} H_{2r-i}(Y)(-r) \cong H^{2m-2r+i}(Y)(m-r),$$

where $cl(\Gamma)$ is considered as an element in $H_{2r}(Z)(-r)$. The lemma follows from the fact that $cl(\Gamma) = 0$ in $H_{2r}(Z)(-r)$. The statement for Γ^* follows by duality. □

2.4.

Let X, Y, Z be smooth irreducible varieties over K and let $[\Gamma_1]$ (resp. $[\Gamma_2]$) be a proper correspondence from X to Y (resp. Y to Z). Analogous to the definition of composition of correspondences [11, Chapter 16], we define $[\Gamma_2] \circ [\Gamma_1] \in PCorr(X, Z)$ by

$$[\Gamma_2] \circ [\Gamma_1] = [p_{XZ*}(p_{XY}{}^*(\Gamma_1) \cdot p_{YZ}{}^*(\Gamma_2))] ,$$

where the p's denote the projection morphisms from $X \times Y \times Z$ and the product is the refined intersection product. Note that $p_{XY}{}^*(\Gamma_1) \cdot p_{YZ}{}^*(\Gamma_2)$ is a cycle which is well defined upto rational equivalence on $|\Gamma_1| \times_Y |\Gamma_2|$. Since $|\Gamma_1| \times_Y |\Gamma_2|$ is proper over Z, its image in $X \times Z$ is also proper over Z, so $[\Gamma_2] \circ [\Gamma_1]$ is a well defined element of $PCorr(X, Z)$.

Lemma 2.3. *Let X, Y, Z and Γ_1, Γ_2 be as above. Then $(\Gamma_2 \circ \Gamma_1)_* = (\Gamma_2)_* \circ (\Gamma_1)_*$ as maps from $H^*(X)$ to $H^*(Z)$ and $(\Gamma_2 \circ \Gamma_1)^* = (\Gamma_1)^* \circ (\Gamma_2)^*$ as maps from $H_c^*(Z)$ to $H_c^*(X)$.*

Proof. The key point is the compatibility of the refined cycle class maps with refined intersection products.

Let $a \in H^*(X)$. Then

$$\begin{aligned}
(\Gamma_2)_* \circ (\Gamma_1)_*(a) &= p_Z^{YZ}{}_*(cl(\Gamma_2) \cdot p_Y^{YZ*}(p_Y^{XY}{}_*(cl(\Gamma_1) \cdot p_X^{XY*}(a)))) \\
&= p_Z^{YZ}{}_*(cl(\Gamma_2) \cdot p_{YZ}^{XYZ}{}_*(p_{XY}^{XYZ*}(cl(\Gamma_1) \cdot p_X^{XY*}(a)))) \\
&= p_Z^{YZ}{}_*(p_{YZ}^{XYZ}{}_*(p_{YZ}^{XYZ*}(cl(\Gamma_2)) \cdot p_{XY}^{XYZ*}(cl(\Gamma_1) \cdot p_X^{XY*}(a)))) \\
&= p_Z^{YZ}{}_*(p_{YZ}^{XYZ}{}_*(p_{YZ}^{XYZ*}(cl(\Gamma_2)) \cdot p_{XY}^{XYZ*}(cl(\Gamma_1)) \cdot p_X^{XYZ*}(a)))) \\
&= p_Z^{XZ}{}_*(p_{XZ}^{XYZ}{}_*(p_{YZ}^{XYZ*}(cl(\Gamma_2)) \cdot p_{XY}^{XYZ*}(cl(\Gamma_1)) \cdot p_X^{XYZ*}(a)))) \\
&= p_Z^{XZ}{}_*(cl(\Gamma_2 \circ \Gamma_1) \cdot p_X^{XZ*}(a)) \\
&= (\Gamma_2 \circ \Gamma_1)_*(a)
\end{aligned}$$

We use the projection formula several times along with compatibility of the cycle class map with smooth pullbacks, products and proper pushforwards. □

2.5.

