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On Strong Multiplicity One for l-adic Representations

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1 Introduction

A basic theme in the study of arithmetical questions is the local-global principle, by which a global arithmetical object is examined by means of its local behavior at the various primes. At the representation theoretic level, the problem of strong multiplicity one is the determination of the multiplicities of the fibres of the global to local map, especially in situations where we do not have the necessary information at all primes. Multiplicity one results have been used by Atkin and Lehner to establish an Euler product indexed by all primes, and the functional equation for newforms. They have also been of use in problems concerning base change, and other functorial questions in automorphic forms and Galois representations. Thus it is of fundamental importance not only to study refinements of strong multiplicity one, but also to understand the possible structural aspects of strong multiplicity one. In this paper, we obtain a fairly complete picture of strong multiplicity one for l-adic representations.

Let K be a global field, let $G_K := \mathrm{Gal}(\bar{K}/K)$ be the Galois group of an algebraic closure $\bar{K}$ of K, and let Σ_K be the set of nonarchimedean places of K. Let F be a nonarchimedean local field of characteristic zero and residue characteristic l. We consider representations $\rho\colon G_K \to GL_r(F)$ satisfying the following hypothesis:

$$\rho \text{ is continuous, unramified outside a finite set } S \subset \Sigma_K, \text{ and semisimple.} \qquad (*)$$

Since we will study the representations through the associated characters, and the characters determine the representation only up to their semisimplification, we assume that

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the representations are semisimple. For $v \notin S$, we have a well-defined Frobenius conjugacy class $\rho(\sigma_v)$ in the image group $\rho(G_K)$.

We recall the notion of upper density. The upper density $ud(P)$ of a set P of primes of K, is defined to be the ratio

$$ud(P) = \limsup_{x\to\infty} \#\{v \in \Sigma_K \mid Nv \leq x,\ v \in P\}/\#\{v \in \Sigma_K \mid Nv \leq x\};$$

here Nv, the norm of v, is the cardinality of the finite set $\mathcal{O}_K/\mathfrak{p}_v$, $\mathcal{O}_K$ is the ring of integers of K, and $\mathfrak{p}_v$ is the prime ideal of $\mathcal{O}_K$ corresponding to the finite place v of K. A set P of primes is said to have a density $d(P)$ if the limit exists as $x \to \infty$ of the ratio $\#\{v \in \Sigma_K \mid Nv \leq x,\ v \in P\}/\#\{v \in \Sigma_K \mid Nv \leq x\}$, and is equal to $d(P)$.

Suppose ρ_1, ρ_2 are representations of G_K into $GL_r(F)$ satisfying (*). Consider the following set:

$$SM(\rho_1, \rho_2) := \{v \in \Sigma_K - S \mid \mathrm{Tr}(\rho_1(\sigma_v)) = \mathrm{Tr}(\rho_2(\sigma_v))\}.$$

We will say two representations ρ_1 and ρ_2 have the *strong multiplicity one property* if the upper density of $SM(\rho_1, \rho_2)$ is positive. We answer in the affirmative the following conjecture due to D. Ramakrishnan ([Ra1]).

Theorem 1. If the upper density λ of $SM(\rho_1, \rho_2)$ is strictly greater than $1 - 1/2r^2$, then $\rho_1 \simeq \rho_2$. □

The result was known for finite groups. There were examples constructed by Serre ([Ra1]), which showed that the above bound is sharp. For unitary, cuspidal automorphic representations on GL_2/K, the corresponding result was established by Ramakrishnan ([Ra2]). The proof was based on the following result of Jacquet and Shalika: If π_1 and π_2 are unitary cuspidal automorphic representations on GL_n, then $\pi_1 \simeq \bar{\pi}_2$ if and only if $L(s, \pi_1 \times \pi_2)$ has a pole at $s = 1$, where $\bar{\pi}_2$ denotes the contragredient of π_2. In analogy, it was expected that the obstruction to the proof of the above theorem lies in the Tate conjectures on the analytical properties of L-functions attached to l-adic cohomologies of algebraic varieties defined over K. However, we will see below that the proof needs only the classical Chebotarev density theorem.

