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On Ramanujan's modular identities

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Abstract. For Ramanujan's modular identities connected with his well-known partition congruences for the moduli 5 or 7, we had given, in an earlier paper, natural and uniform proofs through the medium of modular forms. Analogous (modular) identities corresponding to the (more difficult) case of the modulus 11 are provided here, with the consequent partition congruences; the relationship with relevant results of N J Fine is also sketched.

Keywords. Ramanujan's modular identities; partition congruences; modular forms.

1. Introduction

For the partition function $p(\cdot)$, Ramanujan exhibited several interesting congruence relations and in particular, $p(5n+4) \equiv 0 \pmod 5$ for $n = 0, 1, 2, \ldots$ etc. Besides a simple proof for the above congruences modulo 5 via the modular equation of degree 5, Ramanujan gave an elegant proof in ([7], p. 5). First, he writes down the beautiful identity

$$x\frac{\prod_{n=1}^{\infty}(1-x^{5n})^5}{\prod_{n=1}^{\infty}(1-x^n)} = \sum_{n=0}^{\infty}\left\{\frac{x^{5n+1}}{(1-x^{5n+1})^2} - \frac{x^{5n+2}}{(1-x^{5n+2})^2} - \frac{x^{5n+3}}{(1-x^{5n+3})^2} + \frac{x^{5n+4}}{(1-x^{5n+4})^2}\right\} \tag{1}$$

and then, for the "allied function" $\prod_{n=1}^{\infty}\{(1-x^n)^5/(1-x^{5n})\}$, the identity

$$\frac{\prod_{n=1}^{\infty}(1-x^n)^5}{\prod_{n=1}^{\infty}(1-x^{5n})} = 1 - 5\left(\frac{x}{1-x} - \frac{2x^2}{1-x^2} - \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{6x^6}{1-x^6} - \frac{7x^7}{1-x^7} - \cdots\right) \tag{2}$$

To derive the above partition congruences modulo 5, Ramanujan applies to both sides of (1) an "averaging operator" U_5 which turns out essentially to be the operator T_5^5 considered by Hecke ([4], p. 839). In connection with the partition congruences above modulo 7, Ramanujan wrote down the identity

$$x\prod_{n=1}^{\infty}(1-x^n)^3\prod_{n=1}^{\infty}(1-x^{7n})^3+8x^2\frac{\prod_{n=1}^{\infty}(1-x^{7n})^7}{\prod_{n=1}^{\infty}(1-x^n)}$$

$$=\sum_{n=0}^{\infty}\left\{\frac{x^{7n+1}(1+x^{7n+1})}{(1-x^{7n+1})^3}+\frac{x^{7n+2}(1+x^{7n+2})}{(1-x^{7n+2})^3}+\frac{x^{7n+4}(1+x^{7n+4})}{(1-x^{7n+4})^3}\right.$$

$$\left.-\frac{x^{7n+3}(1+x^{7n+3})}{(1-x^{7n+3})^3}-\frac{x^{7n+5}(1+x^{7n+5})}{(1-x^{7n+5})^3}-\frac{x^{7n+6}(1+x^{7n+6})}{(1-x^{7n+6})^3}\right\}. \tag{3}$$

He also gave an 'allied' identity

$$49x\prod_{n=1}^{\infty}(1-x^n)^3\prod_{n=1}^{\infty}(1-x^{7n})^3+8\frac{\prod_{n=1}^{\infty}(1-x^n)^7}{\prod_{n=1}^{\infty}(1-x^{7n})}$$

$$=8-7\left(\frac{x}{1-x}+\frac{4x^2}{1-x^2}-\frac{9x^3}{1-x^3}+\frac{16x^4}{1-x^4}-\frac{25x^5}{1-x^5}-\frac{36x^6}{1-x^6}+\frac{64x^8}{1-x^8}+\cdots\right). \tag{4}$$

For identities (1) and (2), there exist proofs due to Bailey ([1], [2]) and we were not able to find any proof for (3) in the literature at the time of writing [6] wherein we had provided a uniform (and what appeared to us as quite a natural) proof for all the identities (1)–(4) through the medium of modular forms of Haupttypus $(-2,5,1)$ and Nebentypus $(-3,7,\varepsilon)$ in the sense of Hecke [4]. We had also indicated at the end of [6] the existence of identities such as (1)–(4) corresponding to the case of modulus 11 in connection with Ramanujan's partition congruences modulo 11. The object of this note is precisely to provide such identities.