The key technical result which allows us to deduce congruences from cycle theoretic information is the following:

Proposition 2.4. *Let X, Y, Γ be as above and assume that $m = n = r$. If $\dim(p_X(|\Gamma|)) < n$ then all the eigenvalues of (the geometric) Frobenius acting on $\Gamma^*(H_c^i(Y)) \subset H_c^i(X)$ are algebraic integers which are divisible by $|k|$.*

Proof. Replacing k by a finite extension does not affect the conclusion of the lemma, so without loss of generality we may assume that Γ is a geometrically irreducible subvariety of $X \times Y$. Using the definition of Γ_* as given in the proof of Lemma 2.2, Γ_* is the composite of the following maps:

$$H^i(X) \xrightarrow{p_X^*} H^i(\Gamma) \longrightarrow H_{2n-i}(\Gamma)(-n) \xrightarrow{p_Y|\Gamma_*} H_{2n-i}(Y)(-n) \cong H^i(Y) ,$$

where we have used the hypothesis that $m = n = r$.

Suppose $\pi : \Gamma' \to \Gamma$ is a proper dominant generically finite morphism with Γ' smooth and geometrically irreducible. Then the projection formula shows that we may replace Γ with Γ' and p_X (resp. p_Y) with $p_X\pi$ (resp. $p_Y\pi$) in the above without changing the image of the composite.

Let W be the Zariski closure of $p_X(\Gamma)$ in X and let $\dim(W) = t$. By the theorem of De Jong [4, 4.1] we may find Γ' as above, $\sigma : W' \to W$ with σ proper dominant generically finite and W' smooth, and $p : \Gamma' \to W'$ such that $p_X\pi = i_W\sigma p$, where $i_W : W \to X$ is the inclusion. Since $p_X\pi = (i_W\sigma)p$ and both Γ' and W' are smooth, it follows from the functoriality of pushforward maps that $(p_X\pi)_* = (i_W\sigma)_* p_*$. By the remarks of the previous paragraph we see that the image of Γ_* is the same as that of the composite of the sequence:

$$H^i(X) \longrightarrow H^i(W') \longrightarrow H^i(\Gamma') \longrightarrow H^i(Y) .$$

Since Γ' and W' are smooth, by duality we get a factorisation of Γ^* as a composite of maps,

$$H_c^j(Y) \longrightarrow H_c^j(\Gamma') \longrightarrow H_c^{2(t-n)+j}(W')(t-n) \longrightarrow H_c^j(X) .$$

By Deligne's integrality theorem [6], Exposé XXI, Corollaire 5.5.3, all the eigenvalues of Frobenius on $H_c^*(W')$ are algebraic integers. Since $n - t > 0$ and the geometric Frobenius acts on $\mathbb{Q}_l(t-n)$ by $|k|^{n-t}$, the proposition follows. □

3. Proofs of the main results

Using the results of the previous section, we now give the proofs of the results stated in the introduction.

3.1.

Proof of Theorem 1.1. If Y is not geometrically irreducible, then $Y(k) = \emptyset$ so there is nothing to prove. If $X_2(k) = \emptyset$ then $X_1(k) = \emptyset$ and the theorem is true. Thus we can assume that $X_2(k)$ is nonempty and this implies that X_2 is geometrically irreducible. If X_1 is not geometrically irreducible, then after a base change to a finite extension of k, X_1 becomes a union of at least two connected components, each of which maps dominantly to the base change of $\overline{X_2}$. Hence there is more than one connected component of Z_1 base changed to $\overline{k(X_1)}$ which maps surjectively to each connected component of Z_2 base changed to $\overline{k(X_1)}$. This contradicts the injectivity of the map on Chow groups. Hence we can assume that X_1 is also geometrically irreducible.