Theorem 1 is still not completely satisfactory, as it does not provide any information on the relationship between ρ_1 and ρ_2 possessing the strong multiplicity one property. The motivating question for this paper was the following: *Suppose* ρ_1 *and* ρ_2 *are 'general' representations of* G_K *into* $GL_2(F)$, *possessing the strong multiplicity one property. Does there exist a Dirichlet character* χ *such that* $\rho_2 \simeq \rho_1 \otimes \chi$*?*

It is this stronger question that provides us with a clue to the solution of this problem. By the Chebotarev density theorem, if C is a closed analytic subset of the image

G of ρ, of strictly smaller dimension than G and stable under conjugation, then $\{v \notin S \mid \rho(\sigma_v) \in C\} = o(x/\log x)$. Conversely, if the density is positive, then $\dim(C) = \dim(G)$. In connection with the question of characterising the representations possessing the strong multiplicity one property, this motivates the introduction of algebraic techniques. The following result can be considered as a qualitative version of strong multiplicity one, and provides a vast strengthening of Theorem 1 in general.

Theorem 2. Suppose that the Zariski closure H_1 of the image $\rho_1(G_K)$ in GL_r is a connected, algebraic group. If the upper density of $SM(\rho_1, \rho_2)$ is positive, then the following hold:

(i) There is a finite Galois extension L of K, such that $\rho_1|_{G_L} \simeq \rho_2|_{G_L}$.

(ii) The connected component H_2^0 of the Zariski closure of the image $\rho_2(G_K)$ in GL_r is conjugate to H_1. In particular, $H_2^0 \simeq H_1$.

(iii) Assume in addition that ρ_1 is absolutely irreducible. Then there is a Dirichlet character, i.e., a character χ of Gal(L/K) into $GL_1(F)$ of finite order, such that $\rho_2 \simeq \rho_1 \otimes \chi$. □

Hence in the 'general case,' the strong multiplicity one property indicates that the representations are Dirichlet twists of each other, and the set of primes for which $\mathrm{Tr}(\rho_1(\sigma_v)) = \mathrm{Tr}(\rho_2(\sigma_v))$ is not some arbitrary set of primes, but are for a finite set of the primes which split in some cyclic extension of K. Morever for any pair of representations satisfying the strong muliplicity one property, the above theorem indicates that the set of primes for which $\mathrm{Tr}(\rho_1(\sigma_v)) = \mathrm{Tr}(\rho_2(\sigma_v))$ has a 'finite' Galois theoretical interpretation.

Motivated by this result, one can raise the following question for automorphic representations: Suppose π_1, π_2 are two irreducible, cuspidal, automorphic representations of "general type" on GL_n/K. Suppose that the set $\{v \in \Sigma_K \mid \pi_{1,v} \simeq \pi_{2,v}\}$ has positive density, where $\pi_{1,v}$ (resp. $\pi_{2,v}$) denotes the local component at v of π_1 (resp. π_2). Does there exist a Dirichlet character χ, such that $\pi_2 \simeq \pi_1 \otimes \chi$? In the case of GL_2, a general type cuspidal automorphic representation should be equivalent to the representation being nonartinian and nondihedral ([MR]). For GL_1, such a result is true and follows from Hecke's results on equidistribution ([R]). However, even for GL_1, the known proofs require the analytic continuation and holomorphicity properties of $L(s, \theta^n)$, for all n, where θ is a grossencharacter of infinite order restricted to the ideles of norm 1. Thus it is surprising that the above results can be established for l-adic representations.

Let N, k be positive integers, and let $\epsilon\colon (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}$ be a character mod N, satisfying $\epsilon(-1) = (-1)^k$. Denote by $M(N, k, \epsilon)$ the space of modular forms on $\Gamma_0(N)$ of weight k, and Nebentypus character ϵ. For modular forms we refer to [DI]. Given $f \in M(N, k, \epsilon)$, we can write $f(z) = \sum_{n=0}^{\infty} a_n(f) e^{2\pi i n z}$, $\mathrm{Im}(z) > 0$, where $a_n(f)$ is the nth Fourier coefficient of f. Denote by $M(N, k, \epsilon)^0$ the set of eigenforms for the Hecke operators T_p, $(p, N) = 1$, with eigenvalue $a_p(f)$. We will define two such forms $f_i \in M(N_i, k_i, \epsilon_i)$, $i = 1, 2$, to be equivalent, denoted by $f_1 \sim f_2$, if $a_p(f_1) = a_p(f_2)$ for almost all primes p. Given any cuspidal eigenform f as above, it follows from the decomposition of $M(N, k, \epsilon)$ into old and new subspaces and by the mulitplicity one theorem, that there exists a unique new form equivalent to f. By a twist of f by a Dirichlet character χ, we mean the form represented by

