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MINIMUM L_1-NORM ESTIMATION FOR FRACTIONAL ORNSTEIN–UHLENBECK TYPE PROCESS

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Abstract. We investigate the asymptotic properties of the minimum L_1-norm estimator of the drift parameter for fractional Ornstein–Uhlenbeck type process satisfying a linear stochastic differential equation driven by a fractional Brownian motion.

1. Introduction

Long range dependence phenomenon is said to occur in a stationary time series $\{X_n, n \geq 0\}$ if the $\operatorname{Cov}(X_0, X_n)$ of the time series tend to zero as $n \to \infty$ and yet the condition

$$\sum_{n=0}^{\infty} |\operatorname{Cov}(X_0, X_n)| = \infty$$

is satisfied. In other words, the $\operatorname{Cov}(X_0, X_n)$ tend to zero but so slowly that their sum diverges. This phenomenon was first observed by the hydrologist Hurst (1951) on projects involving the design of reservoirs along the Nile river (cf. Montanari (2003)) and by others in hydrological time series. It was recently observed that a similar phenomenon occurs in problems concerning traffic patterns of packet flows in high-speed data networks such as the Internet (cf. Willinger et al. (2003), Norros (2003)). The long range dependence pattern is also observed in macroeconomics and finance (cf. Henry and Zafforoni (2003)). Long range dependence is also related to the concept of self-similarity for a stochastic process. A stochastic process $\{X(t), t \in \mathbf{R}\}$ is said to be H-self-similar with index $H > 0$ if for every $a > 0$, the processes $\{X(at), t \in \mathbf{R}\}$ and the process $\{a^H X(t), t \in \mathbf{R}\}$ have the same finite-dimensional distributions. Suppose a self-similar process has stationary increments. Then the increments form a stationary time series which exhibits long range dependence. A Gaussian H-self-similar process with stationary increments with $0 < H < 1$ is called a fractional Brownian motion. A recent monograph by Doukhan et al. (2003) discusses the theory and applications of long range dependence and properties of fractional Brownian motion (Taqqu (2003)). If $H = \frac{1}{2}$, then the fractional Brownian motion reduces to the standard Brownian motion also called the Wiener process.

Diffusion processes and diffusion type processes satisfying stochastic differential equations driven by Wiener processes are used for stochastic modelling in wide variety of sciences such as population genetics, economic processes, signal processing as well as for modeling sunspot activity and more recently in mathematical finance. Statistical inference for diffusion type processes satisfying stochastic differential equations driven by

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Wiener processes have been studied earlier, and a comprehensive survey of various methods is given in Prakasa Rao (1999). There has been a recent interest in studying similar problems for stochastic processes driven by a fractional Brownian motion to model processes involving long range dependence. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein–Uhlenbeck process. Such processes play a potentially important role in the modelling of financial time series. The fractional Ornstein–Uhlenbeck process is a fractional analogue of the Ornstein–Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = X_0 + \theta \int_0^t X_s \, ds + \sigma W_t^H, \qquad t \geq 0. \tag{1.1}$$

The above-mentioned authors investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

Parametric estimation for more general classes of stochastic processes satisfying the linear stochastic differential equations driven by fractional Brownian motion, observed over a fixed period of time T, is studied in Prakasa Rao (2003a,b). It is well known that the sequential estimation methods might lead to equally efficient estimators from the process observed possibly over a shorter expected period of observation time. We have investigated the conditions for such a phenomenon for estimating the drift parameter of a fractional Ornstein–Uhlenbeck type process in Prakasa Rao (2004). Novikov (1972) investigated the asymptotic properties of a sequential maximum likelihood estimator for the drift parameter in the Ornstein–Uhlenbeck process.

In spite of the fact that maximum likelihood estimators (MLE) are consistent and asymptotically normal and also asymptotically efficient in general, they have some shortcomings at the same time. Their calculation is often cumbersome as the expressions for MLE involve stochastic integrals which need good approximations for computational purposes. Furthermore, MLE are not robust in the sense that a slight perturbation in the noise component will change the properties of MLE substantially. In order to circumvent such problems, the minimum distance approach is proposed. Properties of the minimum distance estimators (MDE) were discussed in Millar (1984) in a general framework.

Our aim in this paper is to obtain the minimum L_1-norm estimates of the drift parameter of a fractional Ornstein–Uhlenbeck type process and investigate the asymptotic properties of such estimators following the work of Kutoyants and Pilibossian (1994).

