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NONPARAMETRIC INFERENCE FOR A CLASS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS BASED ON DISCRETE OBSERVATIONS

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SUMMARY. Consider the stochastic partial differential equations of the type

$$du_\epsilon(t,x) = (\triangle u_\epsilon(t,x) + u_\epsilon(t,x))dt + \epsilon\ \theta(t)\ dW_Q(t,x),\ \ 0 \le t \le T$$

and

$$du_\epsilon(t,x) = \triangle u_\epsilon(t,x)dt + \epsilon\ \theta(t)\ (I-\triangle)^{-1/2}\ dW(t,x),\ \ 0 \le t \le T$$

where $\triangle = \frac{\partial^2}{\partial x^2}, \theta \in \Theta$ and Θ is a class of positive valued functions such that $\theta^2(t) \in L^2(R)$. We obtain an estimator for the function $\theta(t)$ based on the Fourier coefficients $u_{i\epsilon}(t), 1 \le i \le N$ of the random field $u_\epsilon(t,x)$ observed at discrete times and study its asymptotic properties.

1. Introduction

Stochastic partial differential equations (SPDE) are used for stochastic modelling , for instance, in the study of neuronal behviour in neurophysiology and in building stochastic models of turbulence (cf. Kallianpur and Xiong, 1995). The theory of SPDE is investigated in Ito (1984), Rozovskii (1990) and De prato and Zabczyk (1992) among others.

Huebner et al. (1993) started the investigation of maximum likelihood estimation of parameters for a class of SPDE and extended their results to parabolic SPDE in Huebner and Rozovskii (1995). Bernstein -von Mises theorems were developed for such SPDE in Prakasa Rao (1998, 2000b) following the techniques in Prakasa Rao (1981). Asymptotic properties of Bayes estimators of parameters for SPDE were discussed in Prakasa Rao (1998, 2000b). Statistical inference for diffusion type processes and semimartingales in general is studied in Prakasa Rao (1999a,b).

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The problem of nonparametric estimation of a linear mutiplier for some classes of SPDE's is discussed in Prakasa Rao (2000a, 2001a) using the methods of nonparametric inference following the approach of Kutoyants (1994). In all the papers cited earlier, it was assumed that a continuous observation of the random field $u_\epsilon(t,x)$ satisfying the SPDE over the region $[0,1]\times[0,T]$ is available. It is obvious that this assumption is not feasible and the problem of interest is to develop methods of parametric and nonparametric inference based on a set of observations of the random field observed at discrete times t and at discrete positions x. Methods of estimation based on such data seem to lead to equations which are computationally difficult to solve. We now consider a simplified problem. Suppose we are able to observe the Fourier coefficients $u_{i\epsilon}(t)$ of $u_\epsilon(t,x)$ at discrete times. Parametric estimation for some classes of SPDE's based on such discrete data is investigated in Prakasa Rao (2000c, 2001b) when the parameter is involved either in the "trend" term of the SPDE or in the "trend" as well as in the "forcing" terms of the SPDE. We now discuss nonparametric estimation of a function $\theta(t)$ involved in the "forcing" term for a class of SPDE's. The problem of estimation of the diffusion coefficient in a SDE from discrete observations has attracted lot of attention recently in view of the applications in mathematical finance especially for modelling interest rates. Our work here deals with a similar probem for a SPDE. A review of recent results on parametric and nonparametric inference for SPDE's is given in Prakasa Rao (2001c).

2. Estimation from Discrete Observations: Example I

2.1 *Preliminaries.* Let $(\Omega,\mathcal{F},P)$ be a probability space and consider the process $u_\epsilon(t,x), 0\leq x\leq 1, 0\leq t\leq T$ governed by the stochastic partial differential equation

$$du_\epsilon(t,x)=(\triangle u_\epsilon(t,x)+u_\epsilon(t,x))dt+\epsilon\ \ \theta(t)\ \ dW_Q(t,x) \tag{2.1}$$

where $\triangle=\frac{\partial^2}{\partial x^2}$. Suppose that $\theta(.)$ is a positive valued function with $\theta(t)\in C^m([0,\infty))$ for some $m\geq 1$. Further suppose that $\theta^2(.)\in L^2(R)$ and that the function $\theta(.)$ has a compact support contained in the interval $[-\epsilon, T+\epsilon]$ for some $\epsilon>0$.

