On Fourier coefficients of Maass cusp forms in 3-dimensional hyperbolic space

S RAGHAVAN and J SENGUPTA
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Bombay 400 005, India

Dedicated to the memory of Professor K G Ramanathan

Abstract. In this article we establish the analogue of a theorem of Kuznetsov (theorem 6 of [3]) in the case of 3-dimensional hyperbolic space. We also consider a generalization of this result for higher dimensional hyperbolic spaces and discuss the relevant ingredients of a proof.

Keywords. Fourier coefficients, Maass cusp forms; 3-dimensional hyperbolic space; Kuznetsov theorem.

Let \( \mathbb{H}_3 = \{ w = z + jy | z = x_1 + ix_2 \in \mathbb{C}, x_1, x_2, y \in \mathbb{R}, y > 0, ij = -ji, j^2 = -1 = i^2 \} \) be the 3-dimensional hyperbolic space. The group \( \text{PSL}(2; \mathbb{C}) \) acts on \( \mathbb{H}_3 \) via the mappings \( w \to \gamma \langle w \rangle := (aw + b)(cw + d)^{-1} \) for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2; \mathbb{C}) \). If \( R \) denotes the ring of integers in an imaginary quadratic field \( K \) over \( \mathbb{Q} \) of discriminant \( d = d_K < 0 \) in \( \mathbb{Z} \), then \( \Gamma := \text{PSL}(2; R) \) is a discontinuous group of homeomorphisms \( w \to \gamma \langle w \rangle \) of \( \mathbb{H}_3 \) onto itself, with a fundamental domain \( \mathcal{F} \). On \( \mathbb{H}_3 \), we have the \( \text{PSL}(2; \mathbb{C}) \)-invariant volume element \( dv = y^{-3} \, dx_1 \, dx_2 \, dy \). Let \( \Gamma_{\infty} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right\} \) and \( \Gamma_{\infty}^\prime := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \Gamma_{\infty} \right\} \).

For \( w \in \mathbb{H}_3 \) and \( \gamma \in \Gamma \), we write \( \gamma \langle w \rangle = z(\gamma \langle w \rangle) + jy(\gamma \langle w \rangle) \).

For \( n \) in \( R^* := \{ 2a/\sqrt{d} | a \in R \} \), \( w \in \mathbb{H}_3 \) and \( s \) in \( \mathbb{C} \) with \( \text{Re}(s) > 2 \), the Poincaré series \( P(w, s; n) \) is defined by

\[
P(w, s; n) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} y(\gamma \langle w \rangle)^n \exp[-2\pi|n|y(\gamma \langle w \rangle) + 2\pi i \text{Re}(z(\gamma \langle w \rangle)\bar{n})].
\]

For \( n = 0 \), \( P(w, s; 0) \) is precisely the Eisenstein series denoted by \( E(w, s) \). While \( P(w, s; n) \) is real analytic on \( \mathbb{H}_3 \) and invariant under \( \Gamma \), it is in \( L^2(\mathcal{F}) \) only for \( n \neq 0 \); on the other hand, it is an eigenfunction of the Laplacian \( \Delta \) on \( \mathbb{H}_3 \) only for \( n = 0 \), with eigenvalue \( s(s - 2) \).

The Eisenstein series \( E(w, s) \) can be continued meromorphically for all \( s \) in \( \mathbb{C} \), the only singularity being a simple pole at \( s = 2 \). The Poincaré series \( P(w, s; n) \), for \( n \neq 0 \), have a meromorphic continuation for all \( \text{Re}(s) > 1 \) and are holomorphic there except possibly at a finite number of points \( s_i \); the finitely many \( \mu_i := s_i(2 - s_i) \) in \( (0, 1) \) are called the 'exceptional' eigenvalues (in the spectrum of \( \Delta \)). The Eisenstein series \( E(w, 1 + it) \) for \( t \) in \( \mathbb{R} \) together with similar series for the cusps of \( \mathcal{F} \) different from \( \infty \) 'span' the continuous spectrum of \( \Delta \); further, within \( [1, \infty) \), there is also the discrete
spectrum of $\Delta$ giving rise to the Maass waveforms $u_l(l=1,2,\ldots)$ which are eigenfunctions for $\Delta$ in $L^2(\Gamma \setminus \mathbb{H}_3)$ with $\Delta u_l + \lambda_l u_l = 0$ and $\lambda_l \geq 1$. For these $u_l$ as well as the eigenfunctions corresponding to $\mu_l$, we have at the infinite cusp, the Fourier expansion

$$u_l(w) = \sum_{0 \neq m \in \mathbb{R}^\times} \rho_l(m) y K_{2l}(2\pi |m| y) \exp[2\pi i \text{Re}(\bar{m} w)]$$  \hspace{1cm} (1)$$

where $K_v(\cdot)$ is the Macdonald function of order $v$ and $\chi_l := (\lambda_l - 1)^{1/2}$ or $(\lambda_l - 1)^{1/2}$.

If $w'$ is the order of the group $R^\times$ of units of $R$, then the index $[\Gamma_\infty : \Gamma_{w'}] = w'/2$. In the case of fields $K$ with class number 1, we know from ([1], Theorem 4.12) that the Eisenstein series $E(w,s)$ has the Fourier expansion

$$E(w,s) = \frac{w'}{2} y^s + \frac{\pi w'}{(s-1) \sqrt{|d_K|}} \zeta_K(s-1) y^{2-s} + \frac{2\pi s}{(s-1) \zeta_K(s) \sqrt{|d_K|}} \Gamma(s) \zeta_K(s)$$ \hspace{1cm} (2)$$

$$\times \sum_{0 \neq m \in \mathbb{R}^\times} |m|^{s-1} \sigma_{1-s}(\frac{m}{2} \sqrt{|d_K|}) y K_{s-1}(2\pi |m| y) \exp[2\pi i \text{Re}(\bar{m} w)]$$

where $\zeta_K$ is the Dedekind zeta function for $K$ and $\sigma_v(m) := \sum_{f \mid m} |f|^{2v}$. We note that the expansion (2) is valid even for $s = 1 + ir$ for $r \in \mathbb{R}$. Let us write $\langle f,g \rangle : = \int_{\mathbb{R}} f \overline{g} \text{dv}$ for $f,g$ (measurable on $\mathcal{F}$) and $v(\mathcal{F}) := \int \text{dv}$.

Our object is to obtain an estimate for the Fourier coefficients $\rho_l(m)$ of $u_l$ in terms of $m$ and $\chi_l$. This estimate can be deduced from an asymptotic formula for

$$A(X) := \sum_{|x| \leq X} \frac{|\rho_l(m)|^2}{\Gamma(1 - i \chi_l)^2}$$  \hspace{1cm} (3)$$

More specifically, we shall prove the following.

**Theorem.** For $A(X;m) = A(X)$ defined by (3), we have, as $X \to \infty$,

$$A(X;m) = \frac{y(\mathcal{F})}{\pi^3 |d_K| c_1} X^3 + O(X^{5/2} |m|^{1+\varepsilon})$$  \hspace{1cm} (4)$$

for any $\varepsilon > 0$ with the $O-$constant depending on $\varepsilon$, where $c_1 := 2 \int_0^\infty \frac{u}{\sinh \pi u} \text{du}$. As a consequence, we obtain

$$\rho_l(m) = O(\chi_l^{3/4} e^{\varepsilon |m|^{1/2}} |m|^{1/2 + \varepsilon})$$  \hspace{1cm} (5)$$

and $O$-constant depending at most on $\varepsilon$.

**Remark.** Surprisingly, the contribution to $A(X)$ from the exceptional eigenvalues is very much under control here (unlike in the problem of sums of Kloosterman sums).