2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ be a stochastic basis satisfying the usual conditions, and suppose that the processes discussed in the following are $(\mathcal{F}_t)$-adapted. Further the natural filtration of a process is understood as the P-completion of the filtration generated by this process.

Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0$, $\mathsf{E}(W_t^H) = 0$ and

$$\mathsf{E}\left(W_s^H W_t^H\right) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right], \qquad t \geq 0,\ s \geq 0. \tag{2.1}$$

Consider a stochastic process $\{X_t, t \geq 0\}$ defined by the stochastic integral equation

$$X_t = x_0 + \theta \int_0^t X(s)\, ds + \varepsilon W_t^H, \qquad 0 \leq t \leq T, \tag{2.2}$$

where θ is an unknown drift parameter. For convenience, we write the above integral equation in the form of a stochastic differential equation

$$dX_t = \theta X(t)\, dt + \varepsilon dW_t^H, \qquad X_0 = x_0,\ 0 \leq t \leq T, \tag{2.3}$$

driven by the fractional Brownian motion W^H. For a discussion of the equivalence of (2.2) and (2.3), see Cheridito et al. (2003). Even though the process X is not a semimartingale, one can associate with it a semimartingale $Z = \{Z_t, t \geq 0\}$, which is called a *fundamental semimartingale*, such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{X}_t)$ of the process X (Kleptsyna et al. (2000)). Define, for $0 < s < t$,

$$k_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right), \tag{2.4}$$

$$\kappa_H(t,s) = k_H^{-1} s^{1/2-H}(t-s)^{1/2-H}, \tag{2.5}$$

$$\lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \tag{2.6}$$

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2.7}$$

$$M_t^H = \int_0^t \kappa_H(t,s)\, dW_s^H, \qquad t \geq 0. \tag{2.8}$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. (1999)), and its quadratic variance is $\langle M_t^H \rangle = w_t^H$. Furthermore, the natural filtration of the martingale M^H coincides with the natural filtration of the fBM W^H. Let

$$K_H(t,s) = H(2H-1)\frac{d}{ds}\int_s^t r^{H-1/2}(r-s)^{H-3/2}\, dr, \qquad 0 \leq s \leq t. \tag{2.9}$$

The sample paths of the process $\{X_t, t \geq 0\}$ are smooth enough so that the process Q defined by

$$Q(t) = \frac{d}{dw_t^H}\int_0^t \kappa_H(t,s) X_s\, ds, \qquad t \in [0,T], \tag{2.10}$$

is well defined, where w^H and k_H are as defined in (2.7) and (2.5) respectively and the derivative is understood in the sense of absolute continuity with respect to the measure generated by w^H. Moreover, the sample paths of the process Q belong to $L^2([0,T], dw^H)$ a.s. $[\mathsf{P}]$. The following theorem due to Kleptsyna et al. (2000) associates with the process X a *fundamental semimartingale* Z such that the natural filtration $(\mathcal{Z}_t)$ coincides with the natural filtration $(\mathcal{X}_t)$ of X.

Theorem 2.1. *Let the process $Z = (Z_t, t \in [0,T])$ be defined by*

$$Z_t = \int_0^t \kappa_H(t,s)\, dX_s, \tag{2.11}$$

where the function $\kappa_H(t,s)$ is defined in (2.5). *Then the following results hold:*

(i) *The process Z is an $(\mathcal{F}_t)$-semimartingale with the decomposition*

$$Z_t = \theta \int_0^t Q(s)\, dw_s^H + \varepsilon M_t^H, \tag{2.12}$$

where M^H is the Gaussian martingale defined by (2.8).

(ii) *The process X admits the representation*

$$X_t = \int_0^t K_H(t,s)\,dZ_s, \tag{2.13}$$

where the function K_H is as defined in (2.9).

(iii) *The natural filtrations of $(\mathcal{Z}_t)$ and $(\mathcal{X}_t)$ coincide.*

Even though the fBm $\{W_t^H, t \geq 0\}$ is not a semimartingale, it is still possible to define stochastic integration with respect to the fBm for deterministic integrands. For instance, for $f \in L_2(\mathbf{R}_+) \cap L_1(\mathbf{R}_+)$, one can define a stochastic integral of the form

$$\int_0^T f(s)\,dW_s^H$$

(cf. Grippenberg and Norris (1996), Norros et al. (1999)). Such a stochastic integral can be represented in terms of another stochastic integral with respect to the fundamental Gaussian martingale M^H. The following result is due to Kleptsyna et al. (2000).