Further suppose the initial and the boundary conditions are given by

$$\begin{cases} u_\epsilon(0,x)=f(x), f\in L_2[0,1]\\ u_\epsilon(t,0)=u_\epsilon(t,1)=0, 0\leq t\leq T\end{cases} \tag{2.2}$$

and Q is the nuclear covariance operator for the Wiener process $W_Q(t,x)$ taking values in $L_2[0,1]$ so that

$$W_Q(t,x) = Q^{1/2}W(t,x)$$

and $W(t,x)$ is a cylindrical Brownian motion in $L_2[0,1]$. Then, it is known that (cf. Rozovskii (1990), Kallianpur and Xiong (1995))

$$W_Q(t,x) = \sum_{i=1}^{\infty} q_i^{1/2} e_i(x) W_i(t) \text{ a.s.} \tag{2.3}$$

where $\{W_i(t), 0 \leq t \leq T\}, i \geq 1$ are independent one - dimensional standard Wiener processes and $\{e_i\}$ is a complete orthonormal system in $L_2[0,1]$ consisting of eigen vectors of Q and $\{q_i\}$ eigen values of Q.

We assume that the operator Q is a special covariance operator Q with $e_k = sin(k\pi x), k \geq 1$ and $\lambda_k = (\pi k)^2, k \geq 1$. Then $\{e_k\}$ is a complete orthonormal system with the eigen values $q_i = (1+\lambda_i)^{-1}, i \geq 1$ for the operator Q and $Q = (I - \triangle)^{-1}$. Note that

$$dW_Q = Q^{1/2} dW. \tag{2.4}$$

We define a solution $u_\epsilon(t,x)$ of (2.1) as a formal sum

$$u_\epsilon(t,x) = \sum_{i=1}^{\infty} u_{i\epsilon}(t) e_i(x) \tag{2.5}$$

(cf. Rozovskii (1990)). It can be checked that the Fourier coefficient $u_{i\epsilon}(t)$ satisfies the stochastic differential equation

$$du_{i\epsilon}(t) = (1-\lambda_i)u_{i\epsilon}(t)dt + \frac{\epsilon}{\sqrt{\lambda_i + 1}}\theta(t)dW_i(t),\ 0 \leq t \leq T \tag{2.6}$$

with the initial condition

$$u_{i\epsilon}(0) = v_i,\ v_i = \int_0^1 f(x)e_i(x)dx. \tag{2.7}$$

2.2 *Estimation.* We now consider the problem of estimation of the function $\theta(t), 0 \leq t \leq T$ based on the observation of the Fourier coefficients $u_{i\epsilon}(t_j), t_j = j2^{-n}, j = 0,1,\ldots,[2^nT], 1 \leq i \leq N$, or equivalently based on the observations $u_\epsilon^{(N)}(t_j,x), t_j = j2^{-n}, j = 0,1,\ldots,[2^nT]$ of the projection of the process $u_\epsilon(t,x)$ onto the subspace spanned by $\{e_1,\ldots,e_N\}$ in $L_2[0,1]$. Here $[x]$ denotes the largest integer less than or equal to x.

We will at first construct an estimator of $\theta(.)$ based on the observations $\{u_{i\epsilon}(t_j), t_j = j2^{-n}, j = 0, 1, \ldots, [2^n T]\}$. Our technique follows the methods in Genon-Catalot et al. (1992).

Let $\{V_j, -\infty < j < \infty\}$ be an increasing sequence of closed subspaces of $L^2(R)$. Suppose the family $\{V_j, -\infty < j < \infty\}$ is an r-regular multiresolution analysis of $L^2(R)$ such that the associated scale function ϕ and wavelet function ψ are compactly supported and belong to $C^r(R)$. For a short introduction to the properties of wavelets and multiresolution analysis, see Prakasa Rao (1999a).

Let W_j be the subspace defined by

$$V_{j+1} = V_j \oplus W_j \tag{2.8}$$

and define

$$\begin{aligned} \phi_{j,k}(x) &= 2^{j/2}\phi(2^j x - k), -\infty < j, k < \infty & (2.9)\\ \psi_{j,k}(x) &= 2^{j/2}\psi(2^j x - k), -\infty < j, k < \infty. & (2.10) \end{aligned}$$

Then (i) for all $-\infty < j < \infty$, the collection of functions $\{\phi_{j,k}, -\infty < k < \infty\}$ is an orthonormal basis of V_j ; (ii) for all $-\infty < j < \infty$, the collection of functions $\{\psi_{j,k}, -\infty < k < \infty\}$ is an orthonormal basis of W_j ; and (iii) the collection of functions $\{\psi_{j,k}, -\infty < j, k < \infty\}$ is an orthonormal basis of $L^2(R)$.