The proof of the theorem follows Kuznetsov [3] and at the same time, hopefully helps in clearing a few obscurities. It rests, of course, on an identity arising from the computation of the inner product

$$\langle P_\varepsilon(2 + it; m), P_\varepsilon(2 + it; n) \rangle$$ for $t$ in $\mathbb{R}$ and $m, n \neq 0$ in $R^*$ \hspace{1cm} (6)$$

in two different ways, namely via the Parseval relation for $L^2(\mathcal{F})$ and also using the
Fourier expansion of the Poincaré series involved. We proceed to give the details of the proof wherein, for the sake of simplicity, we shall restrict ourselves to the case when the field $K$ has class number 1.

Let us first establish that for $\text{Res} > 2, r \in \mathbb{R}$ and $m \neq 0$ in $R^*$,

$$
\langle P(., s; m), E(., 1 + ir) \rangle = 
\frac{2^{1 + 2ir} \pi^{3/2}}{\sqrt{|d_K| (4\pi m)^{s-1 + ir}}} \frac{\Gamma(s - 1 + ir) \Gamma(s - 1 - ir) \sigma_m \left( \frac{m}{2} \sqrt{d_K} \right)}{\Gamma(1 - ir) \Gamma(s - \frac{1}{2}) \zeta_K(1 - ir)}
$$

(7)

The left hand side is precisely

$$
\int_{\mathbb{R} \setminus \mathbb{H}_3} \sum_{\gamma \in \Gamma \backslash \Gamma_0} \gamma \langle y \langle w \rangle \rangle^* \exp[-2\pi|y|\langle y \langle w \rangle \rangle + 2\pi i \text{Re}(y \langle w \rangle \langle m \rangle)] \bar{E}(w, 1 + ir) \, dv
$$

$$
= \int_{\Gamma \setminus \mathbb{H}_3} y^* \exp[-2\pi|y|^2] \bar{E}(w, 1 + ir) \exp[2\pi i \text{Re}(z \langle m \rangle)] \, dv
$$

$$
= \int_0^\infty y^{s-3} \exp[-2\pi|m|y] \, dy \int_{\mathbb{R} \setminus \mathbb{C}} \bar{E}(w, 1 + ir) \exp[-2\pi i \text{Re}(z \langle m \rangle)] \, dx, \, dx_2
$$

$$
= \int_0^\infty y^{s-3} \exp[-2\pi|m|y] \left( \frac{2\pi^{1 - ir}|m|^{-ir} \sigma_m \left( \frac{m}{2} \sqrt{d_K} \right)}{\sqrt{|d_K|} \Gamma(1 - ir) \zeta_K(1 - ir)} \right) \, dy, \text{by (2)}
$$

$$
= \frac{2\sqrt{\pi} \Gamma(s - 1 + ir) \Gamma(s - 1 - ir) \pi^{1 - ir} \sigma_m \left( \frac{m}{2} \sqrt{d_K} \right)}{|d_K|^{1/2} (4\pi m)^{s-1} \Gamma(s - 1/2) |m|^r \Gamma(1 - ir) \zeta_K(1 - ir)}
$$

which proves (7). Using now (7) with $m, n \neq 0$ in $R^*$,

$$
\int_{-\infty}^\infty \langle P(., s; m), E(., 1 + ir) \rangle \langle P(., s; n), E(., 1 + ir) \rangle \, dr
$$

$$
= \frac{4\pi^3 (4\pi)^{2 - s - r}}{|m|^{s-1} |n|^{s-1} |d_K|}
$$

$$
\times \int_0^\infty \left| \frac{\Gamma(s - 1 + ir) |\Gamma(s - 1 - ir)|^2 \sigma_m \left( \frac{m}{2} \sqrt{d_K} \right) \sigma_n \left( \frac{n}{2} \sqrt{d_K} \right)}{|\Gamma(1 - ir)|^2 |\Gamma(s - 1/2)|^2 \zeta_K(1 + ir)^2} \right| \, dr.
$$

(8)

On the other hand, we know from Sarnak [4], that, for $t$ in $\mathbb{R}$,

$$
\langle P(., 2 + it; m), u_t(.) \rangle \langle P(., 2 + it; n), u_t(.) \rangle
$$

$$
= \left| d_K \rho_j(m) \rho_j(n) \frac{|\Gamma(1 + i(t + \chi))|^2 |\Gamma(1 + i(t - \chi))|^2}{16\pi |m|^{1+it} |n|^{1-it} |\Gamma(3/2 + it)|^2} \right|^2.
$$

(9)
Using (8) and (9), the Parseval relation gives us immediately that the inner product in (6) is nothing but

\[
\frac{|d_K|}{16\pi|m|^{1+i\epsilon}|n|^{1-i\epsilon}} \sum_{\ell=1}^{\infty} \rho_\ell(m) \rho_\ell(n) \frac{|\Gamma(1+i(t+\chi))|^2|\Gamma(1+i(t-\chi))|^2}{|\Gamma(3/2+it)|^2} 
\]
\[
+ \frac{4\pi^2|\Gamma(3/2+it)|^{-2}}{(4\pi^2|d_K||m|^{1+i\epsilon}|n|^{1-i\epsilon})} \times \int_{-\infty}^{\infty} \frac{n^{|i\epsilon}}{m} \left|\frac{|\Gamma(1+i(t+r))|^2|\Gamma(1+i(t-r))|^2}{|\Gamma(1-i\epsilon)|^2} \right| \sigma_{\nu}
\left(\frac{m\sqrt{d_K}}{n} \right) \sigma_{\nu}
\left(\frac{n\sqrt{d_K}}{m} \right) dr
\]
\[
\times \frac{\zeta_K(1-i\epsilon)^2}{\zeta_K(1-i\epsilon)^2}
\]
(10)

Using the Fourier expansion of the Poincaré series \(P(\cdot, 2+it, \cdot)\) from [4], we see that the inner product in (6) also equals

\[
\frac{\delta_m v(\mathcal{F}_R)}{4\pi^2(|m|+|n|)^2} + \sum_{0 \neq \alpha \in \mathbb{R} \mod \pm 1} \frac{S(m, n; c)}{|c|^{4+2it}} \int_0^\infty \frac{dy}{y^{1+2it}} \int_{\mathbb{R}^2} \left[ -2\pi i \left( \frac{|m|}{y|c|^2(1+|z|^2)} \right) + 2\pi i \text{Re} \left( \frac{z\bar{m}}{c^2 y(1+|z|^2)} - y\bar{z} \right) \right. 
\]
\[
\left. \times \frac{dx_1 dx_2}{(1+|z|^2)^{2+i\epsilon}} \right] 
\]
(11)

where

\[
S(m, n; c) = \sum_{a \equiv \pm 1 (\mod c)} \exp[2\pi i \text{Re}((\bar{m}a + \bar{n}\bar{a})/c)].
\]

For the Kloosterman sums \(S(m, m; c)\) we have from [2], the estimate similar to Weil's, viz. for every \(\epsilon > 0\),

\[
|S(m, m; c)| \leq |c|^{1+\epsilon} |(m, \sqrt{d_K}, c)| \sigma_0(c/(m, \sqrt{d_K}, c))
\]
(12)

On the same lines as in Kuznetsov [3], let us introduce

\[
H(r, t) = \frac{|\Gamma(1+i(t+r))|^2|\Gamma(1+i(t-r))|^2}{\pi |\Gamma(1-i\epsilon)|^2}
\]

and

\[
h_Y(r) = \int_0^Y \cosh(\pi t) H(r, t) dt, \text{ for } Y > 0 \text{ and } r \in \mathbb{C} \text{ with } |Imr| < 1.
\]