$$\sum_{n=0}^{\infty} \chi(n) a_n(f) e^{2\pi i n z}.$$

We recall the notion of CM forms ([Ri]). f is said to be a CM form, if f is a cusp form of weight $k \geq 2$, and the Fourier coefficients $a_p(f)$ vanish for all primes p inert in some quadratic extension of $\mathbf{Q}$.

Corollary 1. Suppose $f_i \in M(N_i, k_i, \epsilon_i)^0$, $i = 1, 2$, and f_1 is a non-CM cusp form of weight $k_1 \geq 2$. Suppose that the set $\{p \in \Sigma_{\mathbf{Q}},\ (p, N_1N_2) = 1 \mid a_p(f_1) = a_p(f_2)\}$ has positive upper density. Then there exists a Dirichlet character χ of $\mathbf{Q}$, such that $f_2 \sim f_1 \otimes \chi$. In particular, f_2 is also a non-CM cusp form of weight $k_2 = k_1$. Hence, apart from finitely many primes, the set of primes at which the Fourier coefficients of f_1 and f_2 agree is the set of primes which split in a cyclic extension of $\mathbf{Q}$. □

Proof. Let K_0 be the number field generated by the values $a_p(f_i)$, $\epsilon_i(p)$, $i = 1, 2$. Fix a rational prime l of $\mathbf{Q}$, which splits completely in K_0. By Eichler and Shimura, Deligne [D], and Deligne and Serre [DS], there exists a continuous l-adic representation $\rho_i\colon G_{\mathbf{Q}} \to GL(2, \mathbf{Q}_l)$, $i = 1, 2$ associated to the forms f_i satisfying the following properties for $i = 1, 2$:

(i) ρ_i is semisimple.

(ii) For primes p satisfying $(p, N_i l) = 1$, ρ_i is unramified at p, and $\mathrm{Tr}(\rho_i(\sigma_p)) = a_p(f_i)$ and $\det(\rho_i(\sigma_p)) = \epsilon_i(p) p^{k_i - 1}$.

By our hypothesis, it is known ([S1, Proposition 17]) that $\rho_1(G_K)$ is an open subgroup inside $GL(2, \mathbf{Q}_l)$. If f_2 is not a non-CM form, then the image $\rho_2(G_K)$ is contained in the normaliser of a Cartan subgroup inside $GL(2)$, provided that $k_2 \geq 2$; if $k_2 = 1$, then the image $\rho_2(G_K)$ is a finite group. Since the associated l-adic representation determines the form uniquely, the corollary follows from Theorem 2. ■

Suppose E_1 and E_2 are elliptic curves defined over a number field K. Let S denote the finite set consisting of the ramified places of E_1 and E_2. For a rational prime l, let $\rho_i\colon G_K \to GL(2, \mathbf{Q}_l)$, $i = 1, 2$, denote, respectively, the l-adic representations associated to the elliptic curves E_i. The representations ρ_i, $i = 1, 2$ are unramified outside S. For $v \in \Sigma_K - S$, $(l, Nv) = 1$, let $a_v(E_i)$ $i = 1, 2$ denote the trace $\mathrm{Tr}(\rho_i(\sigma_v))$ of the Frobenius σ_v at v. We have the following corollary.

Corollary 2. Suppose E_1 is a non-CM elliptic curve. Suppose that the set $\{v \in \Sigma_K - S \mid a_v(E_1) = a_v(E_2)\}$ has positive upper density. Then there exists a quadratic Dirichlet character χ of G_K such that E_2 is isogeneous to $E_1 \otimes \chi$; i.e., E_2 is isogeneous to a K-form of E_1. In particular, E_2 is also a non-CM elliptic curve. □

Proof. By Faltings' proof of the Tate conjectures ([F]), the l-adic representation of an elliptic curve determines the elliptic curve up to isogeny. By [S2], the image of the Galois group is known to be an open subgroup inside $GL(2, \mathbf{Q}_l)$ if E does not have CM, and is contained in the normaliser of a split torus if E has CM. Since the only automorphisms of a non-CM elliptic curve are multiplication by $\{\pm 1\}$, the corollary follows from Theorem 2.