In view of the earlier assumptions made on the function $\theta(t)$, it follows that the function $\theta(t)$ belongs to the Sobolev space $H^m(R)$. Let $j(n)$ be an increasing sequence of positive integers tending to infinity as $n \to \infty$. The space $L^2(R)$ has the following decomposition:

$$L^2(R) = V_{j(n)} \oplus (\oplus_{j \geq j(n)} W_j). \tag{2.11}$$

The function $\theta^2(t)$ can be represented in the form

$$\theta^2(t) = \sum_{k=-\infty}^{\infty} \mu_{j(n),k}\phi_{j(n),k}(t) + \sum_{j \geq j(n), -\infty < k < \infty} \nu_{j,k}\psi_{j,k}(t) \tag{2.12}$$

where

$$\mu_{j,k} = \int_R \theta^2(t)\phi_{j,k}(t)dt \tag{2.13}$$

and

$$\nu_{j,k} = \int_R \theta^2(t)\psi_{j,k}(t)dt. \tag{2.14}$$

We will now define estimators of the coefficients $\mu_{j,k}$ based on the observations $\{u_{i\epsilon}(t_r), t_r = r2^{-n}, j = 0, 1, \ldots, [2^n T]\}$. Define

$$\hat{\mu}^{(i)}_{j,k} = \frac{\lambda_i + 1}{\epsilon^2} \sum_{r=0}^{M-1} \phi_{j,k}(t_r)(u_{i\epsilon}(t_{r+1}) - u_{i\epsilon}(t_r))^2 \tag{2.15}$$

where $M = [2^n T]$.

The subspace V_j is not finite dimensional. However, the functions θ^2 and the functions ϕ are compactly supported. Hence, for each resolution j, the set of all k such that $\mu_{j,k} \neq 0$ and the set of all k such that $\hat{\mu}_{j,k} \neq 0$ is a finite set L_j depending only on the constant T and the support of ϕ and the cardinality of the set is $O(2^j)$.

Define the estimator of $\theta^2(t)$ by

$$\hat{\theta}^2_i(t) = \sum_{k \in L_{j(n)}} \hat{\mu}^{(i)}_{j(n),k} \phi_{j(n),k}(t) \tag{2.16}$$

$$= \sum_{-\infty < k < \infty} \hat{\mu}^{(i)}_{j(n),k} \phi_{j(n),k}(t). \tag{2.17}$$

Note that for any function f such that

$$\int_0^T f(t)\theta^2(t)dt < \infty,$$

it can be shown that

$$\sum_{r=0}^{M-1} f(t_r)(u_{i\epsilon}(t_{r+1}) - u_{i\epsilon}(t_r))^2 \xrightarrow{p} \frac{\epsilon^2}{\lambda_i + 1} \int_0^T f(t)\theta^2(t)dt \;\text{ as }\; n \to \infty.$$

Hence

$$\hat{\mu}^{(i)}_{j,k} \xrightarrow{p} \mu_{j,k} \;\text{ as }\; n \to \infty. \tag{2.18}$$

Let $h(.)$ be a continuous function on $[0, T]$ with compact support contained in $(0, T)$ and belonging to the Sobolev space $H^{m'}(R)$ with $m' > \frac{1}{2}$. Let h_j be the projection of h on the space V_j. Further more suppose that

$$r \wedge m + r \wedge m' > 2, j(n) = [\alpha n] \tag{2.19}$$

with

$$(2(r \wedge m + r \wedge m'))^{-1} \leq \alpha < \frac{1}{4}. \tag{2.20}$$

Note that r is the regularity of the multiresolution analysis, m is the exponent of the Soblev space to which θ^2 belongs to and m' is the exponent of the

Soblev space to which h belongs to. Applying the Proposition 3.1 of Genon-Catalot et al. (1992), we obtain that the following representation holds:

$$\begin{aligned} J_{in} &\equiv 2^{n/2}\int_0^T h(t)(\hat{\theta}_i^2(t)-\theta^2(t))dt \\ &= 2^{n/2}\sum_{r=0}^{M-1} h_{j(n)}(t_r)[(\int_{t_r}^{t_{r+1}}\theta(s)\ \ dW_i(s))^2-\int_{t_r}^{t_{r+1}}\theta^2(s)\ \ ds]+R_{in} \end{aligned}$$

where $R_{in}=o_p(1)$ as $n\to\infty$. Further more

$$J_{in}\xrightarrow{\mathcal{L}}\mathcal{N}(0,2\int_0^T h^2(t)\theta^4(t)\ \ dt)\ \text{ as }\ n\to\infty \tag{2.21}$$

by Theorem 3.1 of Genon-Catalot et al. (1992). Note the estimators $\{\hat{\theta}_i(t), i\geq 1\}$ are independent estimators of $\theta(t)$ for any fixed t since the processes $\{W_i, i\geq 1\}$ are independent Wiener processes.