Moreover, let us multiply the two equal expressions (10) and (11) for the inner product in (6) with \(m = n\), throughout by

\[
\frac{16\pi^2|m|^2}{|d_K|} \left( \frac{1}{4} + r^2 \right) = \frac{16\pi^2|m|^2}{|d_K|} \cosh(\pi t)|\Gamma(3/2+it)|^2
\]

and then integrate both the resulting expressions with respect to \(t\) from 0 to \(Y\). We
then obtain
\[
\sum_{l} \frac{\rho_{l}(m)^{2}}{1/\Gamma(1-\delta_{l})} \cdot h_{y}(x_{l}) + \frac{\pi}{|d_{K}|} \int_{0}^{\infty} \cosh(\pi t) dt \int_{-\infty}^{\infty} \frac{H(r, L)\sigma_{\nu}(\frac{m}{2}\sqrt{d_{K}})}{|\zeta_{K}(1+ir)|^{2}} dr =
\]
\[
v(\mathcal{F}_{K}) \left( \frac{1}{3} y^{3} + \frac{1}{4} y \right) + \frac{16\pi|m|^{2}}{|d_{K}|} \sum_{c \equiv 1 \mod 1} |c|^{-1} S(m, m; c) \times
\]
\[
\times \int_{0}^{\infty} \int_{0}^{\infty} \int_{\mathbb{R}^{3}} \left( \frac{1}{4} + t^{2} \right) (y^{2}|\delta_{1}|)^{-1/2}
\]
\[
\exp[-2\pi|m|/(y + 1/(y|\delta_{1}|))] - 2\pi i \text{Re}(\bar{m}z(y + 1/(y|\delta_{1}|)))] dt \frac{dy}{y} \frac{dx_{1} dx_{2}}{y (1 + x_{1}^{2} + x_{2}^{2})^{2}}
\]
using the abbreviation \( \delta_{1} \) for \( c^{2}(1 + x_{1}^{2} + x_{2}^{2}) \) and recalling that \( z = x_{1} + ix_{2} \).

To derive an asymptotic formula for \( A(X) \) as \( X \) tends to infinity, we first need to estimate \( h_{y}(r) \) before employing (13). We see actually that
\[
h_{y}(r) = \int_{0}^{\infty} \frac{r^{2} - r^{2}}{2r} \left( \frac{1}{\sinh(\pi t - \pi r)} - \frac{1}{\sinh(\pi t + \pi r)} \right) dt
\]
and then it is not difficult to find that indeed
\[
h_{\infty}(r) = 2 \int_{0}^{\infty} \frac{u}{\sinh(\pi u)} du
\]
(which positive constant we denote by \( c_{1} \)). Again, as in [2], we have for \( 1 \leq r \leq Y - \log Y \),
\[
c_{1} > h_{y}(r) = c_{1} + O(Y^{-\pi/2} + \exp(-\pi r))
\]
(14)

Further, for \( r \geq Y + \log Y \),
\[
h_{y}(r) \ll (r - Y)\exp(-\pi(r - Y)).
\]
(15)

We next estimate the series
\[
\frac{16\pi|m|^{2}}{|d_{K}|} \sum_{c} |c|^{-4} S(m, m; c) I(c)
\]
on the right hand side of (13), denoting the relevant integral therein by \( I(c) \). This integral may be rewritten as
\[
I(c) = \int_{0}^{\infty} \frac{dy}{y} \int_{0}^{\infty} \frac{r dr}{(1 + r^{2})^{2}} \int_{0}^{2\pi} \times \exp[-2\pi|m|\delta(y + y^{-1})] - 2\pi i \text{Re}(\bar{m}\delta(y + y^{-1})\exp(-2it log y))] \times
\]
\[
\times \int_{0}^{\infty} \left( \frac{1}{4} + t^{2} \right) \exp[-2it \log y] dt.
\]
(16)
(after replacing $y/\delta$ by $y$, with $\delta := 1/|\delta_1|^{1/2} = 1/(|c|(1 + r^2)^{1/2})$. The integral with respect to $t$ occurring in (16) is trivially $O(Y^2)$; however, for $y \neq 1$, it can be seen to be $O(Y^2/|\log y|)$, on using integration by parts. The integration with respect to $y$ can be broken up into integration over $A_1 := (0, 1/w_Y), A_2 := [1/w_Y, w_Y]$ and $A_3 := (w_Y, \infty)$ writing $w_Y$ for $\exp(1/Y^2)$, for any fixed $\varepsilon > 0$. We are then led to the estimation

$$
I(c) \ll Y^2 \int_0^\infty \frac{rdr}{(1 + r^2)^2} \left( \int_{A_1} + \int_{A_2} \right) \exp[-2\pi|m|\delta(y + y^{-1})] \frac{dy}{y|\log y|} + 
$$

$$
+ Y^3 \int_0^\infty \frac{rdr}{(1 + r^2)^2} \int_{A_3} \exp[-2\pi|m|\delta(y + y^{-1})] \frac{dy}{y} 
$$

$$
\ll Y^2 \int_0^\infty \frac{r}{(1 + r^2)^2} Y^2 \left( \frac{|c|(1 + r^2)^{1/2}}{|m|} \right) Y^{-\varepsilon} \left( \frac{|c|(1 + r^2)^{1/2}}{|m|} \right) \frac{dr}{r^{1-\beta}} 
$$

for any $\beta$ in $(0, 1)$. Choosing now $\varepsilon = 1/2$, we obtain that

$$
I(c) \ll \frac{Y^{5/2}}{|m|/|c|^{1-\beta}} \int_0^\infty \frac{rdr}{(1 + r^2)^{3+\beta/2}} \ll Y^{5/2} (|c|/|m|)^{1-\beta} 
$$

for any $\beta$ in $(0, 1)$. This together with the estimate (12) implies, for the second term on the right hand side of (13), the bound, for any $\beta$ in $(0, 1)$:

$$
\frac{16\pi|m|^2}{|d_x|} \sum_{0 \neq \omega \in \mathbb{R}} |c|^{-4} S(m, m; c) I(c) \ll Y^{5/2} \sigma_{(1+ \beta)/2} (m, \sqrt{d_x}). 
$$

(17)

For the contribution to the first term on the left hand side of (13) from all the exceptional $\chi_i := (\mu - 1)^{1/2}$, we derive the following estimate, namely

$$
\sum_{\chi_i := (\mu - 1)^{1/2}} |\rho_1(m)|^2 h_Y(\chi_i) |\Gamma(1 - i\chi_i)|^2 \ll |m|^{1+\varepsilon} \text{ for any } \varepsilon > 0. 
$$

(18)

For this purpose, we note that these $\chi_i$ are all purely imaginary and further know from Sarnak ([4], Theorem 3.1) that $|\chi_i| \leq 1/2$; hence $H(\chi, t)$ and $h_Y(\chi_i)$ are well-defined. The asymptotic formula $\Gamma(x + iy) \sim \sqrt{2\pi} \exp[-\pi|y|/2]|y|^{-1/2}$ for $|y| \to \infty$ implies that $H(\chi, t) \sim 2\pi |t|^2 \exp[-2\pi|t|]/|\Gamma(1 - i\chi)|^2$ as $t \to \infty$. Consequently, for every $Y > 0$, we have

$$
h_Y(\chi_i)/|1/|\Gamma(1 - i\chi_i)|^2| \ll \int_0^Y t^2 \exp[-2\pi t|\cosh(\pi t)|dt < 4/\pi^3. 
$$

Now, for $Y \geq 1$,

$$
\min_{\chi_i := (\mu - 1)^{1/2}} \frac{h_Y(\chi_i)}{1/|\Gamma(1 - i\chi_i)|^2} \geq \min_{\chi_i := (\mu - 1)^{1/2}} \frac{h_Y(\chi_i)}{1/|\Gamma(1 - i\chi_i)|^2} =: C(> 0) 
$$
and so
\[ C \sum_{x_i = (\mu - 1)^{1/2}} |\rho_i(m)|^2 \leq \sum_{x_i = (\mu - 1)^{1/2}} \frac{|\rho_i(m)|^2 h_1(x_i)}{1/|\Gamma(1 - i x_i)|^2} \ll |m|^{1 + \varepsilon}, \] by (13) and (17).