We remark that this corollary also follows from [S2, Lemma 7] and [S1, Theorem 10]. ■

The method of proof of the theorems is in essence a suitable 'algebraization' of the Chebotarev density theorem. The proof of the theorems is an algebraic adaptation of the methods developed by Serre to deal with lacunarity questions of the coefficients of L-series attached to l-adic representations of G_K ([S1, Section 6]). It is to be expected that this method of algebraization would be of use in studying other distribution-related questions concerning motives.

2 Proofs

Let F be a nonarchimedean local field of characteristic zero and residue characteristic l. Let M be an algebraic group defined over F. Suppose

$$\rho\colon\ G_K \to M(F)$$

is a continuous representation of the Galois group G_K. We will always assume our l-adic representations are such that there exists a finite set S of nonarchimedean places of K, and that ρ is unramified at all places $v \in \Sigma_K - S$. Denote by G the image $\rho(G_K)$. G is then an l-adic, compact, analytic subgroup of M(F). For $v \in \Sigma_K - S$, we have a well-defined Frobenius conjugacy class $\rho(\sigma_v)$ in the image G.

Suppose C is a subset of G, closed under conjugation by G. Define for a real number $x \geq 2$,

$$\pi_C(x) := \#\{v \in \Sigma_K - S,\ Nv \leq x \mid \rho(\sigma_v) \subset C\}.$$

We are interested in the growth of $\pi_C(x)$ as $x \to \infty$. The basic result we need is the following ([S3]): Suppose C is also a closed subset of Haar measure zero in G; then

$$\pi_C(x) = o\left(\frac{x}{\log\ x}\right) \text{ as } x \to \infty. \quad (1)$$

Remark. It is known that if C is a closed analytic subset of G, of strictly smaller dimension than the dimension of G, then C has no interior points, and consequently is of Haar measure zero. Conversely, let C be such that $\{v \notin S \mid \rho(\sigma_v) \in C\}$ is of positive density. By the above result, we have $\dim(C) = \dim(G)$. If everything were algebraic, then we can conclude that C consists essentially of connected components of G. This motivates us to introduce algebraic methods in the proof. The key point of the proof is to interpret the density as counting the number of connected components of an algebraic group, which allows us to change fields and work over $\mathbf{C}$.

Suppose X is an algebraic subscheme of M defined over F, and stable under the adjoint action of M on itself. Let

$$C = X(F) \cap \rho(G_K).$$

C is then a closed, analytic subset of G, stable under conjugation by G. Let H denote the algebraic envelope of G inside M, i.e., the smallest algebraic subgroup H of M, defined over F, such that $G \subset H(F)$. H is also the Zariski closure of G inside M. Let H^0 be the identity component of H, and let $\Phi = H/H^0$ be the finite group of connected components of H. For $\phi \in \Phi$, let H^ϕ denote the corresponding connected component of H, $G^\phi = G \cap H^\phi(F)$, and $C^\phi = C \cap G^\phi$.

Theorem 3. With notation as above, let $\Psi = \{\phi \in \Phi \mid H^\phi \subset X\}$. Then

$$\pi_C(x) = \frac{|\Psi|}{|\Phi|}\frac{x}{\log x} + o\left(\frac{x}{\log x}\right), \qquad \text{as } x \to \infty.$$

Hence the density of the set of primes v of K with $\rho(\sigma_v) \in C$ is precisely $|\Psi|/|\Phi|$. □

Proof. Suppose $C \cap G^\phi$ contains an open subset of G^ϕ. By Zariski density, the Zariski closure of $C \cap G^\phi$ is precisely H^ϕ. Since X is an algebraic subscheme of M, such that $X(F)$ contains C, we have $H^\phi \subset X$. Hence $C \cap G^\phi = G^\phi$. Decompose $C = C_{in} \cup C'$, where

$$C_{in} = \cup_{\phi \in \Psi} G^\phi, \text{ and}$$
$$C' = \cup_{\phi \in \Phi - \Psi} C^\phi.$$