It has been pointed out to us kindly by Professor N J Fine in a recent letter (subsequent to the Ramanujan Centenary Conference at Urbana-Champaign) that both the identities (3) and (4) had already been proved by him in [3] using identities involving elliptic functions, "without much difficulty". One finds in [3] also the identity

$$\sum_{0}^{\infty}p(11m+6)x^{m+1}=11x^5\frac{\prod_{n=1}^{\infty}(1-x^{11n})^{11}}{\prod_{n=1}^{\infty}(1-x^n)^{12}}(11^3+8.11\alpha+\alpha^2+11\beta) \tag{5}$$

stemming from "lengthy" calculations, where α,β are modular functions for the congruence subgroup $\Gamma_0(11)$ with integral coefficients in their Fourier expansions. In the sequel, we shall exhibit an identity involving $\sum_0^\infty p(11m+6)x^{m+1}$ and modular forms of Nebentypus $(-3, 11, \varepsilon)$; this will turn out to be the same as (5) after proper identification of α,β above with ratios of modular forms of Nebentypus $(-3,11,\varepsilon)$. An "allied' identity involving

$$\prod_{n=1}^{\infty}(1-x^n)^{11}\Big/\prod_{n=1}^{\infty}(1-x^{11n}) \quad \text{in lieu of} \quad x^5\prod_{n=1}^{\infty}(1-x^{11n})^{11}\Big/\prod_{n=1}^{\infty}(1-x^n)$$

is also derived later and this gives rise to a classical congruence $p_{11}(11n) \equiv p_1(n) \pmod{11}$ for $n = 1, 2, \ldots$, where $p_r(n)$ is the coefficient of x^n in $\prod_{n=1}^{\infty}(1-x^n)^r$.

We are convinced more than ever that an approach to the partition congruences modulo 5, 7 or 11 through modular forms is not only natural but also has an unmistakable simplicity of its own. For example, identities (4.14), (4.16), (4.17), (4.12) in [3] are transparent linear relations between modular forms of Nebentypus $(-3, 7, \varepsilon)$; the arguments in this context used in [3], appear, on the other hand, to be more involved.

2. Notation and preliminaries

A modular form of Nebentypus $(-k, p, \varepsilon)$ in the notation of Hecke ([4], p. 809) is a holomorphic function f of a complex variable z in the upper half-plane H such that for every $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{Z})$ with $c \equiv 0 \pmod p$, $f((az+b)(cz+d)^{-1}) = \varepsilon(d)\ (cz+d)^k f(z)$ for a character ε on $(\mathbb{Z}/p\mathbb{Z})^\times$ and further such that $f((qz+r)(sz+t)^{-1})(sz+t)^{-k}$ is bounded at infinity for every $\begin{pmatrix} q & r \\ s & t \end{pmatrix}$ in $SL_2(\mathbb{Z})$. The latter condition is equivalent to saying that f is bounded in a fundamental domain for the congruence subgroup $\Gamma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod p \right\}$ in H. For prime p, $\Gamma_0(p)$ has exactly two cusps, namely at $0, \infty$. We know from Hecke again that for $p = 11$ in particular and odd $k \geqslant 1$, the Eisenstein series

$$E_{1,k}(z) := \sum_{n=1}^{\infty} \left(\sum_{1 \leqslant d \mid n} d^{k-1} \left(\frac{n/d}{11} \right) \right) \exp(2\pi i n z),$$

$$E_{2,k}(z) := \gamma_k + \sum_{n=1}^{8} \left(\sum_{1 \leqslant d \mid n} d^{k-1} \left(\frac{d}{11} \right) \right) \exp(2\pi i n z)$$

with a well-defined rational number γ_k (e.g. $\gamma_3 = -3$, $\gamma_5 = 1275/11$) are of Nebentypus $(-k, 11, \varepsilon)$ with $\varepsilon(r) := (r/11)$, the Legendre symbol, while the "theta series with Grossencharacter"

$$\vartheta_k(z) := \sum_{\mu \in \mathbb{Z} + \mathbb{Z}\omega} \mu^{k-1} \exp(2\pi i z \mu \bar{\mu}) \quad (\text{with } \omega := \tfrac{1}{2} + \sqrt{-11})$$

is a cusp form of Nebentypus $(-k, 11, \varepsilon)$ for odd $k \geqslant 1$. Let $\eta(z)$ be Dedekind's function defined by