This proves (18).

For the proof of our theorem, we shall, in the light of (18), totally ignore the presence of the \( \mu_i \)'s, in the sequel.

Setting \( Y = X + \log X \) in (13), the left hand side of (13) is \( \geq (c_1 - c_2 X^{-\pi/2}) A(X) - c_3 \sum_{|x_i| \leq X} |\rho_i(m)|^2 \exp[-\pi x_i] |\Gamma(1 - i x_i)|^2 \) for certain positive constants \( c_2 \) and \( c_3 \), on using (14) as well as the positivity of the terms on the left hand side of (13). Hence, for any \( \varepsilon > 0 \), we obtain, with a constant \( c_4 > 0 \),
\[
\frac{\pi^3 |d_K|}{\nu(\mathcal{F}_K)} c_1 A(X) \leq X^3 + c_4 X^{5/2} |m|^{1 + \varepsilon}.
\] (19)

We next substitute \( Y = X - \log X \) in (13) and noting that \( Y + \log Y \approx X \) and \( Y - \log Y \gg X - 2 \log X \), we get by (14) that
\[
\sum_{x_i \leq X - 2 \log X} |\rho_i(m)|^2 \frac{(c_1 + O((X - \log X)^{-\pi/2} + \exp(-\pi x_i)))}{1/|\Gamma(1 - i x_i)|^2} + \\
+ \sum_{X - 2 \log X < x_i \leq X} |\rho_i(m)|^2 h_Y(x_i) + \\
+ O \left( \sum_{x_i > X} \exp(-\pi(x_i - X + \log X)) \frac{|\rho_i(m)|^2 (x_i - X + \log X)}{1/|\Gamma(1 - i x_i)|^2} \right) + I_1 = \\
= \frac{\nu(\mathcal{F}_K)}{\pi |d_K|} \left\{ \frac{(X - \log X)^3}{3} + \frac{X - \log X}{4} \right\} + O((X - \log X)^{5/2} |m|^{1 + \varepsilon})
\]
where \( I_1 \) denotes the second term on the left hand side of (13). From (14), (15) and the estimate \([\xi_K (1 + ir)^{-1}] = O(|r|^\varepsilon)\) for every \( \varepsilon > 0 \) as \( |r| \to \infty \), we see that \( I_1 = O(X^{1+\varepsilon} |r|^{\varepsilon}) \), while the right hand side of the equality above is \( \geq \frac{\nu(\mathcal{F}_K)}{\pi^3 |d_K|} X^3 - c_4 X^{5/2} |m|^{1 + \varepsilon} \) for a positive constant \( c_4 \). But now, by (13) and (17),
\[
\sum_{X - 2 \log X \leq x_i \leq X} |\rho_i(m)|^2 \frac{h_Y(x_i)}{1/|\Gamma(1 - i x_i)|^2} = O(X^3 - (X - 2 \log X)^3) + \\
+ O(X^{5/2} |m|^{1 + \varepsilon})
\]
Therefore
\[ c_1 A(X) \geq c_1 A(X - 2 \log X) = \frac{\nu(\mathcal{F}_K)}{3 |d_K| \pi} X^3 - c_6 X^{5/2} |m|^{1 + \varepsilon}, \]
for a positive constant \( c_6 \). This together with (19) proves the Theorem. The estimate (5) follows on applying the usual difference argument to (4).
Remarks (i) In principle, it should be possible to derive the estimate (5) through the general theory of automorphic forms on $GL(2)$ but our proof is elementary.

(ii) An estimate similar to (5) with $|m|^{1+\varepsilon}$ in place of $|m|^{1/2+\varepsilon}$ may be obtained in the case of Maass cusp forms on 4-dimensional hyperbolic space (the order of integral Hurwitz quaternions replacing the ring $R$).

Appendix

In this appendix we indicate how one can proceed to generalise the result of the theorem above to higher-dimensional hyperbolic spaces. We will utilise the exposition in [5] for our basic set up in this case.

Let $q$ be a non-degenerate quadratic form on a $k$-dimensional vector space $E$ over $\mathbb{Q}$ and $\mathfrak{O}(q)$, the associated Clifford algebra (see [5]), identifying $\mathbb{Q}$ and $E$ with their canonical images in $\mathfrak{O}(q)$; for $k=0$, $\mathfrak{O}(q) : = \mathbb{Q}$. Taking an orthogonal basis $\{e_1, \ldots, e_k\}$ for $E$ over $\mathbb{Q}$ with respect to $q$, we have

$$e_i^2 = q(e_i)(p = 1, 2, \ldots, k), \quad e_i e_m = -e_m e_i (1 \leq l \neq m \leq k).$$

For any subset $\{e_{i_1}, \ldots, e_{i_r}\}$ of the basis $\{e_1, \ldots, e_k\}$, define $e_{i_1} \cdots e_{i_r}$ and $e_\phi : = 1$ for the empty set $\phi$. Then these $e_M$ form a basis for $\mathfrak{O}(q)$ over $\mathbb{Q}$. We have three $\mathbb{Q}$-linear involutions $x \mapsto x^*$, $x \mapsto \bar{x}$ and $x \mapsto x^*$ on $\mathfrak{O}(q)$ reducing to the identity map on $\mathbb{Q}$ such that for any $e_M$ as above $e_M = (-1)^r e_M$, $\bar{e}_M = (-1)^{(r+1)n/2} e_M$ and $e^*_M = (-1)^{r(n-1)/2} e_M$. Further for any $x, y$ in $\mathfrak{O}(q)$,

$$\bar{xy} = y\bar{x} \quad \text{and} \quad (xy)^* = y^*x^*.$$

We have a trace map $\text{tr}: \mathfrak{O}(q) \to \mathfrak{O}(q)$ defined by $\text{tr}(x) = x + \bar{x}$. When $q = -I_k$, the negative of the unit quadratic form $I_k$ on $E$, we have on $V_{k+1} : = \mathbb{Q}.1 \oplus E \subset \mathfrak{O}(q)$, a scalar product $\langle v, w \rangle : = \frac{1}{2} \text{tr}(vw)$ for all $v, w$ in $V_{k+1}$, so that $\{1, e_1, \ldots, e_k\}$ becomes an orthonormal basis for $V_{k+1}$. For any $x = \sum_M \lambda_M e_M \in \mathfrak{O}(q)$ with $\lambda_M \in \mathbb{Q}$ or more generally

$$x = \sum_M \lambda_M e_M = \sum_M \lambda_M e_M \otimes 1 \in \mathfrak{O}(q) \otimes \mathbb{R} \quad \text{with} \quad \lambda_M \in \mathbb{R},$$

we know that $|x| : = \left( \sum_M \lambda_M^2 \right)^{1/2}$ defines the Euclidean norm of $x$. Further we have $|v|^2 = \bar{v} v = \bar{v} v$ whenever $v \neq 0$ in $\mathfrak{O}(q) \otimes \mathbb{R}$ has the property that there exists a $\mathbb{Q}$-linear automorphism $\varphi_v : V_{k+1} \to V_{k+1}$ such that $v x = \varphi_v(x) v'$ for all $x$ in $V_{k+1}$. We denote the algebra $\mathfrak{O}(q) \otimes \mathbb{R}$ and the vector space $V_{k+1} \otimes \mathbb{R}$ resulting by base change respectively from $\mathfrak{O}(q)$ and $V_{k+1}$ also by $\mathfrak{O}(q)$ and $V_{k+1}$ again. Given a lattice $L$ in $V_{k+1}$, the dual lattice $L^\#$ is defined by $L^\# : = \{ y \in V_{k+1} | \langle x, y \rangle \in \mathbb{Z} \text{ for every } x \in L \}$.