C' defines a closed, analytic subset of G, having no interior point, and stable under conjugation. By (1), it follows that

$$\pi_{C'}(x) = o\left(\frac{x}{\log x}\right) \qquad \text{as } x \to \infty.$$

Moreover, $C_{in} = \cup_{\phi \in \Psi} G^\phi$. Let $\phi = 0$ correspond to the identity component of H. Let E be the fixed field of G^0 acting on $\bar{K}$. E is a finite Galois extension of K, with Galois group isomorphic to G/G^0. Let $\bar{C}_{in}$ denote the image of C_{in} inside G/G^0. By the above result and by Chebotarev density theorem, we obtain for $x \to \infty$,

$$\begin{aligned}\pi_C(x) &= \pi_{C_{in}}(x) + \pi_{C'}(x)\\ &= \pi_{\bar{C}_{in}}(x) + o\left(\frac{x}{\log\, x}\right)\\ &= \frac{|\bar{C}_{in}|}{|G/G^0|}\frac{x}{\log\, x} + o\left(\frac{x}{\log\, x}\right).\end{aligned}$$

Since $|\bar{C}_{in}|/|G/G^0| = |\Psi|/|\Phi|$, the theorem follows. ∎

Suppose $\rho_i\colon G_K \to GL_r(F)$, $i = 1, 2$ are semisimple representations of G_K, satisfying (*). Let ρ denote the semisimple representation $\rho_1 \times \rho_2\colon G_K \to GL_r(F) \times GL_r(F)$. Let H (resp. H_1, H_2) be the algebraic envelope of $G := \rho(G_K)$ (resp. $G_i = \rho_i(G_K)$, $i = 1, 2$). By our hypothesis, H (resp. H_1, H_2) are reductive subgroups of $GL_r \times GL_r$ (resp. GL_r). Let p_1, p_2 be, respectively, the projections to the first and second factors of $GL_r \times GL_r$. $H_1 \times H_2$ is an algebraic subgroup of $GL_r \times GL_r$ such that $G \subset (H_1 \times H_2)(F)$. Hence $H \subset H_1 \times H_2$. Since $p_i(G) = G_i$ for $i = 1, 2$, it follows from the definition of H_i that $p_i(H) = H_i$, $i = 1, 2$.

We apply the theorem proved above with $M = GL_r \times GL_r$, and $X = \{(g_1, g_2) \in GL_r \times GL_r \mid Tr(g_1) = Tr(g_2)\}$. We see that $\lambda = |\Psi|/|\Phi|$ is the density of $SM(\rho_1, \rho_2)$, where

$$\Psi = \{\phi \in H/H^0 \mid H^\phi \subset X\}.$$

To count the connected components of the algbraic group H by the Lefschetz principle, we can work over **C**. Let J be a maximal compact subgroup of H(**C**). For $\phi \in \Phi$, let $J^\phi = H^\phi \cap J$. J^ϕ is a connected component of J, and we have $\Phi \simeq H/H^0 \simeq J/J^0$. Since H is reductive, J^ϕ is Zariski dense in H^ϕ, and so

$$\Psi = \{\phi \in \Phi \mid J^\phi \subset X\}.$$

Proof of Theorem 1. Any nonempty open subset of J^ϕ is Zariski dense in H^ϕ. Hence for $\phi \in \Phi - \Psi$, we have that $J^\phi \cap X$ is a proper, real analytic subvariety of the connected component J^ϕ, and is of Haar measure zero. Let $d\mu$ be the normalised Haar measure on J. If $\rho_1 \neq \rho_2$, then by orthogonality relations,

$$\int_J |Tr(\rho_1(j)) - Tr(\rho_2(j))|^2 d\mu(j) \geq 2. \tag{2}$$

But for $\phi \in \Psi,\ j \in J^{\phi}$, $\mathrm{Tr}(\rho_1(j)) = \mathrm{Tr}(\rho_2(j))$. Hence

$$\int_J |\mathrm{Tr}(\rho_1(j)) - \mathrm{Tr}(\rho_2(j))|^2 d\mu(j) = \sum_{\phi \in \Phi - \Psi} \int_J |\mathrm{Tr}(\rho_1(j)) - \mathrm{Tr}(\rho_2(j))|^2 d\mu(j) \leq (1-\lambda)4r^2. \tag{3}$$