For any measurable function f on $[0,T]$ and for $t \in [0,T]$, define

$$K_H^f(t,s) = -2H\frac{d}{ds}\int_s^t f(r)r^{H-1/2}(r-s)^{H-3/2}\,dr, \qquad 0 \leq s \leq t, \tag{2.14}$$

where the derivative exists in the sense of absolute continuity with respect to Lebesgue measure (cf. Samko et al. (1993)).

Lemma 2.2. *Let M^H be the fundamental martingale associated with the fBm W^H. Then the following equality holds a.s.* $[\mathsf{P}]$*:*

$$\int_0^T f(s)\,dW_s^H = \int_0^T K_H^f(t,s)\,dM_s^H, \qquad t \in [0,T], \tag{2.15}$$

provided both integrals on both sides are well defined.

Alos et al. (2001) used the stochastic calculus of variations or Malliavin calculus to develop a stochastic calculus with respect to Gaussian processes, in particular, for a fractional Brownian motion. The fractional Brownian motion W_t^H can be represented in the form

$$W_t^H = \int_0^t K^H(t,s)\,dW_s,$$

where $\{W_t, t \geq 0\}$ is a Brownian motion and the kernel K^H is singular if $H < \frac{1}{2}$ and regular if $H > \frac{1}{2}$. For the definition of singularity and regularity of a kernel, see Alos et al. (2001). They develop an Itô formula in the regular case and in the singular case for $H > \frac{1}{4}$. We will not go into more discussion here.

The following lemma due to Grippenberg and Norris (1996) gives the covariance between the two stochastic integrals

$$\int_0^T f(s)\,dW_s^H \quad \text{and} \quad \int_0^T g(s)\,dW_s^H.$$

Lemma 2.3. *For $f, g \in L_2(\mathbf{R}_+) \cap L_1(\mathbf{R}_+)$,*

$$\begin{aligned} &\mathsf{E}\left(\int_0^\infty f(s)\,dW_s^H \int_0^\infty g(s)\,dW_s^H\right) \\ &\quad = H(2H-1)\int_0^\infty\int_0^\infty f(s)g(t)|s-t|^{2H-2}\,dt\,ds. \end{aligned} \tag{2.16}$$

The result in Lemma 2.3 can be proved under weaker conditions. Pipiras and Taqqu (2000) showed that for $H \in (\frac{1}{2}, 1)$ and for any two real-valued measurable functions

$$f, g \in \left\{ f\colon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u)| \cdot |f(v)| \cdot |u-v|^{2H-2}\, du\, dv < \infty \right\},$$

the result in Lemma 2.3 holds.

3. Minimum L_1-norm estimation

We now consider the problem of estimation of the parameter θ based on the observation of a fractional Ornstein–Uhlenbeck type process $X = \{X_t, 0 \le t \le T\}$ satisfying the stochastic differential equation

$$dX_t = \theta X(t)\, dt + \varepsilon\, dW_t^H, \qquad X_0 = x_0,\ 0 \le t \le T, \tag{3.1}$$

for a fixed time T, where $\theta \in \Theta \subset \mathbf{R}$, and study its asymptotic properties as $\varepsilon \to 0$.

Let $x_t(\theta)$ be the solution of the above differential equation with $\varepsilon = 0$. It is obvious that

$$x_t(\theta) = x_0 e^{\theta t}, \qquad 0 \le t \le T. \tag{3.2}$$

Let

$$S_T(\theta) = \int_0^T |X_t - x_t(\theta)|\, dt. \tag{3.3}$$

We define θ_ε^* to be a *minimum L_1-norm estimator* if there exists a measurable selection θ_ε^* such that

$$S_T(\theta_\varepsilon^*) = \inf_{\theta \in \Theta} S_T(\theta). \tag{3.4}$$

Conditions for the existence of a measurable selection are given in Lemma 3.1.2 in Prakasa Rao (1987). We assume that there exists a measurable selection θ_ε^* satisfying the above equation. An alternative way of defining the estimator θ_ε^* is by the relation

$$\theta_\varepsilon^* = \arg \inf_{\theta \in \Theta} \int_0^T |X_t - x_t(\theta)|\, dt. \tag{3.5}$$

Consistency. Let

$$W_T^{H*} = \sup_{0 \le t \le T} |W_t^H|.$$

The self-similarity of the fractional Brownian motion W_t^H implies that the random variables W_{at}^H and $a^H W_t$ have the same probability distribution for any $a > 0$. Furthermore, it follows from the self-similarity that the supremum process W^{H*} has the property that the random variables W_{at}^{H*} and $a^H W_t^{H*}$ have the same probability distribution for any $a > 0$. Hence we have the following observation due to Novikov and Valkeila (1999).

