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WHITTLE TYPE INEQUALITY FOR DEMISUBMARTINGALES

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ABSTRACT. A Whittle type inequality for demisubmartingales is derived and a strong law of large numbers for functions of a demisubmartingale is obtained.

1. Introduction

Whittle ([12]) proved an inequality for real valued random variables generalizing the Kolmogorov inequality, the inequality of Hajek-Renyi ([3]) and the inequality of Dufresnoy ([2]). An application of this result for Hilbert space valued random elements $\{Z_k, k \geq 1\}$ such that the family $\{\phi_k(Z_k), k \geq 1\}$ is a real valued submartingale is given in Rao ([8]). An application of this result to obtain a lower bound for the probability of a simultaneous confidence region in multivariate analysis is given in Rao ([8]) sharpening the bound given in Sen ([10]). Recently Shixin ([11]) proved a Hajek-Renyi type inequality for Banach space valued martingales. A Whittle type inequality for Banach space valued martingales was given in Prakasa Rao ([6]) from which the results in Shixin ([11]) follow as special cases.

We now derive a Whittle type inequality for demisubmartingales. This result generalises the recent results on Hajek-Renyi type inequality for demimartingales proved by Christofides ([1]) and the Hajek-Renyi type inequality for associated sequences proved by Prakasa Rao ([5]).

2. Preliminaries

Let $S_i, i \geq 1$, be a sequence of integrable random variables such that

$$E\{(S_{j+1} - S_j) f(S_1, \ldots, S_j)\} \geq 0, j \geq 1, \tag{2.1}$$

for every componentwise nondecreasing function f such that the expectation is defined. Then the sequence $\{S_j, j \geq 1\}$ is called a *demimartingale* (cf. Newman and Wright ([4])). If condition (2.1) holds for every componentwise nonnegative nondecreasing function f such that the expectation is defined, then the sequence $\{S_j, j \geq 1\}$ is called a *demisubmartingale*.

A collection of random variables $X_i, 1 \leq i \leq n$, is said to be *associated* if

$$Cov(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0 \tag{2.2}$$

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for any two componentwise nondecreasing functions f and g such that the covariance exists. An infinite sequence of random variables $\{X_n, n \geq 1\}$ is said to be *associated* if every finite subset of $\{X_n, n \geq 1\}$ is associated.

If $X_i, 1 \leq i \leq n$, is an associated sequence of random variables with $E(X_i) = 0, 1 \leq i \leq n$, then the sequence of partial sums $S_i = X_1 + \cdots + X_i, 1 \leq i \leq n$, forms a demimartingale (cf. Newman and Wright [4]).

For an extensive review of the probabilistic properties of associated sequences of random variables and related statistical inference problems, see Prakasa Rao and Dewan ([7]) and Roussas ([9]).

3. Whittle type inequality

Let $S_n, n \geq 1$, be a demisubmartingale and $\phi(.)$ be a nondecreasing convex function. Then the sequence $\phi(S_n), n \geq 1$, is a demisubmartingale by Lemma 2.1 of Christofides ([1]).

We now state our main theorem.

Theorem 3.1. *Let the sequence of random variables $\{S_n, n \geq 1\}$ be a demisubmartingale and $\phi(.)$ be a nonnegative nondecreasing convex function such that $\phi(S_0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$. Let A_n be the event that $\phi(S_k) \leq \psi(u_k), 1 \leq k \leq n$, where $0 = u_0 < u_1 \leq \cdots \leq u_n$. Then*

$$P(A_n) \geq 1 - \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}. \tag{3.1}$$

If, in addition, there exist nonnegative real numbers $\Delta_k, 1 \leq k \leq n$, such that

$$\begin{aligned} 0 &\leq E[(\phi(S_k) - \phi(S_{k-1}))f(\phi(S_1), \ldots, \phi(S_{k-1}))] \\ &\leq \Delta_k E[f(\phi(S_1), \ldots, \phi(S_{k-1}))], 1 \leq k \leq n, \end{aligned}$$

for all componentwise nonnegative nondecreasing functions f such that the expectation is defined and

$$\psi(u_k) \geq \psi(u_{k-1}) + \Delta_k, 1 \leq k \leq n,$$

then

$$P(A_n) \geq \prod_{k=1}^{n} (1 - \frac{\Delta_k}{\psi(u_k)}). \tag{3.2}$$

Remarks. The above result is an analogue of the inequality in Whittle ([12]) for real valued random variables. A version of Theorem 3.1 for a sequence of Hilbert space valued random elements was proved in Rao ([8]) and an application to Banach space valued martingales is given in Prakasa Rao ([6]).