Hence $\lambda \leq 1 - 1/2r^2$. ■

Remark ([S1, Section 6]). Let $L(s,\rho) = \sum_{\mathfrak{m} \in M_K} a(\mathfrak{m}) N\mathfrak{m}^{-s}$ be the formal Dirichlet series associated to ρ, where M_K is the set of all nonzero ideals in $\mathcal{O}_K$. We have $a(\mathfrak{p}_v) = \mathrm{Tr}(\rho(\sigma_v))$ for $v \in \Sigma_K - S$. Let

$$P_\rho(x) = \{v \in \Sigma_K - S \mid a(\mathfrak{p}_v) = 0\},$$
$$M_\rho(x) = \{\mathfrak{m} \in M_K \mid a(\mathfrak{m}) \neq 0\}.$$

It is shown in [S1] that if there is some $\lambda < 1$ and $\delta > 0$ such that

$$|P_\rho(x)| = \lambda x/\log\ x + O\left(x/(\log\ x)^{1+\delta}\right), \qquad \text{as } x \to \infty, \text{ then}$$
$$|M_\rho(x)| \sim \gamma_\rho x/(\log\ x)^{\lambda} \qquad \text{as } x \to \infty,$$

where γ_ρ is a positive constant. Applying Theorem 3 to the case $M = GL_r,\ X = \{g \in GL_r \mid \mathrm{Tr}(g) = 0\}$, and arguing as above, we can conclude that $\lambda \leq 1 - 1/r^2$ ([S1, Proposition 16]).

Suppose that the Zariski closure of the image $\rho(G_K)$ is a connected algebraic group. Since $\mathrm{Tr}(1) \neq 0$, we can conclude from Theorem 3 that $\{v \in \Sigma_K - S \mid \mathrm{Tr}(\rho(\sigma_v)) = 0\}$ is of zero density. This is Corollary 2, Proposition 15 of [S1]. Thus, in this case, we can conclude that the associated Dirichlet series $L(s,\rho)$ is 'not lacunar'; i.e., there is a positive proportion of Fourier coefficients $a(\mathfrak{m})$ which are nonzero.

Example. By equating the inequalities corresponding to (2) and (3) in the proof of the above remark, it can be deduced that the sharp bound of $1 - 1/r^2$ is attained precisely when the representation ρ is absolutely irreducible, and the image of ρ inside $PGL_r(F)$ is a finite group of cardinality r^2 ([S1, Section 6.5]). Consider the representations $\rho_1 = \rho \otimes \mathrm{Id}$ and $\rho_2 = \rho \otimes sgn$ of the group $G \times \{\pm 1\}$, which can again be considered as representations of G_K. It can be seen that $SM(\rho_1, \rho_2)$ is of density precisely $1 - 1/2r^2$. This was the example given by Serre to illustrate the sharpness of the bound. See [Ra1, p. 442]. Conversely, it can be seen by examining the proof of the theorem that these are the only examples where the above sharp bound is attained.

Proof of Theorem 2. (i) Suppose the upper density λ of $SM(\rho_1, \rho_2)$ is positive. Then there exists $\phi \in \Phi$, such that the corresponding connected component $J^\phi \subset X$. Since we have assumed H_1 is connected, the projection onto the first factor of H^ϕ is H_1. Hence an element of the form $(1, j)$ is in J^ϕ, for some j belonging to the maximal compact subgroup J_2 of $H_2(\mathbf{C})$, which we can assume to be contained in some group U_r of unitary matrices inside $GL_r(\mathbf{C})$. But there is precisely only one element in U_r which has trace r, and that is the identity matrix 1_r! Hence $(1_r, 1_r) \in J^\phi$, and so J^ϕ is the identity component of J. Thus the elements $(j_1, j_2) \in J^0$ satisfy the relation $\mathrm{Tr}(j_1) = \mathrm{Tr}(j_2)$, and by Zariski density, $H^0 \subset X$.