Let W be an irreducible subvariety of X_1 which maps generically finitely and dominantly to X_2 and let d be the degree of W over X_1. Let Γ_g be the graph of g in $X_1 \times_k X_2$, let Γ_W be the transpose of the graph of $g|_W$ embedded in $X_2 \times_k X_1$ and let $\Gamma_1 = (\Gamma_W \circ \Gamma_g)/d$. Since W is a subvariety of X_1, Γ_W is a proper correspondence, and Γ_g is a proper correspondence since g is proper. Hence Γ_1 is a proper correspondence of dimension $n_1 = \dim X_1$. Since $f_2 \circ g = f_1$, the correspondence Γ_1 is naturally represented by a cycle supported in $X_1 \times_Y X_1$.

Let V_2 be the open subset of X_2 over which $g|_W$ is finite and let $V_1 = g^{-1}(V_2)$. By the construction of W, $p_{12}^*(\Gamma_g)$ and $p_{23}^*(\Gamma_W)$, which are subvarieties of $X_1 \times_k X_2 \times_k X_1$, meet properly when pulled back to $V_1 \times_k X_2 \times_k X_1$. It follows that one can write $\Gamma_1 = \Gamma_1' + \Gamma_1''$ in $PCorr(X_1, X_1)$ where $(Id_{X_1} \times g)_*(\Gamma_1') = \Gamma_g$ in $PCorr(X_1, X_2)$ and $p_1(|\Gamma_1''|)$ is a proper subvariety of X_1.

Let Δ_{X_1} be the diagonal in $X_1 \times_k X_1$ and consider the proper correspondence $\Gamma_2 := \Delta_{X_1} - \Gamma_1$. Note that Γ_2 is represented by a cycle supported in $X_1 \times_Y X_1$. Hence we may consider its restriction (i.e. flat pullback) γ_2 to $Z_1 \times_{k(Y)} Z_1$. Since $(Id_{X_1} \times g)_*(\Delta_{X_1}) = \Gamma_g$, it follows from the previous paragraph that $(Id_{Z_1} \times g|_{Z_1})_*(\gamma_2)$ can be represented by a cycle on $Z_1 \times_{k(Y)} Z_2$ which becomes zero when restricted to $k(Z_1) \times_{k(Y)} Z_2$. Since the map $g_* : CH_0(Z_{1\overline{k(X_1)}})_{\mathbb{Q}} \to CH_0(Z_{2\overline{k(X_1)}})_{\mathbb{Q}}$ is injective, it follows that γ_2 can be represented by a cycle on $Z_1 \times_{k(Y)} Z_1$ whose support maps to a proper subvariety of Z_1 by the projection to the first factor. By taking the Zariski closures in $X_1 \times_Y X_1$, it follows that Γ_2 can be represented by a proper correspondence on $X_1 \times_k X_1$ whose support maps to a proper subvariety of X_1 by the projection to the first factor.

By Proposition 2.4 all the eigenvalues of Frobenius acting on the image of Γ_2^* in $H_c^*(X_1)$ are divisible by $|k|$. It follows from Lemma 2.3 and the definition of Γ_1, that the image of Γ_1^* is contained in the image of g^*. By the definition of Γ_2, we conclude that all the eigenvalues of Frobenius acting on the cokernel of $g^* : H_c^*(X_2) \to H_c^*(X_1)$ are divisible by $|k|$. Note that since g is dominant, the map g^* is injective.

The hypotheses of the theorem, and the above discussion, are not affected if we replace Y by an open subvariety U and X_i by $f_i^{-1}(U)$, $i = 1, 2$, so we may assume that Y has only one rational point y. Applying the Grothendieck-Lefschetz trace formula and the statement on eigenvalues above, we conclude that the $|X_1(k)| \equiv |X_2(k)| \mod |k|$. Each rational point of X_i must lie over the unique rational point y of Y, hence the theorem follows. □

Corollary 3.1. *Let $f : X \to Y$ be a proper dominant generically finite morphism of smooth irreducible varieties over a finite field k such that the extension* *of function fields $k(Y) \to k(X)$ is purely inseparable. Then for any $y \in Y(k)$,* $|f^{-1}(y)(k)| \equiv 1 \mod |k|$.