Let $H^{k+2}$ be the $(k + 2)$-dimensional hyperbolic space given by the upper half-space

$$\{ w = x_0 + x_1 e_1 + x_2 e_2 + \cdots + x_k e_k + x_{k+1} e_{k+1} \mid \quad x_0, x_1, x_2, \ldots, x_k + e_{k+1} \in \mathbb{R}, x_{k+1} > 0 \}. $$

We write $w = Z + e_{k+1}$ or more precisely, for $P = (x_0, x_1, \ldots, x_{k+1})$ corresponding to $w = x_0 + x_1 e_1 + \cdots + x_k e_k + x_{k+1} e_{k+1}$ in $H^{k+2}$, $Z = Z(P) = x_0 + x_1 e_1 + \cdots + x_k e_k$ and $r = r(P) = x_{k+1} > 0$. Then $|W(P)|^2 = |Z(P)|^2 + (r(P))^2$ or simply $|w|^2 = |Z|^2 + r^2$. 
We have on $\mathbb{H}^{k+2}$, a Riemannian metric $ds^2 = x_{k+1}^2(dx_0^2 + \cdots + dx_k^2)$ and associated volume element $dv := x_{k+1}^{2(k+2)-d}dx_0\Lambda dx_1\Lambda dx_{k+1}$; the corresponding Laplace-Beltrami operator $\Delta$ is given by

$$\Delta := x_{k+1}^2 \left( \frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_{k+1}^2} \right) - kx_{k+1} \frac{\partial}{\partial x_{k+1}},$$

For the Clifford algebra $\mathfrak{C}(q)$ over the base field $K = \mathbb{R}$ or $\mathbb{Q}$ and $q = -I_K$, the Vahlen group $SV_k(K)$ is defined by

$$SV_k(K) := \left\{ \begin{array}{c} \alpha, \beta, \gamma, \delta \in \mathfrak{C}(q) \text{ with } \alpha\delta^* - \beta\gamma^* = 1 \\
(i) \\
(ii) \alpha\beta^* = \beta\alpha^*, \gamma\delta^* = \delta\gamma^* \\
(iii) a\tilde{a}, b\tilde{b}, \gamma\delta, \delta\gamma \in K \\
(iv) a\delta, b\delta \in V_{k+1} \\
v) a\beta + b\tilde{a} + \gamma\tilde{\delta} + \delta\tilde{\gamma} \in K, \forall x \in V_{k+1} \\
(vi) a\delta + b\tilde{a} + \gamma\tilde{\delta} + \delta\tilde{\gamma} \in V_{k+1}, \forall x \in V_{k+1} \end{array} \right\}$$

For $K = \mathbb{R}$, $SV_0 = SL_2(\mathbb{R})$ and $SV_1 = SL_2(\mathbb{C})$. Let $J$ be a $\mathbb{Z}$-order in the $\mathbb{Q}$-algebra $\mathfrak{C}(q)$ (i.e., a subring containing 1 and a $\mathbb{Q}$-basis of $\mathfrak{C}(q)$, with the underlying additive group finitely generated) which is stable under the involutions $^*$ and $'$ of $\mathfrak{C}(q)$. By $\Gamma := SV_k(J)$, we mean the subgroup $\left\{ \left( \begin{array}{cc} \alpha & \beta \\
\gamma & \delta \end{array} \right) \in SV_k(\mathbb{Q}) | \alpha, \beta, \gamma, \delta \in J \right\}$. The Vahlen group $SV_k(\mathbb{R})$ acts on $\mathbb{H}^{k+2}$ as orientation preserving isometries through the maps

$$P \mapsto \sigma P := (\alpha P + \beta)(\gamma P + \delta)^{-1} \text{ for } \sigma = \left( \begin{array}{cc} \alpha & \beta \\
\gamma & \delta \end{array} \right) \in SV_k(\mathbb{R}).$$

For any $w = w(P) = Z(P) + r(P)e_{k+1} \in \mathbb{H}^{k+2}$, we have correspondingly for $w(\sigma P) = Z(\sigma P) + r(\sigma P)e_{k+1}$,

$$Z(\sigma P) = \frac{(\alpha Z + \beta)(\gamma Z + \delta) + \alpha\gamma^2}{|\gamma Z + \delta|^2 + |\gamma|^2 r^2}, \text{ with } Z := Z(P), \quad r := r(P)$$

$$= \alpha \gamma^{-1} - (\gamma^*)^{-1} \frac{(\gamma Z + \delta)}{|\gamma Z + \delta|^2 + |\gamma|^2 r^2} \text{ whenever } \gamma \neq 0,$$

and $r(\sigma P) = r(P)/(|\gamma Z + \delta|^2 + |\gamma|^2 r^2)$. By $\Gamma_\infty$, we mean the subgroup $\left\{ \left( \begin{array}{cc} 1 & w \\
0 & 1 \end{array} \right) \in \Gamma \right\}$. We also assume, for simplicity, that $\Gamma$ has only one cusp viz. $\infty$, for the above action on $\mathbb{H}^{k+2}$.

If $\wedge := J \cap V_{k+1}$ for a given $\mathbb{Z}$-order $J$ in $\mathfrak{C}(q)$ over $\mathbb{Q}$, then $\wedge$ is a lattice in $V_{k+1}$. Then, for any $\mu$ in the dual lattice $\wedge^\#$ and $s$ in $\mathbb{C}$ with Re$(s) > k + 1$, the Poincaré series $U_\mu(s, s)$ is defined by

$$U_\mu(P, s) := \sum_{\sigma \in \Gamma \setminus \Gamma} r(\sigma P)^se(i|\mu|r(\sigma P) + \langle Z(\sigma P), \mu \rangle)$$

(1)

where $e(\theta) := \exp(2\pi i\theta)$ for $\theta \in \mathbb{C}$ and $i = \sqrt{-1} \in \mathbb{C}$ (with arg $i = \pi/2$). (see [5]). This series
converges absolutely, uniformly on compact subsets of \( H^{k+2} \times \{ s \in \mathbb{C} | \text{Re}(s) > k + 1 \} \)
and \( U_0(P, s) \) is actually an Eisenstein series. If \( \mu \neq 0 \) in \( \wedge^\# \) and \( \text{Re}(s) > k + 1 \),
\( U_\mu(s) \in L^2(\Gamma \backslash H^{k+2}) \). For \( \text{Re}(s) > k + 1 \) again, \( U_\mu(s) \) satisfies the differential equation
\[
(-\Delta - s(k+1-s)) U_\mu(s) = 2\pi |\mu|(2s-k) U_\mu(s,s+1)
\]  
which implies immediately that \( U_\mu(s) \) has a meromorphic continuation to the domain
given by \( \text{Re}(s) > k \); further, it has no pole at \( s = k+1 \) and indeed, from ([5],
Theorem 10.1) it even follows that \( U_\mu(s) \) is holomorphic in \( s \) for \( \text{Re}(s) > k + 1/2 \).
The possible poles of \( U_\mu(s) \) in \((k+1/2, k+1)\) correspond to the values of \( s > (k+1)/2 \)
for which \( s(k+1-s) \) is an \( \"\text{exceptional}\" \) eigenvalue of \( -\Delta \).