Let $\gamma(t)$ be a nonnegative continuous function with support contained in the interval $[0,T]$. Define

$$Q_{in}=E\{\int_0^T\gamma(t)(\hat{\theta}_i^2(t)-\theta^2(t))^2dt\}. \tag{2.22}$$

Note that Q_{in} is the integrated mean square error of the estimator $\hat{\theta}_i^2(t)$ of the function $\theta^2(t)$ corresponding to the weight function $\gamma(t)$. It can be written in the form

$$Q_{in}=B_{in}^2+V_{in} \tag{2.23}$$

where

$$B_{in}^2=\int_0^T\gamma(t)(E\hat{\theta}_i^2(t)-\theta^2(t))^2dt \tag{2.24}$$

is the integrated square of the bias term with the weight function $\gamma(t)$ and

$$V_{in}=E\{\int_0^T\gamma(t)(\hat{\theta}_i^2(t)-E\hat{\theta}_i^2(t))^2dt\} \tag{2.25}$$

is the integrated square of the variance term with the weight function $\gamma(t)$. Let

$$D_{in}=E\{\int_0^T(\hat{\theta}_i^2(t)-E\hat{\theta}_i^2(t))^2dt\} \tag{2.26}$$

and suppose that $\sup\{\gamma(t):t\in[0,T]\}\leq K$. Further suppose that $j(n)-\frac{n}{2}\to-\infty$. Then it follows, by Theorem 4.1 of Genon-Catalot et al. (1992), that there exists a constant C_i depending on ϵ, λ_i and the functions ϕ, γ and θ^2 such that

$$B_{in}^2 \leq C_i(2^{4j(n)-2n} + 2^{-2j(n)(m\wedge r)} + 2^{-n}) \tag{2.27}$$

and

$$D_{in} = 2^{j(n)-n}\ 2\int_0^T \theta^4(t)dt + o(2^{j(n)-n}). \tag{2.28}$$

Further more

$$V_{in} \leq KD_{in}. \tag{2.29}$$

Let

$$\tilde{\theta}_N^2(t) = \frac{1}{N}\sum_{i=1}^N \hat{\theta}_i^2(t). \tag{2.30}$$

It is obvious that, for any function h satisfying the conditions stated above, and for any fixed integer $N \geq 1$,

$$\begin{aligned}
&2^{n/2}\int_0^T h(t)(\tilde{\theta}_N^2(t) - \theta^2(t))dt \\
&= N^{-1}\sum_{i=1}^N J_{in} \\
&= N^{-1}\sum_{i=1}^N\{2^{n/2}\sum_{r=0}^{M-1} h_{j(n)}(t_r)[(\int_{t_r}^{t_{r+1}}\theta(s)\ dW_i(s))^2 \\
&\quad -\int_{t_r}^{t_{r+1}}\theta^2(s)\ ds]\} + N^{-1}\sum_{i=1}^N R_{in} \\
&= 2^{n/2}\sum_{r=0}^{M-1} h_{j(n)}(t_r)\{N^{-1}\sum_{i=1}^N[(\int_{t_r}^{t_{r+1}}\theta(s)\ dW_i(s))^2 \\
&\quad -\int_{t_r}^{t_{r+1}}\theta^2(s)\ ds]\} + N^{-1}\sum_{i=1}^N R_{in}.
\end{aligned}$$

From the independence of the estimators $\hat{\theta}_i(t), 1 \leq i \leq N$, it follows from the Theorem 3.1 of Genon-Catalot et al. (1992) that

$$2^{n/2}\int_0^T h(t)(\tilde{\theta}_N^2(t) - \theta^2(t))dt \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2N^{-1}\int_0^T h^2(t)\theta^4(t)\ dt) \text{ as } n \to \infty. \tag{2.31}$$

We have the following theorem.