Lemma 3.1. *Let $T > 0$ and $\{W_t^H, 0 \le t \le T\}$ be an fBm with Hurst index H. Let $W_T^{H*} = \sup_{0 \le t \le T} W_t^H$. Then*

$$\mathsf{E}\left(W_T^{H*}\right)^p = K(p, H) T^{pH} \tag{3.6}$$

for every $p > 0$, where $K(p, H) = \mathsf{E}(W_1^{H*})^p$.

Let θ_0 denote the true parameter. For any $\delta > 0$, define

$$g(\delta) = \inf_{|\theta - \theta_0| > \delta} \int_0^T |X_t(\theta) - x_t(\theta_0)|\, dt. \tag{3.7}$$

Note that $g(\delta) > 0$ for any $\delta > 0$.

Theorem 3.2. *For every $p > 0$, there exists a constant $K(p, H)$ such that for every $\delta > 0$,*

$$\mathsf{P}_{\theta_0}^{(\varepsilon)}\{|\theta_\varepsilon^* - \theta_0| > \delta\} \leq 2^p T^{pH+p} K(p, H) e^{|\theta_0|Tp} (g(\delta))^{-p} \varepsilon^p = O((g(\delta))^{-p} \varepsilon^p). \tag{3.8}$$

Proof. Let $\|\cdot\|$ denote the L_1-norm. Then

$$\begin{aligned}
\mathsf{P}_{\theta_0}^{(\varepsilon)}\{|\theta_\varepsilon^* - \theta_0| > \delta\} &= \mathsf{P}_{\theta_0}^{(\varepsilon)}\left\{\inf_{|\theta-\theta_0|\leq\delta} \|X - x(\theta)\| > \inf_{|\theta-\theta_0|>\delta} \|X - x(\theta)\|\right\} \\
&\leq \mathsf{P}_{\theta_0}^{(\varepsilon)}\left\{\inf_{|\theta-\theta_0|\leq\delta} (\|X - x(\theta_0)\| + \|x(\theta) - x(\theta_0)\|)\right\} \\
&< \inf_{|\theta-\theta_0|>\delta} (\|x(\theta) - x(\theta_0)\| - \|X - x(\theta_0)\|) \\
&= \mathsf{P}_{\theta_0}^{(\varepsilon)}\left\{2\|X - x(\theta_0)\| > \inf_{|\theta-\theta_0|>\delta} \|x(\theta) - x(\theta_0)\|\right\} \\
&= \mathsf{P}_{\theta_0}^{(\varepsilon)}\left\{\|X - x(\theta_0)\| > \frac{1}{2} g(\delta)\right\}.
\end{aligned} \tag{3.9}$$

Since the process X_t satisfies the stochastic differential equation (3.2), it follows that

$$\begin{aligned}
X_t - x_t(\theta_0) &= x_0 + \theta_0 \int_0^t X_s\, ds + \varepsilon W_t^H - x_t(\theta_0) \\
&= \theta_0 \int_0^t (X_s - x_s(\theta_0))\, ds + \varepsilon W_t^H
\end{aligned} \tag{3.10}$$

since $x_t(\theta) = x_0 e^{\theta t}$. Let $U_t = X_t - x_t(\theta_0)$. Then it follows from the above equation that

$$U_t = \theta_0 \int_0^t U_s\, ds + \varepsilon W_t^H. \tag{3.11}$$

Let $V_t = |U_t| = |X_t - x_t(\theta_0)|$. The above relation implies that

$$V_t = |X_t - x_t(\theta_0)| \leq |\theta_0| \int_0^t V_s\, ds + \varepsilon \left|W_t^H\right|. \tag{3.12}$$

Applying the Gronwall–Bellman lemma, we obtain that

$$\sup_{0\leq t\leq T} |V_t| \leq \varepsilon e^{|\theta_0 T|} \sup_{0\leq t\leq T} |W_t^H|. \tag{3.13}$$