Proof. Since the sequence $\{S_n, n \geq 1\}$ is a demisubmartingale by hypothesis and the function $\phi(.)$ is a nondecreasing convex function, it follows that the sequence $\{\phi(S_n), n \geq 1\}$ forms a demisubmartingale by Lemma 2.1 of Christofides ([1]). Hence

$$E\{(\phi(S_{n+1}) - \phi(S_n))f(\phi(S_1), \ldots, \phi(S_n))\} \geq 0, n \geq 1, \tag{3.3}$$

for every nonnegative componentwise nondecreasing function f such that the expectation is defined.

Let χ_j be the indicator function of the event $[\phi(S_j) \leq \psi(u_j)]$ for $1 \leq j \leq n$.

Note that

$$\chi_n \geq (1 - \frac{\phi(S_n)}{\psi(u_n)})$$

and hence

$$\begin{aligned} P(A_n) &= E(\prod_{i=1}^{n} \chi_i) = E(\{\prod_{i=1}^{n-1} \chi_i\}\chi_n) \\ &\geq E(\{\prod_{i=1}^{n-1} \chi_i\}(1 - \frac{\phi(S_n)}{\psi(u_n)})). \end{aligned}$$

Therefore

$$\begin{aligned} & E[\{\prod_{i=1}^{n-1} \chi_i\}\{(1 - \frac{\phi(S_n)}{\psi(u_n)}) - (1 - \frac{\phi(S_{n-1})}{\psi(u_n)})\} + \frac{\phi(S_n) - \phi(S_{n-1})}{\psi(u_n)}] \\ & \qquad = E[(1 - \prod_{i=1}^{n-1} \chi_i)(\frac{\phi(S_n) - \phi(S_{n-1})}{\psi(u_n)})] \geq 0 \end{aligned}$$

since the function $1 - \prod_{i=1}^{n-1} \chi_i$ is a nonnegative componentwise nondecreasing function of $\phi(S_i), 1 \leq i \leq n-1$. Hence

$$\begin{aligned} P(A_n) &\geq E(\{\prod_{i=1}^{n-1} \chi_i\}(1 - \frac{\phi(S_{n-1})}{\psi(u_n)})) - \frac{E\{\phi(S_n)\} - E\{\phi(S_{n-1})\}}{\psi(u_n)} \\ &\geq E(\{\prod_{i=1}^{n-2} \chi_i\}(1 - \frac{\phi(S_{n-1})}{\psi(u_{n-1})})) - \frac{E\{\phi(S_n)\} - E\{\phi(S_{n-1})\}}{\psi(u_n)}. \end{aligned}$$

The last inequality follows from the observation that the sequence $\psi(u_n), n \geq 1$, is positive and nondecreasing.

Applying this inequality repeatedly, we get that

$$P(A_n) \geq 1 - \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}, \tag{3.4}$$

completing the proof of the first part of the theorem.

Note that

$$\begin{aligned} & E\{\prod_{i=1}^{n-1} \chi_i(1 - \frac{\phi(S_n)}{\psi(u_n)}) - (1 - \frac{\Delta_n}{\psi(u_n)})(1 - \frac{\phi(S_{n-1})}{\psi(u_{n-1})}) \prod_{i=1}^{n-1} \chi_i\} \\ & \qquad \geq E\{\frac{\phi(S_{n-1})}{\psi(u_n)\psi(u_{n-1})}[\psi(u_n) - \psi(u_{n-1}) - \Delta_n] \prod_{i=1}^{n-1} \chi_i\} \end{aligned}$$

and the last term is nonnegative by hypothesis. Hence

$$P(A_n) \geq (1 - \frac{\Delta_n}{\psi(u_n)})E(\{\prod_{i=1}^{n-2} \chi_i\}(1 - \frac{\phi(S_{n-1})}{\psi(u_{n-1})})). \tag{3.5}$$

Applying this inequality repeatedly, we obtain that

$$P(A_n) \geq \prod_{k=1}^{n}(1 - \frac{\Delta_k}{\psi(u_k)}). \tag{3.6}$$

4. Applications

Suppose $\{S_n, n \geq 1\}$ is a demisubmartingale. Then $\{(S_n^+)^p, n \geq 1\}$ and $\{(S_n^-)^p, n \geq 1\}$ are demisubmartingales by Corollary 2.1 of Christofides ([1]). Furthermore $|S_n|^p = (S_n^+)^p + (S_n^-)^p$ for all $p \geq 1$.