Proof. Since $(Z_{\overline{k(X)}})_{red}$ is isomorphic to $\mathrm{Spec}(\overline{k(X)})$, the hypothesis on CH_0 is trivially satisfied. □

Remark 3.2. Since we only use intersection theory (resp. cohomology groups) with rational (resp. $\mathbb{Q}_l$) coefficients, the above results hold even when X and Y are quotients of smooth varieties by finite groups.

3.2.

Proof of Corollary 1.4. As in the proof of Theorem 1.1, we may assume that X and Y are geometrically irreducible and that Y has a unique rational point.

By a result of De Jong [5, 5.15] there exists an irreducible variety X' over k which is the quotient of a smooth variety by a finite group and a proper dominant generically finite morphism $\pi : X' \to X$ such that the extension of function fields $k(X) \to k(X')$ is purely inseparable. Let $f' = f\pi : X' \to Y$ and let Z' be the generic fibre of f'. Since X' is irreducible over k, Z' is irreducible over $k(Y)$. Z is geometrically irreducible since it is smooth over $k(Y)$; since the extension of function fields induced by the map $Z' \to Z$ is $k(X) \to k(X')$, Z' is also geometrically irreducible. The induced morphism $(Z'_{\overline{k(X)}})_{red} \to Z_{\overline{k(X)}}$ thus satisfies the assumptions of Lemma 3.6 below, so $CH_0(Z'_{\overline{k(X')}})_{\mathbb{Q}} = \mathbb{Q}$. Applying Corollary 1.2 (cf. Remark 3.2) to the morphism $f' : X' \to Y$ we see that $X'(k) \neq \emptyset$. Thus $X(k) \neq \emptyset$. □

Remark 3.3. For singular X it is not always true that $|f^{-1}(y)(k)| \equiv 1 \mod |k|$, however the only examples we know where this fails are non-normal.

Remark 3.4. The following example[3] gives an example of a proper non-smooth variety over a finite field which has no rational point and such that the Chow group of zero cycles of degree 0 is trivial: let C be a smooth projective geometrically ireducible curve over k, a finite field, with $C(k) = \emptyset$ and let p be any closed point of C. Let t be a closed point of $\mathbf{P}^1_k$ of degree greater than one, and let X' be the blowup of $C \times_k \mathbf{P}^1_k$ with centre the closed subscheme $p \times_k t$. Let X be the normal variety obtained by blowing down the strict transform of $C \times_k t$ in X'. Then X is rationally chain connected but $X(k) = \emptyset$.

Remark 3.5. $X(k) \neq \emptyset$ if X is a proper variety over a finite field k which is rationally connected i.e. any two general points of $X(\Omega)$ are contained in an irreducible rational curve in X defined over Ω, where $\Omega \supset k$ is a universal domain.

[3] We learnt of such an example, due to J. Kollár (unpublished), from a talk by J. Iyer at the University of Essen in May 2003.

(To prove this we may replace X by X' as in the proof of Corollary 1.4. Since the extension of function fields $k(X) \to k(X')$ is purely inseparable, it follows that X' is also rationally connected, so $CH_0(X'_{\overline{k(X')}})_{\mathbb{Q}} = \mathbb{Q}$. By Remark 3.2 it follows that $X'(k) \equiv 1 \mod |k|$, hence $X'(k) \neq \emptyset$. Thus $X(k) \neq \emptyset$.) However, a degeneration of a rationally connected variety is in general only rationally chain connected, so one cannot use this to prove Corollary 1.4 even if Z is rationally connected.