THEOREM 2.1. Under the conditions stated above , the estimator $\tilde{\theta}^2_N(t)$ of $\theta^2(t)$ satisfies the following property for any function $h(t)$ as defined earlier:

$$2^{n/2}\int_0^T h(t)(\tilde{\theta}^2_N(t)-\theta^2(t))dt \xrightarrow{\mathcal{L}} \mathcal{N}(0,2N^{-1}\int_0^T h^2(t)\theta^4(t)\ \ dt) \text{ as } n\to\infty. \tag{2.32}$$

Let $\gamma(t)$ be a nonnegative continuous function with support contained in the interval $[0,T]$. Define

$$Q_n = E\{\int_0^T \gamma(t)(\tilde{\theta}^2_N(t)-\theta^2(t))^2 dt\}. \tag{2.33}$$

Note that Q_n is the integrated mean square error of the estimator $\tilde{\theta}^2_N(t)$ of the function $\theta^2(t)$ corresponding to the weight function $\gamma(t)$. It can be written in the form

$$Q_n = B_n^2 + V_n \tag{2.34}$$

where

$$B_n^2 = \int_0^T \gamma(t)(E\tilde{\theta}^2_N(t)-\theta^2(t))^2 dt \tag{2.35}$$

is the integrated square of the bias term with the weight function $\gamma(t)$ and

$$V_n = E\{\int_0^T \gamma(t)(\tilde{\theta}^2_N(t)-E\tilde{\theta}^2_N(t))^2 dt\} \tag{2.36}$$

is the integrated square of the variance term with the weight function $\gamma(t)$. Let

$$D_n = E\{\int_0^T (\tilde{\theta}^2_N(t)-E\tilde{\theta}^2_N(t))^2 dt\}. \tag{2.37}$$

We have the following theorem from the estimates on $\{B_{in}, 1\le i\le N\}$ and on $\{D_{in}, 1\le i\le N\}$ given above.

THEOREM 2.2. Suppose that $j(n)-\frac{n}{2}\to-\infty$. Then there exists a constant C_N depending on N,ϕ,γ,θ^2 such that

$$B_n^2 \le C_N\ \ (2^{4j(n)-2n}+2^{-2j(n)(m\wedge r)}+2^{-n}) \tag{2.38}$$

and

$$D_n = N^{-1}2^{j(n)-n}\ 2\int_0^T \theta^4(t)dt + o(N^{-1}\ 2^{j(n)-n}). \tag{2.39}$$

Further more

$$V_n \le KD_n \tag{2.40}$$

where $K=\sup\{\gamma(t): 0\le t\le T\}$.

3. Estimation from Discrete Observations: Example II

3.1 Preliminaries. Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the process $u_\epsilon(t,x), 0 \leq x \leq 1, 0 \leq t \leq T$ governed by the stochastic partial differential equation

$$du_\epsilon(t,x) = \triangle u_\epsilon(t,x)dt + \epsilon \;\; \theta(t) \;\; (I - \triangle)^{-1/2} \;\; dW(t,x) \tag{3.1}$$

where $\triangle = \frac{\partial^2}{\partial x^2}$. Suppose that $\theta(.)$ is a positive valued function with $\theta(t) \in C^m([0,\infty])$ for some $m \geq 1$. Further suppose that $\theta^2(.) \in L^2(R)$ and that the function $\theta(.)$ has a compact support contained in the interval $[-\epsilon, T+\epsilon]$ for some $\epsilon > 0$.

Further suppose the initial and the boundary conditions are given by

$$\begin{cases} u_\epsilon(0,x) = f(x), f \in L_2[0,1] \\ u_\epsilon(t,0) = u_\epsilon(t,1) = 0, 0 \leq t \leq T. \end{cases} \tag{3.2}$$

We define a solution $u_\epsilon(t,x)$ of (3.1) as a formal sum

$$u_\epsilon(t,x) = \sum_{i=1}^{\infty} u_{i\epsilon}(t)e_i(x) \tag{3.3}$$

(cf. Rozovskii, 1990). Following the arguments given in the Section 2, it can be checked that the Fourier coefficient $u_{i\epsilon}(t)$ satisfies the stochastic differential equation

$$du_{i\epsilon}(t) = -\lambda_i u_{i\epsilon}(t)dt + \frac{\epsilon}{\sqrt{\lambda_i + 1}}\theta(t)dW_i(t), \; 0 \leq t \leq T \tag{3.4}$$

with the initial condition

$$u_{i\epsilon}(0) = v_i, \; v_i = \int_0^1 f(x)e_i(x)dx. \tag{3.5}$$

3.2 Estimation. We now consider the problem of estimation of the function $\theta(t), 0 \leq t \leq T$ based on the observation of the Fourier coefficients $u_{i\epsilon}(t_j), t_j = j2^{-n}, j = 0, 1, \ldots, [2^nT], 1 \leq i \leq N$, or equivalently based on discrete observations $u_\epsilon^{(N)}(t_j, x), t_j = j2^{-n}, j = 0, 1, \ldots, [2^nT]$ of the projection of the process $u_\epsilon(t,x)$ onto the subspace spanned by $\{e_1, \ldots, e_N\}$ in $L_2[0,1]$.