By going to a normal subgroup of finite index in G_K, corresponding to a finite Galois extension L of K, we can assume that $\rho(G_L) \subset H^0(F)$. It follows then for $g \in G_L$ that $\mathrm{Tr}(\rho_1(g)) = \mathrm{Tr}(\rho_2(g))$. Since ρ_1 and ρ_2 are assumed to be semisimple, $\rho_1|_{G_L}$ is conjugate to $\rho_2|_{G_L}$ inside $GL_r(F)$.

(ii) Since $\rho_1(G_L)$ is open inside G_1, and H_1 is connected, it follows that $\rho_1(G_L)$ is Zariski dense inside H_1. By (i) above, we have that H_2^0 is conjugate to H_1.

(iii) Suppose in addition that ρ_1 is absolutely irreducible. Since H_1 is connected, the Zariski closure of any open subgroup inside G is again H_1. It follows that $\rho_1|_{G_L}$ is absolutely irreducible. We can also assume that $\rho_1|_{G_L} = \rho_2|_{G_L}$. By Schur's lemma, the commutant of $\rho_1(G_L)$ inside the algebraic group GL_r/F is a form of GL_1. By Hilbert's theorem 90, the commutant of $\rho_1(G_L)$ inside $GL_r(F)$ consists of precisely the scalar matrices.

For $\sigma \in G_K$, let $T(\sigma) = \rho_1(\sigma)^{-1}\rho_2(\sigma)$. Since $\rho_1|_{G_L} = \rho_2|_{G_L}$, $T(\sigma) = \mathrm{Id}$ for $\sigma \in G_L$. It can also be checked that $T(\sigma)$ is equivariant with respect to the representation $\rho_1|_{G_L}$, and hence is given by a scalar matrix $\chi(\sigma)$. Since $\chi(\sigma)$ is a scalar matrix, it follows that for $\sigma, \tau \in G_K$, $\chi(\sigma\tau) = \chi(\sigma)\chi(\tau)$; i.e., χ is a character of $\mathrm{Gal}(L/K)$ into the group of invertible elements F^* of F, and $\rho_2(\sigma) = \chi(\sigma)\rho_1(\sigma)$ for all $\sigma \in G_K$. ∎

Remark. In [L, p. 210], Langlands introduces the 'motivic' group H over the complex numbers associated to an isobaric automorphic representation π of GL_n/K. Langlands further suggests that distribution questions concerning π can be studied in terms of the image of the normalised Haar measure of J on the space X of conjugacy classes of J, where J is a maximal compact subgroup of $H(\mathbf{C})$. With respect to this measure, X has total measure one. In analogy with the Chebotarev density theorem and Sato-Tate conjectures, it is to be expected that for a given motive, the Frobenius conjugacy classes in the corresponding space X are equidistributed with respect to this measure. It is interesting to note that to conclude the proof of the above theorems, we have to finally make recourse to the compact Lie group J, which should correspond to the group introduced by Langlands. Moreover, since it is the group of connected components of H which is of interest in the above results for questions related to distribution properties of motives or automorphic representations, we need the full motivic Galois group H, rather than just the Mumford-Tate group corresponding to a motive.

We have the following result, which can be considered as a converse to Corollary 2, Proposition 15 of [S1], and also as a generalisation of the example of CM forms ([Ri]).

Theorem 4. Suppose ρ satisfies the hypothesis (*) and is absolutely irreducible. Assume moreover that there is a finite extension L of K such that $\rho|G_L$ is not absolutely irreducible. Then the set $\{v \in \Sigma_K - S \mid \mathrm{Tr}(\rho(\sigma_v)) = 0\}$ is of positive density. □

Proof. We apply a theorem of Burnside ([Fe, Theorem 6.9, p. 36]), i.e., given an irreducible representation ρ of a finite group G into $GL(r, E)$, $r > 1$, E a field of characteristic zero. Then there exist elements $g \in G$, such that $\mathrm{Tr}(\rho(g)) = 0$. We divide the proof into the following three cases.

Case (i): Image of G_K is finite, or, equivalently, $H^0 = 1$. By hypothesis, $r > 1$, and the theorem follows from the result of Burnside and the Chebotarev density theorem.