The proof of the proposed generalization rests as before, on an identity arising
from the computation of the inner product
\[
\langle U_\mu(s), k+1+it \rangle, U_\mu(s, k+1+it) \rangle \quad \text{for } \mu \neq \alpha \text{ in } \wedge^\#, \quad t \in \mathbb{R}
\]
in two different ways, namely through the Parseval relation in \( L^2(\Gamma \backslash H^{k+2}) \) or by
using the Fourier expansion of the given Poincaré series as described below.

Let \( C_{\mu \nu} \), for given \( \mu \nu \in \wedge^\#, \) denote the number of \( \alpha \) in \( J \) such that \( \alpha \alpha = 1 \) and
\( \varphi_\alpha^*(\mu) = \nu \) where \( \varphi_\alpha^* \) is the map dual (with respect to \( \langle, \rangle \)) to the map \( \varphi_\alpha \) defined by
\( \alpha x = \varphi_\alpha (x) \alpha' \) for every \( x \in V_{k+1} \otimes \mathbb{R} \). We note that \( C_{\mu \nu} \) is \( O(1) \) for all \( \mu, \nu \). For all \( s \in \mathbb{C} \) with \( \text{Re}s > k+1 \), we have from [5] the following Fourier expansion
\[
v(\mathcal{F}_\wedge) U_\mu(w, s) = \sum_{\nu \in \wedge^\#} e(\langle Z, \nu \rangle) \left\{ C_{\mu \nu} r^s \exp\left(-2\pi |\mu| r\right) v(\mathcal{F}_\wedge) + \right.
\]
\[+ \frac{\kappa^{k+1-s}}{\nu \neq 0} \sum_{\gamma \neq 0} S(\mu, \nu, \gamma) \int_{V_{k+1}} e\left(-\frac{2\pi s}{r(1+|Z|^2)} \right) e\left(-\frac{\langle Z, \nu \rangle}{r(1+|Z|^2)} \right) dZ \right\} \]  
\[
\]  
where \( v(\mathcal{F}_\wedge) \) is the volume of fundamental domain \( \mathcal{F}_\wedge = V_{k+1} \wedge, \), denotes the
Kronecker delta and the generalized Kloosterman sum \( S(\mu, \nu, \gamma) \) for \( \gamma \neq 0 \) in \( J \) is defined by
\[
S(\mu, \nu, \gamma) := \sum_{(\kappa, \nu) \in D(\gamma)} e(\langle \gamma y^{-1}, \mu \rangle + \langle y^{-1} \gamma, \nu \rangle)
\]
with \( D(\gamma) := \{ (\alpha, \delta) | \sigma = \begin{pmatrix} \alpha & \beta \\ -\gamma & \delta \end{pmatrix} \} \) for fixed \( \gamma \) lie in distinct double cosets of \( \Gamma \) modulo
\( \Gamma' \). For \( (\alpha, \delta) \in D(\gamma) \), we have \( \alpha \delta \ast -1 \in J \gamma \ast \) by definition.

Let \( G(r, \nu) \) denote the series occurring as the coefficient of \( e(\langle Z, \nu \rangle) \) in the Fourier expansion (3). \( \mu, \nu \neq 0 \), the integral in \( G(r, \nu) \) may be seen to be \( O(|r|^{-N}) \) for
any \( N \in \mathbb{N} \) with the \( O \)-constant independent of \( \gamma \) while, for \( \mu = 0 \), the integral can be evaluated ([5], (9.10) et seq.) and is well-behaved at infinity as a function of \( r \). Therefore
the series \( \sum_{\gamma} G(r, \nu) e(\langle Z, \nu \rangle) \) is absolutely convergent for \( \text{Re}(s) > k + 1/2 \), since, from
[5], we know that the Linnik–Selberg series \( Z(\mu, \nu, s) := \sum_{\gamma} S(\mu, \nu, \gamma) \) converges
3-dimensional hyperbolic space

absolutely for \( \text{Re}(s) > k + 1/2 \). We may thus obtain the continuation of \( U_\mu(,s) \) for \( \text{Re}(s) > k + 1/2 \). Since, for any \( \mu \) in \( \Lambda_0^k \), the growth of \( U_\mu(,s) \) as \( r \) tends to infinity, is governed by that of \( (\max(\text{Re}(s), \text{Re}(k + 1/2 - n))) \times \exp(-2\pi |\mu| r) \), the inner product \( \langle U_\mu(,s_1), U_\nu(,s_2) \rangle \) is well-defined for \( \text{Re}s_1, \text{Re}s_2 > k + 1/2 \) whenever at least one of \( \mu, \nu \) is not zero. We now proceed to compute this inner product first for \( \mu \neq 0, \nu = 0 \). In fact unfolding the same yields

\[
\langle U_\mu(,s_1), E(,s_2) \rangle = \frac{2\pi^{2} |\mu|^{s_2} \Gamma(k + 1/2)}{\Gamma(s_2)} \sum_{\gamma \neq 0} \frac{\bar{S}(0, \mu, \gamma)}{(\gamma \bar{\gamma})^{s_2}} \times
\]

\[
\times \int_{0}^{\infty} r^{s_1 - (k + 3)/2} \exp(-2\pi |\mu| r) K_{s_2 - (k + 1)/2}(2\pi |\mu| r) dr
\]

If we now set (formally) \( s_2 = (k + 1)/2 - it \) (with \( t \in \mathbb{R} \)) and \( s_1 = k + 1 + it \) (with real \( t \)), then we obtain

\[
\langle U_\mu(,k + 1 + it), E(,(k + 1)/2 + it) \rangle = \frac{2\pi^{(k + 1)/2 - it} |\mu|^{-it}}{\Gamma((k + 1)/2 - it)} \times
\]

\[
\times \sum_{\gamma \neq 0} \frac{\bar{S}(0, \mu, \gamma)}{(\gamma \bar{\gamma})^{(k + 1)/2 + it}} \int_{0}^{\infty} r^{(k - 1)/2 + it} \exp(-2\pi |\mu| r) K_{it}(2\pi |\mu| r) dr
\]

\[
= \frac{2\pi^{k + 1/2 - it} |\mu|^{-it}}{\Gamma\left(\frac{k + 1}{2} - it\right)} \times
\]

\[
\times \sum_{\gamma \neq 0} \frac{\bar{S}(0, \mu, \gamma) \sqrt{\pi} \Gamma((k + 1)/2 + it)}{(\gamma \bar{\gamma})^{(k + 1)/2 + it} (4\pi |\mu|)^{(k + 1)/2 + it} \Gamma((k + 2)/2 + it)}
\]

since

\[
\int_{0}^{\infty} \exp(-ax) K_{\nu}(ax) x^{-1} dx = \frac{\sqrt{\pi} \Gamma(s + \nu) \Gamma(s - \nu)}{(2\pi)^{s/2} \Gamma(s + 1/2)} (\text{Re}(s) > 0, \text{Re}(s) > |\text{Re}(\nu)|).
\]