We will at first construct an estimator of $\theta(.)$ based on the observations $\{u_{i\epsilon}(t_j), t_j = j2^{-n}, j = 0, 1, \ldots, [2^nT]\}$. Our technique again follows the methods in Genon-Catalot et al. (1992) using the method of wavelets. We adopt the same notation as in Section 2.

In view of the earlier assumptions made on the function $\theta(t)$, it follows that the function $\theta(t)$ belongs to the Sobolev space $H^m(R)$. Let $j(n)$ be an increasing sequence of positive integers tending to infinity as $n \to \infty$. The space $L^2(R)$ has the following decomposition:

$$L^2(R) = V_{j(n)} \oplus (\oplus_{j \geq j(n)} W_j). \tag{3.6}$$

The function $\theta^2(t)$ can be represented in the form

$$\theta^2(t) = \sum_{k=-\infty}^{\infty} \mu_{j(n),k}\phi_{j(n),k}(t) + \sum_{j \geq j(n), -\infty < k < \infty} \nu_{j,k}\psi_{j,k}(t) \tag{3.7}$$

where

$$\mu_{j,k} = \int_R \theta^2(t)\phi_{j,k}(t)dt \tag{3.8}$$

and

$$\nu_{j,k} = \int_R \theta^2(t)\psi_{j,k}(t)dt. \tag{3.9}$$

We will now define estimators of the coefficients $\mu_{j,k}$ based on the observations $\{u_{i\epsilon}(t_r), t_r = r2^{-n}, j = 0, 1, \ldots, [2^nT]\}$. Define

$$\hat{\mu}_{j,k}^{(i)} = \frac{\lambda_i + 1}{\epsilon^2} \sum_{r=0}^{M-1} \phi_{j,k}(t_r)(u_{i\epsilon}(t_{r+1}) - u_{i\epsilon}(t_r))^2 \tag{3.10}$$

where $M = [2^nT]$.

The subspace V_j is not finite dimensional. However, the functions θ^2 and the functions ϕ are compactly supported. Hence, for each resolution j, the set of all k such that $\mu_{j,k} \neq 0$ and the set of all k such that $\hat{\mu}_{j,k} \neq 0$ is a finite set L_j depending only on the constant T and the support of ϕ and the cardinality of the set is $O(2^j)$.

Define the estimator of $\theta^2(t)$ by

$$\hat{\theta}_i^2(t) = \sum_{k \in L_{j(n)}} \hat{\mu}_{j(n),k}^{(i)}\phi_{j(n),k}(t) \tag{3.11}$$

$$= \sum_{-\infty < k < \infty} \hat{\mu}_{j(n),k}^{(i)}\phi_{j(n),k}(t). \tag{3.12}$$

Note that for any function f such that

$$\int_0^T f(t)\theta^2(t)dt < \infty,$$

it can be shown that

$$\sum_{r=0}^{M-1} f(t_r)(u_{i\epsilon}(t_{r+1}) - u_{i\epsilon}(t_r))^2 \xrightarrow{p} \frac{\epsilon^2}{\lambda_i + 1}\int_0^T f(t)\theta^2(t)dt \text{ as } n \to \infty.$$

Hence

$$\hat{\mu}_{j,k}^{(i)} \xrightarrow{p} \mu_{j,k} \text{ as } n \to \infty. \tag{3.13}$$