$$\eta(z) := \exp(2\pi i z/24) \prod_{n=1}^{\infty} (1 - \exp(2\pi i n z))$$

and $\eta_N(z) := \eta(Nz)$ for every integer $N \geqslant 1$. We know, from Hecke, that $\eta^2 \eta_{11}^2$ is a modular form of Haupttypus $(-2, 11, 1)$. For $k = 3$, the three modular forms $E_{1,3}$, $E_{2,3}, \vartheta_3$ form a basis for the space of modular forms of Nebentypus $(-3, 11, \varepsilon)$. Further a basis for modular forms of Nebentypus $(-5, 11, \varepsilon)$ is provided by $E_{1,5}, E_{2,5}$, $\eta^2 \eta_{11}^2 E_{1,3}, \eta^2 \eta_{11}^2 E_{2,3}$ and $\eta^2 \eta_{11}^2 \vartheta_3$, as may be easily verified. One may find in Hecke ([4], pp. 827, 828) tables for the constants term γ_k of the Eisenstein series above or for the dimensions of the spaces of modular forms of Haupttypus or Nebentypus for small weights k and small prime stufe p.

We now furnish the coefficients $a_n(\cdot)$ of $\exp(2\pi inz)$ from $n=0$ to $n=33$ for the modular forms $\eta^2\eta_{11}^2, E_{1,3}, E_{2,3}$ and ϑ_3 since it was necessary to know the coefficients of q^{11}, q^{22}, q^{33} in the expansions of $\eta^2\eta_{11}^2 E_{1,3}, \eta^2\eta_{11}^2 E_{2,3}, \eta^2\eta_{11}^2\vartheta_3$ for our subsequent working. The coefficients $a_n(\cdot)$ are listed against the respective modular form in increasing order of n from 0 to 33:

$a_n(\eta^2\eta_{11}^2)$: $0, 1, -2, -1, 2, 1, 2, -2, 0, -2, -2, 1, -2, 4, 4, -1, -4, -2, 4, 0, 2, 2, -2, -1, 0, -4, -8, 5, -4, 0, 2, 7, 8, -1.$

$a_n(E_{1,3})$: $0, 1, 3, 10, 13, 26, 30, 48, 51, 91, 78, 121, 130, 168, 144, 260, 205, 288, 273, 360, 338, 480, 363, 530, 510, 651, 504, 820, 624, 840, 780, 962, 819, 1210.$

$a_n(E_{2,3})$: $-3, 1, -3, 10, 13, 26, -30, -48, -51, 91, -78, 1, 130, -168, 144, 260, 205, -288, -273, -360, 338, -480, -3, 530, -510, 651, 504, 820, -624, -840, -780, 962, -819, 10.$

$a_n(\vartheta_3)$: $0, 2, 0, -10, 8, -2, 0, 0, 0, 32, 0, -22, -40, 0, 0, 10, 32, 0, 0, 0, -8, 0, 0, 70, 0, -48, 0, -70, 0, 0, 0, -74, 0, 110.$

Writing q for $\exp(2\pi iz)$, we also highlight those Fourier coefficients in the expansions of $E_{1,5}, E_{2,5}, f_1 := \eta^2\eta_{11}^2 E_{1,3}, f_2 := \eta^2\eta_{11}^2 E_{2,3}$ and $f_3 := \eta^2\eta_{11}^2\vartheta_3$ which are relevant to our proofs in the sequel:

$$\begin{aligned}
E_{1,5}(z) &= q + 15q^2 + 82q^3 + 241q^4 + \cdots + (11)^4 q^{11} + \cdots,\\
E_{2,5}(z) &= \frac{1275}{11} + q - 15q^2 + 82q^3 + 241q^4 + \cdots + q^{11} + \cdots,\\
f_1(z) &= q^2 + q^3 + 3q^4 + \cdots - 11q^{11} + \cdots - 55q^{22} + \cdots + 154q^{33} + \cdots,\\
f_2(z) &= -3q + 7q^2 - 2q^3 + 9q^4 + \cdots - 308q^{11} + \cdots - 1177q^{22} + \cdots\\
&\quad -407q^{33} + \cdots,\\
f_3(z) &= 2q^2 - 4q^3 - 12q^4 + \cdots - 88q^{11} + \cdots + 286q^{22} + \cdots - 220q^{33} + \cdots.
\end{aligned} \tag{6}$$