Hence

$$\begin{aligned}
\mathsf{P}_{\theta_0}^{(\varepsilon)}\left\{\|X - x(\theta_0)\| > \frac{1}{2} g(\delta)\right\} &\leq \mathsf{P}\left\{\sup_{0\leq t\leq T} |W_t^H| > \frac{e^{-|\theta_0 T|} g(\delta)}{2\varepsilon T}\right\} \\
&= \mathsf{P}\left\{W_T^{H*} > \frac{e^{-|\theta_0 T|} g(\delta)}{2\varepsilon T}\right\}.
\end{aligned} \tag{3.14}$$

Applying Lemma 3.1 to the estimate obtained above, we get that

$$\begin{aligned}
P_{\theta_0}^{(\varepsilon)}\{|\theta_\varepsilon^* - \theta_0| > \delta\} &\leq 2^p T^{pH+p} K(p, H) e^{|\theta_0 T| p} (g(\delta))^{-p} \varepsilon^p \\
&= O\left((g(\delta))^{-p} \varepsilon^p\right).
\end{aligned}$$

□

Remarks. As a consequence of the above theorem, we obtain that θ_ε^* converges in probability to θ_0 under $P_{\theta_0}^{(\varepsilon)}$-measure as $\varepsilon \to 0$. Furthermore, the rate of convergence is of order $O(\varepsilon^p)$ for every $p > 0$.

Asymptotic distribution. We will now study the asymptotic distribution if any of the estimator θ_ε^* after suitable scaling. It can be checked that

$$X_t = e^{\theta_0 t}\left\{x_0 + \int_0^t e^{-\theta_0 s}\varepsilon\, dW_s^H\right\} \tag{3.15}$$

or equivalently

$$X_t - x_t(\theta_0) = \varepsilon e^{\theta_0 t}\int_0^t e^{-\theta_0 s}\, dW_s^H. \tag{3.16}$$

Let

$$Y_t = e^{\theta_0 t}\int_0^t e^{-\theta_0 s}\, dW_s^H. \tag{3.17}$$

Note that $\{Y_t, 0 \le t \le T\}$ is a Gaussian process and can be interpreted as the "derivative" of the process $\{X_t, 0 \le t \le T\}$ with respect to ε. Applying Lemma 2.2, we obtain that, P-a.s.,

$$Y_t e^{-\theta_0 t} = \int_0^t e^{-\theta_0 s}\, dW_s^H = \int_0^t K_H^f(t,s)\, dM_s^H, \qquad t \in [0,T], \tag{3.18}$$

where $f(s) = e^{-\theta_0 s}$, $s \in [0,T]$, and M^H is the fundamental Gaussian martingale associated with the fBm W^H. In particular it follows that the random variable $Y_t e^{-\theta_0 t}$ and hence Y_t has normal distribution with mean zero, and furthermore, for any $h \ge 0$,

$$\begin{aligned} \operatorname{Cov}(Y_t, Y_{t+h}) &= e^{2\theta_0 t + \theta_0 h}\mathsf{E}\left[\int_0^t e^{-\theta_0 u}\, dW_u^H \int_0^{t+h} e^{-\theta_0 v}\, dW_v^H\right] \\ &= e^{2\theta_0 t + \theta_0 h} H(2H-1)\int_0^t\int_0^t e^{-\theta_0(u+v)}|u-v|^{2H-2}\, du\, dv \\ &= e^{2\theta_0 t + \theta_0 h}\gamma_H(t) \quad \text{(say)}. \end{aligned} \tag{3.19}$$

In particular

$$\operatorname{Var}(Y_t) = e^{2\theta_0 t}\gamma_H(t). \tag{3.20}$$

Hence $\{Y_t, 0 \le t \le T\}$ is a zero mean Gaussian process with $\operatorname{Cov}(Y_t, Y_s) = e^{\theta_0(t+s)}\gamma_H(t)$ for $s \ge t$.