Lemma 3.6. *Let $f : X \to Y$ be a proper dominant morphism of irreducible varieties over an algebraically closed field K. Assume that Y is smooth, f is generically finite and the extension of function fields $K(Y) \to K(X)$ is purely inseparable. Then $f_* : CH_0(X)_{\mathbb{Q}} \to CH_0(Y)_{\mathbb{Q}}$ is an isomorphism.*

Proof. f_* is surjective because f is surjective. Using the refined intersection theory of [11] and the hypothesis on Y, we see that there is a natural map $f^* : CH_0(Y)_{\mathbb{Q}} \to CH_0(X)_{\mathbb{Q}}$. By the assumptions on f there exists an open subset U of Y such that for $V = f^{-1}(U)$, $f|_V : V \to U$ is a bijection. By the moving lemma (which is trivial for zero cycles) $CH_0(X)$ (resp. $CH_0(Y)$) is generated by the closed points in V (resp. U). This shows that f^* is a surjection since for $y \in U$, $f^*([y]) = [f^{-1}(y)]$. Since $f_* f^*$ is multiplication by $\deg(f)$, it follows that f_* is an isomorphism. □

3.3.

Proof of Corollary 1.2. Let $Y = \prod_i H^0(P, L_i)$ and let $X \subset Y \times P$ be the zero scheme of the universal section of $\bigoplus_i p_P^* L_i$. Since the L_i are basepoint free, the map $X \to P$ is smooth, hence X is also smooth. Since the L_i are assumed to be very ample, Bertini's theorem implies that the generic fibre Z of the projection $f : X \to Y$ is smooth. The assumptions on the L_i and the adjunction formula imply that Z is a Fano variety. By a theorem of Campana [2] and Kollar-Miyaoka-Mori [15] Z is rationally chain connected, so $CH_0(Z_{\overline{k(X)}})_{\mathbb{Q}} = \mathbb{Q}$. The proof is concluded by applying Corollary 1.2 to f. □

Remark 3.7. The proof shows that instead of assuming that the L_i are very ample it suffices to assume that they are basepoint free and that Z is smooth.

4. Further questions

It seems likely that the following mixed characteristic analogue of Corollary 1.4 has a positive answer:

Question 4.1. Let K be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ its ring of integers, k the residue field and X a smooth proper variety over K such that $CH_0(X_{\overline{K(X)}})_{\mathbb{Q}} = \mathbb{Q}$. If

$\mathcal{X} \to Spec(\mathcal{O})$ is a proper scheme with generic fibre isomorphic to X and closed fibre X_0, then is $X_0(k) \neq \emptyset$?

From our proofs we do not obtain any information about the valuations of the eigenvalues of Frobenius acting on the étale cohomology of the fibres of f. One is thus lead to ask the following:

Question 4.2. Let $f : X \to Y$ be a proper dominant morphism of smooth irreducible varieties over a finite field k. Let Z be the generic fibre of f and assume that $CH_0(Z_{\overline{k(X)}})_{\mathbb{Q}} = \mathbb{Q}$. Then for all $y \in Y(k)$ and $i > 0$, does $|k|$ divide all the eigenvalues of Frobenius acting on $H^i((f^{-1}(y))_{\overline{k}}, \mathbb{Q}_l)$ $(l \neq char(k))$?

Theorem 1.1 has the following Hodge theoretic analogue:

Theorem 4.3. *Let $f_i : X_i \to Y$, $i = 1, 2$, be proper dominant morphisms of smooth irreducible varieties over $\mathbb{C}$ and let $g : X_1 \to X_2$ be a dominant morphism over Y. Let Z_i be the generic fibre of f_i and assume that $g_* : CH_0(Z_{1\overline{\mathbb{C}(X_1)}})_{\mathbb{Q}} \to CH_0(Z_{2\overline{\mathbb{C}(X_1)}})_{\mathbb{Q}}$ is an isomorphism. Then $gr_0^F(Coker(g^* : H_c^i(X_1, \mathbb{C}) \to H_c^i(X_2, \mathbb{C}))) = 0$ for all i, where $F^\bullet$ denotes the Hodge filtration of Deligne.*

The proof is essentially the same as that of Theorem 1.1 so we omit the details: one only needs to replace Proposition 2.4 by its Hodge theoretic counterpart.