Let $h(.)$ be a continuous function on $[0,T]$ with compact support contained in $(0,T)$ and belonging to the Sobolev space $H^{m'}(R)$ with $m' > \frac{1}{2}$. Let h_j be the projection of h on the space V_j. Further more suppose that

$$r \wedge m + r \wedge m' > 2, j(n) = [\alpha n] \tag{3.14}$$

with

$$(2(r \wedge m + r \wedge m'))^{-1} \leq \alpha < \frac{1}{4}. \tag{3.15}$$

Note that r is the regularity of the multiresolution analysis, m is the exponent of the Soblev space to which θ^2 belongs to and m' is the exponent of the Soblev space to which h belongs to. Applying the Proposition 3.1 of Genon-Catalot et al. (1992), we obtain that the following representation holds:

$$\begin{aligned}
\tilde{J}_{in} &\equiv 2^{n/2}\int_0^T h(t)(\hat{\theta}_i^2(t) - \theta^2(t))dt \\
&= 2^{n/2}\sum_{r=0}^{M-1} h_{j(n)}(t_r)[(\int_{t_r}^{t_{r+1}} \theta(s)\ dW_i(s))^2 - \int_{t_r}^{t_{r+1}} \theta^2(s)\ ds] + \tilde{R}_{in}
\end{aligned}$$

where $\tilde{R}_{in} = o_p(1)$ as $n \to \infty$. Further more

$$\tilde{J}_{in} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2\int_0^T h^2(t)\theta^4(t)\ dt) \text{ as } n \to \infty \tag{3.16}$$

by Theorem 3.1 of Genon-Catalot et al. (1992). Note the estimators $\{\hat{\theta}_i(t), i \geq 1\}$ are independent estimators of $\theta(t)$ for any fixed t since the processes $\{W_i, i \geq 1\}$ are independent Wiener processes.

Let $\gamma(t)$ be a nonnegative continuous function with support contained in the interval $[0,T]$. Define

$$\tilde{Q}_{in} = E\{\int_0^T \gamma(t)(\hat{\theta}_i^2(t) - \theta^2(t))^2 dt\}. \tag{3.17}$$

Note that $\tilde{Q}_{in}$ is the integrated mean square error of the estimator $\hat{\theta}_i^2(t)$ of the function $\theta^2(t)$ corresponding to the weight function $\gamma(t)$. It can be written in the form

$$\tilde{Q}_{in} = \tilde{B}_{in}^2 + \tilde{V}_{in} \tag{3.18}$$

where

$$\tilde{B}_{in}^2 = \int_0^T \gamma(t)(E\hat{\theta}_i^2(t) - \theta^2(t))^2 dt \tag{3.19}$$

is the integrated square of the bias term with the weight function $\gamma(t)$ and

$$\tilde{V}_{in} = E\{\int_0^T \gamma(t)(\hat{\theta}_i^2(t) - E\hat{\theta}_i^2(t))^2 dt\} \tag{3.20}$$

is the integrated square of the variance term with the weight function $\gamma(t)$. Let

$$\tilde{D}_{in} = E\{\int_0^T (\hat{\theta}_i^2(t) - E\hat{\theta}_i^2(t))^2 dt\} \tag{3.21}$$

and suppose that $\sup\{\gamma(t) : t \in [0,T]\} \leq K$. Further suppose that $j(n) - \frac{n}{2} \to -\infty$. Then it follows, by Theorem 4.1 of Genon-Catalot et al. (1992), that there exists a constant $\tilde{C}_i$ depending on ϵ, λ_i and the functions ϕ, γ and θ^2 such that

$$\tilde{B}_{in}^2 \leq \tilde{C}_i(2^{4j(n)-2n} + 2^{-2j(n)(m\wedge r)} + 2^{-n}) \tag{3.22}$$

and

$$\tilde{D}_{in} = 2^{j(n)-n}\, 2\int_0^T \theta^4(t)dt + o(2^{j(n)-n}). \tag{3.23}$$

Further more

$$\tilde{V}_{in} \leq K\tilde{D}_{in}. \tag{3.24}$$

Let

$$\tilde{\theta}_N^2(t) = \frac{1}{N}\sum_{i=1}^N \hat{\theta}_i^2(t). \tag{3.25}$$

It is obvious that, for any function h satisfying the conditions stated above, and for any fixed integer $N \geq 1$,

$$\begin{aligned} & 2^{n/2}\int_0^T h(t)(\tilde{\theta}_N^2(t) - \theta^2(t))dt \\ = \quad & N^{-1}\sum_{i=1}^N \tilde{J}_{in} \end{aligned}$$

$$\begin{aligned}
&= N^{-1}\sum_{i=1}^{N}\{2^{n/2}\sum_{r=0}^{M-1} h_{j(n)}(t_r)[(\int_{t_r}^{t_{r+1}} \theta(s)\ \ dW_i(s))^2 \\
&\quad -\int_{t_r}^{t_{r+1}} \theta^2(s)\ \ ds]\} + N^{-1}\sum_{i=1}^{N}\tilde{R}_{in} \\
&= 2^{n/2}\sum_{r=0}^{M-1} h_{j(n)}(t_r)\{N^{-1}\sum_{i=1}^{N}[(\int_{t_r}^{t_{r+1}} \theta(s)\ \ dW_i(s))^2 \\
&\quad -\int_{t_r}^{t_{r+1}} \theta^2(s)\ \ ds]\} + N^{-1}\sum_{i=1}^{N}\tilde{R}_{in}.
\end{aligned}$$