Let

$$\zeta = \arg\inf_{-\infty<u<\infty}\int_0^T \left|Y_t - utx_0 e^{\theta_0 t}\right|\, dt. \tag{3.21}$$

Theorem 3.3. *As $\varepsilon \to 0$, the random variable $\varepsilon^{-1}(\theta_\varepsilon^* - \theta_0)$ converges in probability to a random variable whose probability distribution is the same as that of ζ under P_{θ_0}.*

Proof. Let $x_t'(\theta) = x_0 t e^{\theta t}$ and let

$$Z_\varepsilon(u) = \left\|Y - \varepsilon^{-1}(x(\theta_0 + \varepsilon u) - x(\theta_0))\right\| \tag{3.22}$$

and

$$Z_0(u) = \|Y - ux'(\theta_0)\|. \tag{3.23}$$

Furthermore, let

$$A_\varepsilon = \{\omega\colon |\theta_\varepsilon^* - \theta_0| < \delta_\varepsilon\}, \qquad \delta_\varepsilon = \varepsilon^\tau,\ \tau \in \left(\frac{1}{2}, 1\right),\ L_\varepsilon = \varepsilon^{\tau-1}. \tag{3.24}$$

Observe that the random variable $u_\varepsilon^* = \varepsilon^{-1}(\theta_\varepsilon^* - \theta_0)$ satisfies the equation

$$Z_\varepsilon(u_\varepsilon^*) = \inf_{|u|<L_\varepsilon} Z_\varepsilon(u), \qquad \omega \in A_\varepsilon. \tag{3.25}$$

Define

$$\zeta_\varepsilon = \arg \inf_{|u|<L_\varepsilon} Z_0(u). \tag{3.26}$$

Observe that, with probability one,

$$\begin{aligned}\sup_{|u|<L_\varepsilon} |Z_\varepsilon(u) - Z_0(u)| &= \left| \left\| Y - ux'(\theta_0) - \frac{1}{2}\varepsilon u^2 x''(\tilde{\theta}) \right\| - \|Y - ux'(\theta_0)\| \right| \\ &\leq \frac{\varepsilon}{2} L_\varepsilon^2 \sup_{|\theta-\theta_0|<\delta_\varepsilon} \int_0^T |x''(\theta)|\, dt \leq C\varepsilon^{2\tau-1}.\end{aligned} \tag{3.27}$$

Here $\tilde{\theta} = \theta_0 + \alpha(\theta - \theta_0)$ for some $\alpha \in (0,1)$. Note that the last term in the above inequality tends to zero as $\varepsilon \to 0$. Furthermore, the process $\{Z_0(u), -\infty < u < \infty\}$ has a unique minimum u^* with probability one. This follows from the arguments given in Theorem 2 of Kutoyants and Pilibossian (1994). In addition, we can choose the interval $[-L, L]$ such that

$$\mathsf{P}_{\theta_0}^{(\varepsilon)}\{u_\varepsilon^* \in (-L, L)\} \geq 1 - \beta g(L)^{-p} \tag{3.28}$$

and

$$\mathsf{P}\{u^* \in (-L, L)\} \geq 1 - \beta g(L)^{-p}, \tag{3.29}$$

where $\beta > 0$. Note that $g(L)$ increases as L increases. The processes $Z_\varepsilon(u)$, $u \in [-L, L]$, and $Z_0(u)$, $u \in [-L, L]$, satisfy the Lipschitz conditions and $Z_\varepsilon(u)$ converges uniformly to $Z_0(u)$ over $u \in [-L, L]$. Hence the minimizer of $Z_\varepsilon(\cdot)$ converges to the minimizer of $Z_0(u)$. This completes the proof. $\square$

Remarks. We have seen earlier that the process $\{Y_t, 0 \leq t \leq T\}$ is a zero mean Gaussian process with the covariance function

$$\mathrm{Cov}(Y_t, Y_s) = e^{\theta_0(t+s)} \gamma_H(t)$$

for $s \geq t$. Recall that

$$\zeta = \arg \inf_{-\infty<u<\infty} \int_0^T \left| Y_t - utx_0 e^{\theta_0 t} \right| dt. \tag{3.30}$$

It is not clear what the distribution of ζ is. Observe that for every u, the integrand in the above integral is the absolute value of a Gaussian process $\{J_t, 0 \leq t \leq T\}$ with the mean function $\mathsf{E}(J_t) = -utx_0 e^{\theta_0 t}$ and the covariance function $\mathrm{Cov}(J_t, J_s) = e^{\theta_0(t+s)}\gamma_H(t)$ for $s \geq t$. It would be interesting to say something about the distribution of ζ through simulation studies even if an explicit computation of the distribution seems to be difficult.

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