The Hodge version of Question 4.2 is:

Question 4.4. Let $f : X \to Y$ be a proper dominant morphism of smooth irreducible varieties over $\mathbb{C}$. Let Z be the generic fibre of f and assume that $CH_0(Z_{\overline{\mathbb{C}(X)}})_{\mathbb{Q}} = \mathbb{Q}$. Then for all $y \in Y(\mathbb{C})$ and $i > 0$, is $gr_0^F(H^i(f^{-1}(y), \mathbb{C})) = 0$?

It seems likely, as was suggested to us by P. Brosnan, that there should be a purely motivic statement from which Corollary 1.2 and Theorem 4.3 should follow after taking realisations. The category of motives over a base of Corti and Hanamura [3] would seem to be a natural choice in which to formulate such a statement.

Remark 4.5. After this paper was submitted both Question 4.2 and (a stronger form of) Question 4.4 were shown to have positive answers by H. Esnault; her results appear in the appendix to this article [8]. A positive answer to Question 4.1 in case $\mathcal{X}$ is regular was also obtained by her in [9]; the results of that paper include a stronger version of Corollary 1.2 in the case that Y is a curve.

Acknowledgements. N.F. thanks Arvind Nair for a remark which convinced him that Corollary 1.2 should be true (before Theorem 1.1 was proved) and Patrick Brosnan, Hélène Esnault, Madhav Nori and V. Srinivas for their comments on the results of this paper. He also thanks Hélène Esnault and Marc Levine for an invitation to visit the University of Essen in May–June 2003 where he was supported by the DFG and via the Wolfgang Paul prize program of the Humboldt Stiftung; lectures of Hélène Esnault that he attended at that time provided part of the motivation for the results and questions in this paper.

We also thank Bruno Kahn for his comments on the first version of this paper and for suggesting Theorem 1.1 as a common generalisation of Corollaries 1.2 and 1.3.

We thank the referee for his suggestions which helped in improving the exposition of the paper.

References

1. Bloch, S., Esnault, H., Levine, M.: Decomposition of the diagonal and eigenvalues of Frobenius for Fano hypersurfaces. math.AG/0302109, To appear in Am. J. Math.
2. Campana, F.: Connexité rationnelle des variétés de Fano. Ann. Sci. École Norm. Sup. **25**(4), 539–545 (1992)
3. Corti, A., Hanamura, M.: Motivic decomposition and intersection Chow groups. I. Duke Math. J., **103**, 459–522 (2000)
4. de Jong, A.J.: Smoothness, semi-stability and alterations. Inst. Hautes Études Sci. Publ. Math. 51–93 (1996)
5. de Jong, A.J.: Families of curves and alterations. Ann. Inst. Fourier (Grenoble), **47**, 599–621 (1997)
6. Deligne, P., Katz, N.: Groupes de monodromie en géométrie algébrique. II, Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie (SGA 7 II), Lecture Notes in Mathematics, Vol. **340** 1967–1969
7. Douady, A., Verdier, J. Séminaire de Géométrie Analytique, Société Mathématique de France, Paris, 1976. Tenu à l'École Normale Supérieure, Paris, pp. 1974–75, Astérisque, No. 36-37
8. Esnault, H.: Appendix to Congruences for rational points on varieties over finite fields. In: Fakhruddin, N. Rajan, C.S.(eds) math.NT/0403265
9. Esnault, H.: Deligne's integrality theorem in unequal characteristic and rational points over finite fields. math.NT/0405318
10. Esnault, H.: Varieties over a finite field with trivial Chow group of 0-cycles have a rational point. Invent. Math. **151**, 187–191 (2003)
11. Fulton, W.: Intersection theory. Springer-Verlag, Berlin, second ed. 1998
12. Greenberg, M.J.: Lectures on forms in many variables. In: Benjamin, W.A.(ed) Inc. New York-Amsterdam, 1969
13. Kahn, B.: Number of points of function fields over finite fields. math.NT/0210202
14. Katz, N.M.: On a theorem of Ax. Amer. J. Math., **93**, 485–499 (1971)
15. Kollár, J., Miyaoka, Y., Mori, S.: Rational connectedness and boundedness of Fano manifolds. J. Differential Geom. **36**, 765–779 (1992)