Suppose $H^0 \neq 1$. Let $G_L := \rho^{-1}(G \cap H^0(F))$, and let L be the corresponding finite, normal extension of K defined as the fixed field of G_L acting on $\bar{F}$. G_L is normal in G_K and $\Phi \simeq H/H^0 \simeq G_K/G_L$. Applying Theorem 3 to the case $M = GL_r$, $X = \{g \in GL_r \mid \mathrm{Tr}(g) = 0\}$, we conclude that in order to prove the theorem, it is enough to exhibit a $\phi \in \Phi$, such that $G^\phi \subset X(F) \subset X(\bar{F})$. We consider the representations over $\bar{F}$, and will continue to denote by ρ the corresponding action of H (or G_K).

By hypothesis, $\rho|G_L$ is not irreducible (absolutely). Let q denote the cardinality of the finite set Q consisting of the isotypical components of $\rho|G_L$.

Case (ii): $q > 1$. It is known that the isotypical components of $\rho|G_L$ are conjugate under the action of G_K. Thus the finite group Φ acts by permutations on the set Q. Since ρ is irreducible, this action is transitive on Q. Let Φ' be the isotropy group of a fixed element in Q. Φ' is a proper subgroup of Φ, as $q > 1$. Since a finite group cannot be written as a union of conjugates of a proper subgroup (a theorem of Jordan), there exists an element $\phi \in \Phi$, which acts without fixed points on Q. It follows that the elements $g \in G^\phi$ act without fixing any of the isotypical components of $\rho|G_L$. Since $q > 1$, this implies that the elements belonging to G^ϕ are of trace zero, and so the theorem is proved in this case.

Case (iii): $q = 1$. We can express $\bar{F}^r = V \otimes W$, where (V, ρ_V) denotes an irreducible representation of G_L (or equivalently of H^0), and $\rho|G_L \simeq \rho_V \otimes \mathrm{Id}$. For $g' \in G_K$, let $\rho^{g'}$ denote the representation of G_K, sending $g \mapsto \rho(g'gg'^{-1})$, $g \in G_K$. By assumption, there exists an automorphism $A(g')$ of V, such that $A(g') \otimes \mathrm{Id}$ gives an equivalence between $\rho|G_L$ and $\rho^{g'}|G_L$. Now $\rho(g')$ also provides an equivalence between $\rho|G_L$ and $\rho^{g'}|G_L$. Since V is irreducible, the map $g' \mapsto \sigma(g') := \rho(g')(A(g') \otimes \mathrm{Id})^{-1}$ can be considered as a projective representation $\bar{\sigma}$ of the finite group $\Phi \simeq G_K/G_L$ into $PGL(W)$. By Schur theory, $\bar{\sigma}$ lifts to a representation $\tilde{\bar{\sigma}}$ of the universal central extension $\tilde{\Phi}$ of Φ. $\tilde{\Phi}$ is finite, and since ρ is irreducible, $\tilde{\bar{\sigma}}$ is also irreducible. By Burnside's theorem, there exists an element $\tilde{\phi_0} \in \tilde{\Phi}$, such that $\mathrm{Tr}(\tilde{\bar{\sigma}}(\tilde{\phi_0})) = 0$. Let $\tilde{\phi_0} \mapsto \phi_0 \in \Phi$. For $g \in G^{\phi_0}$, $\mathrm{Id} \otimes \tilde{\bar{\sigma}}(\tilde{\phi_0})$ and $\sigma(g)$ differ only up to a scalar matrix, and so $\mathrm{Tr}(\sigma(g)) = 0$. Since $\rho(g) = \sigma(g)(A(g) \otimes \mathrm{Id})$, and Tr is multiplicative on tensor products, we obtain that for $g \in G^{\phi_0}$, $\mathrm{Tr}(\rho(g)) = 0$. This concludes the proof of the theorem. ■

Remark. In Case (ii) above, ρ will be isomorphic to an induced representation from a proper subgroup of G_K, after an extension by scalars of the field F. Choose an isotypical component ρ' of $\rho|G_L$, and let F' be a finite extension of F over which ρ' is defined. Let G_M be the subgroup of finite index in G_K, fixing ρ', and let ρ_M denote the corresponding representation of G_M. Then G_M is of index q in G_K and $\rho \simeq I_M^K(\rho_M)$, where $I_M^K(\rho_M)$ denotes the representation of G_K obtained by induction from the representation ρ_M of G_M.

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