Thus

\[
\langle U_\mu(,k + 1 + it), E(,(k + 1)/2 + it) \rangle =
\]

\[
= \frac{2\pi^{(k + 2)/2 - it} |\mu|^{-it}}{(4\pi |\mu|)^{(k + 1)/2 + it} \Gamma\left(\frac{k + 1}{2} + it\right) \Gamma\left(\frac{k + 2}{2} + it\right)} \times
\]

\[
\times \sum_{\gamma \neq 0} \frac{\bar{S}(0, \mu, \gamma)}{(\gamma \bar{\gamma})^{(k + 1)/2 - it}} \Gamma\left(\frac{k + 1}{2} + it\right) \Gamma\left(\frac{k + 2}{2} + it\right)
\]

\[
= \frac{2\pi^{(k + 2)/2 - it} |\mu|^{-it}}{(4\pi |\mu|)^{(k + 1)/2 + it} \Gamma\left(\frac{k + 1}{2} + it\right) \Gamma\left(\frac{k + 2}{2} + it\right)} \times
\]

\[
\times \bar{Z}\left(0, \mu, \frac{k + 1}{2} + it\right).
\]
We recall that for \( \mu \neq 0 \) in \( \Lambda^\# \), \( U_\mu(., s) \) has a meromorphic continuation for Re\((s) > k \) and is actually holomorphic for Re\((s) > k + 1/2 \); its finitely many poles may come from \( s_i \) in \( \mathbb{R} \) such that \( \mu_i = s_i(k + 1 - s_i) \) is an “exceptional” eigenvalue for \(- \Delta\) i.e. \( 0 < s_i \leq k + 1/2 \) (cf. [5]). In addition to the eigenfunctions \( \psi_i \) corresponding to these “exceptional” eigenvalues, the discrete spectrum for \(- \Delta\) in \( L^2(\Gamma \setminus \mathbb{H}^{k+2}) \) consisting of \( \{0\} \cup \{\lambda_j \mid j = 1, 2, \ldots \} \) gives rise to corresponding eigenfunctions \( u_0 = \text{constant} \) and \( \{u_1, u_2, \ldots \} \) respectively. If we define

\[
\tilde{\chi}_i = i\sqrt{(k + 1)^2/4 - \mu_i}, \quad \chi_j = \sqrt{\lambda_j - (k + 1)^2/4}
\]

corresponding respectively to eigenvalues \( \mu_i \) that are “exceptional” and to “non-exceptional” eigenvalues \( \lambda_j \), then any eigenfunction \( u_\rho \) for \( \rho = i\tilde{\chi}_i \) or \( i\chi_j \) corresponding to \( \mu_i \) or \( \lambda_j \) has the Fourier expansion

\[
u_\rho(w) = b_\rho(0) r^{(k+1)/2} \sum_{0 \leq \mu \in \Lambda^\#} a_\mu(\rho) r^{(k+1)/2} K_\rho(2\pi|\mu|r)e(\langle \mu, Z \rangle)
\]

with a constant \( b_\rho(0) \) possibly non-zero for \( \rho = i\tilde{\chi}_i \).

For any eigenfunction \( U_\rho \), we see that

\[
\langle u_\rho, U_\mu(., k + 1 + it) \rangle = 2\sqrt{\pi(4\pi|\mu|)^{(k+1)/2} - (k+1-it)v(1, \mu)} a_\mu(\mu) \times \frac{\Gamma(k+1-it-(k+1/2)\Gamma(k+1-it-(k+1/2))}{\Gamma(k+1-it-k/2)}
\]

By analytic continuation, the Parseval relation now gives us

\[
\langle U_\mu(., k + 1 + it), U_\mu(., k + 1 + it) \rangle =
\]

\[
= \frac{\nu^2(1, \mu)}{(4\pi|\mu|)^{(k+1)/2} \Gamma(k+1-it)} \sum_{\rho} a_\mu(\mu)^2 \left| \Gamma\left(\frac{k+1}{2} - it + \rho\right) \right|^2 \times
\]

\[
\times \left| \Gamma\left(\frac{k+1}{2} + it - \rho\right) \right|^2 + \frac{4\pi^{k+2}}{(4\pi|\mu|)^{(k+1)}} \times
\]

\[
\times \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{k+1}{2} + it + \tau\right) \right|^2 \left| \Gamma\left(\frac{k+1}{2} + it - \tau\right) \right|^2 \left| \Gamma\left(\frac{k+1}{2} + i\tau\right) \right|^2 \left| \Gamma\left(\frac{k+1}{2} + i\tau\right) \right|^2 \left| Z\left(0, \mu, \frac{k+1}{2} + it\right) \right|^2 \frac{d\tau}{4\pi}
\]

(4)

where \( \rho \) is summed over the non-zero part of the discrete spectrum of \(- \Delta\). This inner product, on using the Fourier expansion of \( U_\mu(., k + 1 + it) \) (and unfolding) becomes

\[
\int_{\Gamma_\mu \setminus \mathbb{H}^{k+2}} U_\mu(w, k + 1 + it) r^{k+1-it} \exp(-2\pi|\mu|r - 2\pi i \langle Z, \mu \rangle) \frac{dZdr}{r^{k+2}}
\]

(5)
3-dimensional hyperbolic space

\[ v(\mathcal{F}) C_{\mu,\nu} \int_0^\infty \exp(-4\pi|\mu| r) r^k dr + \int_0^\infty \frac{k+1}{r^{k+2}} \exp(-2\pi|\mu| r) r^k dr \times \]

\[ \times \sum_{r \neq 0} \frac{S(\mu, \mu, \gamma)}{(\gamma^*)^{k+1+it} \Gamma_{k+1}^{-1} (1+|Z|^2)^{k+1+it} } \exp\left( -\frac{2\pi|\mu| r^2}{r|\gamma|^2 (1+|Z|^2)^{k+1+it} } \right) \exp\left( -\frac{(\gamma^*)^{-1} Z^{-1}}{r(1+|Z|^2)^{k+1+it} } \right) - r<Z, \mu> dZ. \]

As in Kuznetso [3] we now define

\[ H_k(\tau, t) := \left| \Gamma\left( \frac{k+1}{2} + i(t+\tau) \right) \right|^2 \left| \Gamma\left( \frac{k+1}{2} + it - \tau \right) \right|^2 \left| \frac{\Gamma\left( \frac{k+1}{2} - it \right)}{\Gamma\left( \frac{k+1}{2} + it \right)} \right|^2 \]

and for \( \tau \in \mathbb{C} \) with \(|\Im(\tau)| < A \) (a constant) and given parameter \( Y > 0 \), we set

\[ h_Y(\tau) := \int_0^Y \sin(\pi(\kappa_k + it)) \exp(\pi i(\kappa_k - 1/2)) H_k(\tau, t) dt \]  \hspace{1cm} (6)

(\( \kappa_k \) being 0 or 1/2 according as \( k \) is even or odd). For real \( \tau \), we note that \( H_k(\tau, t) = \varphi_k(\tau, t) H_{2k}(\tau, t) \) where \( H_0(\tau, t) \) is precisely the kernel \( H(\tau, t) \) in [3],

\[ \cosh \pi t H_1(\tau, t) = \frac{t^2 - \tau^2}{2\tau} \left( \frac{1}{\sinh(\pi(t-\tau))} - \frac{1}{\sinh(\pi(t+\tau))} \right) \quad \text{and} \]

\[ \varphi_k(\tau, t) = \begin{cases} \prod_{j=1}^{k/2} \frac{(j+1/2)^2 + (t+\tau)^2((j+1/2)^2 + (t-\tau)^2)}{(j+1/2)^2 + \tau^2} (k \text{ even}) \\ \prod_{j=1}^{(k-1)/2} \frac{(j+1/2)^2 + (t+\tau)^2(j^2 + (t-\tau)^2)}{(j^2 + \tau^2)} (k \text{ odd}) \end{cases} \]

Further, for \( \tau \) real and for \( t \geq 0 \), we readily see that

\[ \varphi_k(\tau, t) \geq c_k := \begin{cases} \prod_{j=1}^{k/2} (j+1/2)^2 (k \text{ even}) \\ \prod_{j=1}^{k/2} j^2 (k \text{ odd}) \end{cases} \]  \hspace{1cm} (7) \]