From the independence of the estimators $\hat{\theta}_i(t), 1 \leq i \leq N$, it follows from the Theorem 3.1 of Genon-Catalot et al. (1992) that

$$2^{n/2}\int_0^T h(t)(\tilde{\theta}_N^2(t)-\theta^2(t))dt \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2N^{-1}\int_0^T h^2(t)\theta^4(t)\ \ dt) \text{ as } n \to \infty. \tag{3.26}$$

We have the following theorem.

THEOREM 3.1. *Under the conditions stated above , the estimator $\tilde{\theta}_N^2(t)$ of $\theta^2(t)$ satisfies the following property for any function $h(t)$ as defined earlier:*

$$2^{n/2}\int_0^T h(t)(\tilde{\theta}_N^2(t)-\theta^2(t))dt \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2N^{-1}\int_0^T h^2(t)\theta^4(t)\ \ dt) \text{ as } n \to \infty. \tag{3.27}$$

Let $\gamma(t)$ be a nonnegative continuous function with support contained in the interval $[0, T]$. Define

$$\tilde{Q}_n = E\{\int_0^T \gamma(t)(\tilde{\theta}_N^2(t)-\theta^2(t))^2 dt\}. \tag{3.28}$$

Note that $\tilde{Q}_n$ is the integrated mean square error of the estimator $\tilde{\theta}_N^2(t)$ of the function $\theta^2(t)$ corresponding to the weight function $\gamma(t)$. It can be written in the form

$$\tilde{Q}_n = \tilde{B}_n^2 + \tilde{V}_n \tag{3.29}$$

where

$$\tilde{B}_n^2 = \int_0^T \gamma(t)(E\tilde{\theta}_N^2(t)-\theta^2(t))^2 dt \tag{3.30}$$

is the integrated square of the bias term with the weight function $\gamma(t)$ and

$$\tilde{V}_n = E\{\int_0^T \gamma(t)(\tilde{\theta}_N^2(t)-E\tilde{\theta}_N^2(t))^2 dt\} \tag{3.31}$$

is the integrated square of the variance term with the weight function $\gamma(t)$. Let

$$\tilde{D}_n = E\{\int_0^T (\tilde{\theta}_N^2(t) - E\tilde{\theta}_N^2(t))^2 dt\}. \tag{3.32}$$

We have the following theorem from the estimates on $\{\tilde{B}_{in}, 1 \leq i \leq N\}$ and on $\{\tilde{D}_{in}, 1 \leq i \leq N\}$ given above.

THEOREM 3.2. *Suppose that* $j(n) - \frac{n}{2} \to -\infty$. *Then there exists a constant* $\tilde{C}_N$ *depending on* $N, \phi, \gamma, \theta^2$ *such that*

$$\tilde{B}_n^2 \leq \tilde{C}_N \ (2^{4j(n)-2n} + 2^{-2j(n)(m\wedge r)} + 2^{-n}) \tag{3.33}$$

and

$$\tilde{D}_n = N^{-1} 2^{j(n)-n} \ 2 \int_0^T \theta^4(t) dt + o(N^{-1} \ 2^{j(n)-n}). \tag{3.34}$$

Further more

$$\tilde{V}_n \leq K\tilde{D}_n \tag{3.35}$$

where $K = \sup\{\gamma(t) : 0 \leq t \leq T\}$.

REMARKS. It can be seen, from the Theorems 2.1 and 2.2 and from the Theorems 3.1 and 3.2, that the limiting behaviour of the estimator $\tilde{\theta}_N^2(t)$ of $\theta^2(t)$ does not depend on the "trend" terms in the SPDE's discussed in both the examples as long as the "trend" terms in the SDE's satisfied by the Fourier coefficients do not depend on the function $\theta(t)$ or any other unknown functions. This has also been pointed out by Genon-Catalot et al. (1992) in their work on the estimation of the diffusion coefficient for SDE's.

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