(1) Let $\psi(u) = u^p, p \geq 1$. Applying Theorem 3.1, we get that

$$P(S_j^+ \leq u_j, 1 \leq j \leq n) \geq 1 - \sum_{j=1}^{n} \frac{E(S_j^+)^p - E(S_{j-1}^+)^p}{u_j^p} \tag{4.1}$$

and

$$P(S_j^- \leq u_j, 1 \leq j \leq n) \geq 1 - \sum_{j=1}^{n} \frac{E(S_j^-)^p - E(S_{j-1}^-)^p}{u_j^p}. \tag{4.2}$$

Hence, for every $\varepsilon > 0$,

$$\begin{aligned} P(\sup_{1\leq j\leq n} \frac{|S_j|}{u_j} \geq \varepsilon) &= P(\sup_{1\leq j\leq n} \frac{|S_j|^p}{u_j^p} \geq \varepsilon^p) \\ &= P(\sup_{1\leq j\leq n} \frac{(S_j^+)^p + (S_j^-)^p}{u_j^p} \geq \varepsilon^p) \\ &= P(\sup_{1\leq j\leq n} \frac{(S_j^+)^p}{u_j^p} \geq \frac{1}{2}\varepsilon^p) \\ &\quad + P(\sup_{1\leq j\leq n} \frac{(S_j^-)^p}{u_j^p} \geq \frac{1}{2}\varepsilon^p) \\ &\leq 2\varepsilon^{-p} \sum_{j=1}^{n} \frac{E(S_j^+)^p - E(S_{j-1}^+)^p}{u_j^p} \\ &\quad + 2\varepsilon^{-p} \sum_{j=1}^{n} \frac{E(S_j^-)^p - E(S_{j-1}^-)^p}{u_j^p} \\ &\leq 2\varepsilon^{-p} \sum_{j=1}^{n} \frac{E|S_j|^p - E|S_{j-1}|^p}{u_j^p}. \end{aligned}$$

In particular for p=2, we have

$$P(\sup_{1\leq j\leq n} \frac{|S_j|}{u_j} \geq \varepsilon) \leq 2\varepsilon^{-2} \sum_{j=1}^{n} \frac{ES_j^2 - ES_{j-1}^2}{u_j^2}, \tag{4.3}$$

which is the Hajek-Renyi type inequality for associated sequences derived in Corollary 2.3 of Christofides ([1]).

Suppose $p = 1$. Let $\phi(x) = max(0, x)$. Then $\phi(x)$ is a nonnegative nondecreasing convex function and it is clear that $S_n \leq S_n^+ = \phi(S_n)$ for every $n \geq 1$. Let $\psi(u) = u$. Then

$$\begin{aligned} P(\sup_{1\leq j\leq n} \frac{S_j}{u_j} \geq \varepsilon) &\leq P(\sup_{1\leq j\leq n} \frac{S_j^+}{u_j} \geq \varepsilon) \\ &\leq \varepsilon^{-1} \sum_{j=1}^{n} \frac{ES_j^+ - ES_{j-1}^+}{u_j} \end{aligned}$$

by Theorem 3.1 which is the Chow type maximal inequality derived in Theorem 2.1 of Christofides ([1]).

(2) Let p=2 again in the above discussion. If

$$E(S_j^2 - S_{j-1}^2) \leq u_j^2 - u_{j-1}^2$$

for $1 \leq j \leq n$, then

$$P(A_n) \geq \prod_{j=1}^{n}(1 - \frac{E(S_j^2) - E(S_{j-1}^2)}{u_j^2})$$

which is an analogue of the Dufresnoy's inequality.

(3) Let $\{S_n, n \geq 1\}$ be a demisubmartingale and $\phi(.)$ be a nonnegative nondecreasing convex function such that $\phi(S_0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$. Then, for any nondecreasing sequence $u_n, n \geq 1$ with $u_0 = 0$,

$$P(\sup_{1\leq j\leq n} \frac{\phi(S_j)}{\psi(u_j)} \geq \varepsilon) \leq \varepsilon^{-1} \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}. \tag{4.4}$$

In particular, for any fixed $n \geq 1$,

$$P(\sup_{k\geq n} \frac{\phi(S_k)}{\psi(u_k)} \geq \varepsilon) \leq \varepsilon^{-1}[E(\frac{\phi(S_n)}{\psi(u_n)}) + \sum_{k=n+1}^{\infty} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}]. \tag{4.5}$$

We now derive a strong law of large numbers for functions of demisubmartingales.

Theorem 4.1. *Let $\{S_n, n \geq 1\}$ be a demisubmartingale and $\phi(.)$ be a nonnegative nondecreasing convex function such that $\phi(S_0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$ such that $\psi(u) \to \infty$ as $u \to \infty$. Further suppose that*

$$\sum_{k=1}^{\infty} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)} < \infty \tag{4.6}$$

for a nondecreasing sequence $u_n \to \infty$ as $n \to \infty$. Then

$$\frac{\phi(S_n)}{\psi(u_n)} \xrightarrow{a.s} 0 \;\; as \;\; n \to \infty. \tag{4.7}$$

Proof of this result follows by the standard arguments following the inequality (4.5) given above. We omit the details.

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