Multiplying both sides of the equation \((4)_A = (5)_A\) by

\[ \left( \frac{4\pi|\mu|^k}{\pi(v(\mathcal{F}))^2} \right) \exp(\pi i(\chi_k - 1/2)) \sin(\pi(\chi_k + it)) \left| \Gamma\left( \frac{k+2}{2} + it \right) \right|^2 \]

and then integrating both sides with respect to \( t \) over \((0, Y)\), we get
\[ 4 \sum_{\rho} \left[ \frac{|a_\rho(i)|^2}{1/2 - \rho} \right]^2 h_\gamma(-i\rho) + \frac{4\pi^{k+1}}{\Gamma(k+1)} \int_0^\infty \exp(\pi i(\chi_k - \frac{1}{2})) \sin(\pi(\chi_k + it)) \times \]

\[ \times \left( 1 \right) \int_{-\infty}^{\infty} H_k(\tau, t) \left( \begin{array}{c} Z(0, \mu, k + 1 + it) \end{array} \right)^2 \frac{dr}{2\pi} \frac{dt}{2\pi} = \frac{\Gamma(k + 1)}{\pi v(\mathcal{S}_k)} C_{\mu, \gamma} \int_0^\infty P_k(t) dt \]

\[ + \left( 4\pi|\mu|^k + 1 \right) \int_0^\infty \frac{S(\mu, \mu, \gamma)}{\pi v(\mathcal{S}_k)} \int_{\gamma \neq 0} \frac{S(\mu, \mu, \gamma)}{(\gamma)^{k+1} + it} \int_0^\infty \int_{\gamma \neq 0} \frac{\gamma^{k+1} + it}{(1 + |Z|^2)^2} \right) \times e \left( - \frac{(y^{*})^{-1} \bar{Z} \gamma^{-1}}{r(1 + |Z|^2), \mu} \right) \]

\[ \times e(-r(\langle Z, \mu \rangle) \int_0^1 dt Z \cdot \) (9)

where \( P_k(t) = \frac{1}{\pi} \exp(\pi i) \left( \chi_k - \frac{1}{2} \right) \sin(\pi(\chi_k + it)) \left( \Gamma(k + 1 + it) \right)^2 \) is a monic polynomial in \( t \) of degree \( k + 1 \).

We are now led to the estimation of

\[ I(\gamma) := \int_0^\infty \frac{P_k(t) dt}{(\gamma)^{k+1}} \int_0^\infty \exp \left[ - 2\pi |\mu| \left( r + \frac{1}{r \delta_1} \right) \right] \]

\[ - 2\pi i \left( \left( \frac{y^{*}-1 \bar{Z} \gamma^{-1}}{r(1 + |Z|^2), \mu} \right) + r(\langle Z, \mu \rangle) \right) \]

\[ \frac{dr}{r^{k+2} \left( 1 + |Z|^2 \right)^{k+1} + it} \]

with \( \delta_1 := |\gamma|^2 (1 + |Z|^2) \) and then of the series

\[ \sum_{\gamma \neq 0} \frac{S(\mu, \mu, \gamma)}{(\gamma)^{k+1}} I(\gamma). \]

We have, for \( k > 0 \),

\[ I(\gamma) \ll Y^{k+3/2} (|\mu|/|\gamma|)^{-1+\beta} \forall \beta \in (0, 1). \] (10)

We now proceed to estimate

\[ h_k(\tau) := \begin{cases} \int_0^\infty \sinh \pi t H_k(\tau, t) dt & \text{for } k \text{ even} \\ \int_0^\infty \cosh \pi t H_k(\tau, t) dt & \text{for } k \text{ odd} \end{cases} \] (6)

first for \( \tau \) satisfying \( 1 \leq \tau \leq Y - \log Y \) with \( Y \) large. In fact,

\[ h_k(\tau) = h_{\infty}(\tau) - \int_0^\infty \sinh \pi t \left( \frac{1}{2\tau} \right) \frac{1}{\sinh(\pi(t - \tau))} \frac{1}{\sinh(\pi(t + \tau))} \phi_k(\tau, t) dt. \]

For odd \( k \geq 1 \),

\[ \int_0^\infty \cosh \pi t H_k(\tau, t) dt = \int_0^\infty \frac{t^2 - \tau^2}{2\tau} \left( \frac{1}{\sinh(\pi(t - \tau))} - \frac{1}{\sinh(\pi(t + \tau))} \right) \phi_k(\tau, t) dt. \]
3-dimensional hyperbolic space

\[
\ll \frac{1}{\tau} \prod_{j=1}^{k-1} \left( j^2 + \tau^2 \right) \left\{ \int_{-\infty}^{\infty} \frac{u(\mu + 2\tau)}{\sinh \pi \mu} \sum_{l,m=0}^{k-1} u'(\mu + 2\tau)^m \, du + \right. \\
+ \exp(-\pi \tau) \int_{-\infty}^{\infty} \exp(-\pi \tau) t^{2k-2} \, dt \left\} \frac{1}{t^k} \int_{-\infty}^{\infty} u^{2k} \, du \right. \\
+ \exp(-\pi \tau) \left[ Y^{-\pi - \eta} + \exp(-\pi \tau) \ll Y^{\pi - \eta} \right]
\]

for every \( \varepsilon > 0 \).

The same estimate

\[ h_\infty(\tau) - h_Y(\tau) \ll Y^{\pi - \eta} + \exp(-\pi \tau) \text{ for } 1 \leq \tau \leq Y - \log Y \]

(11)

holds also for even \( k \geq 0 \).

On the other hand, for \( \tau \geq Y + \log Y \), we claim that

\[ h_Y(\tau) \ll \exp(-\tau - \varepsilon)(\tau - Y) \text{ for every } \varepsilon > 0. \]

(12)

It is easy to get lower bounds for \( h_\infty(\tau) \) for all \( \tau \geq 1 \). Actually, from (7)_A,

\[ h_\infty(\tau) \geq c_k \left\{ \begin{array}{ll}
\int_0^\infty \sinh(\pi t) H_0(\tau, t) \, dt & \text{for } k \text{ even} \\
\int_0^\infty \cosh(\pi t) H_1(\tau, t) \, dt & \text{for } k \text{ odd}
\end{array} \right. \]

(13)_A

The integral in (13)_A for even \( k \) is

\[ \geq \frac{1}{2} \int_0^\infty \sinh(\pi t) \exp(\pi t) \exp(\pi t + \tau - \pi |t - t|) \, dt \geq (1 - e^{-\pi})/2\pi \]

while, for odd \( k \), the integral equals \( 2 \int_0^\infty u/\sinh(\pi u) \, du \) which is clearly independent of \( \tau \). We thus obtain

\[ h_\infty(\tau) \geq c_1' = \min\left( \frac{1 - \exp(-\pi)}{2\pi}, \ 2 \int_0^\infty \frac{u}{\sinh \pi u} \, du \right) \]

(13)'

Now, as before, let us define

\[ A(X; \mu) = A(X) = \sum_{\rho \leq X} \frac{|a_\rho(\mu)|^2}{1/\Gamma\left(\frac{k+1}{2} - \rho\right)^2} \]

We need only an analogue of Weil's classical estimate for the generalized Kloosterman sums \( S(\mu, \nu; \gamma) \) as well as an appropriate estimate for \( |Z(0, \mu, k + 1/2 + i\tau)|^2 \) (due to Elstrodt-Grunewald-Mennicke and unpublished as yet) in order to prove the following formula for \( A(X) \), carrying out the remaining steps exactly as in the case \( k = 1 \):

For \( k \geq 2 \) and any \( \varepsilon > 0 \), we have

\[ A(X; \mu) = A(X) = B_k X^{k+2} + O(X^{k+3/2} |\mu|^{k+\varepsilon}) \]
as $X$ tends to infinity, with an explicit constant $B_k$ depending on $k$ and the $O$-constant depending at most on $\varepsilon$ and $k$.

References