In connection with the derivation of partition congruences for the moduli $\rho = 5, 7, 11$ etc., Ramanujan [7] applied to the relevant power-series

$$f(q) = \sum_{n=0}^{\infty} a_n q^n,$$

an operator U_ρ taking f to $U_\rho(f)$ defined by $(U_\rho(f))(q) := 1/\rho \sum_\zeta f(q^{1/\rho}\zeta)$ where the summation is over all the ρth roots of unity ζ; actually, $(U_\rho(f))(q)$ is nothing but $\sum_{n=0}^{\infty} a_{\rho n} q^n$. One may note that this operator U_ρ is essentially T_ρ^ρ considered (much later) by Hecke ([4], p. 839) on modular forms for $\Gamma_0(\rho)$. In the case of modular forms of Nebentypus $(-k, \rho, \varepsilon)$ in particular, the operator U_ρ maps the subspace generated by the two Eisenstein series into itself and the subspace of cusp forms into itself. For the Eisenstein series $E_{1,5}$, $E_{2,5}$ of Nebentypus $(-5, 11, \varepsilon)$, we may thus write

$$U_{11}(E_{1,5}) = \alpha_1 E_{1,5} + \beta_1 E_{2,5}, \quad U_{11}(E_{2,5}) = \alpha_2 E_{1,5} + \beta_2 E_{2,5}$$

with constants α_i, β_j. On the other hand, by directly applying U_{11} to the Fourier expansions of $E_{1,5}$, $E_{2,5}$ in (6), we see that

$$U_{11}(E_{1,5}) = (11)^4 q + \cdots, \; U_{11}(E_{2,5}) = \tfrac{1275}{11} + q + \cdots$$

and a comparison of Fourier coefficients gives us immediately that $\alpha_1 = (11)^4$, $\beta_1 = 0$, $\alpha_2 = 0$, $\beta_2 = 1$ i.e.

$$U_{11}(E_{1,5}) = (11)^4 E_{1,5}, \quad U_{11}(E_{2,5}) = E_{2,5}. \tag{7}$$

From (6), it is clear that

$$\begin{aligned} U_{11}(f_1)(z) &= -11q - 55q^2 + 154q^3 + \cdots \\ U_{11}(f_2)(z) &= -308q - 1177q^2 - 407q^3 + \cdots \\ U_{11}(f_3)(z) &= -88q + 286q^2 - 220q^3 + \cdots . \end{aligned} \tag{8}$$

On the other hand, since f_1, f_2, f_3 generate the subspace of cusp forms of Nebentypus $(-5, 11, \varepsilon)$ which is preserved by U_{11}, we have, for suitable constants $\gamma_{i,j}$,

$$U_{11}(f_i) = \gamma_{i,1} f_1 + \gamma_{i,2} f_2 + \gamma_{i,3} f_3, \quad i = 1, 2, 3. \tag{9}$$

We solve for the constants $\gamma_{i,j}$ from the following equations obtained by equating the corresponding coefficients of q^n in $U_{11}(f_i)$ in the light of (9), (6) and (8):

$$\begin{aligned} &-3\gamma_{1,2} = -11, \gamma_{1,1} + 7\gamma_{1,2} + 2\gamma_{1,3} = -55, \gamma_{1,1} - 2\gamma_{1,2} - 4\gamma_{1,3} = 154 \\ &-3\gamma_{2,2} = -308, \gamma_{2,1} + 7\gamma_{2,2} + 2\gamma_{2,3} = -1177, \gamma_{2,1} - 2\gamma_{2,2} - 4\gamma_{2,3} = -407 \\ &-3\gamma_{3,2} = -88, \; \gamma_{3,1} + 7\gamma_{3,2} + 2\gamma_{3,3} = 286, \; \gamma_{3,1} - 2\gamma_{3,2} - 4\gamma_{3,3} = -220. \end{aligned}$$

We then obtain

$$\begin{aligned} &\gamma_{1,1} = 0, \quad \gamma_{1,2} = 11/3, \quad \gamma_{1,3} = -121/3 \\ &\gamma_{2,1} = -1331, \quad \gamma_{2,2} = 308/3, \quad \gamma_{2,3} = -847/3 \\ &\gamma_{3,1} = 0, \quad \gamma_{3,2} = 88/3, \quad \gamma_{3,3} = 121/3 \end{aligned}$$

and therefore

$$\begin{aligned} &U_{11}(f_1) = \tfrac{11}{3} f_2 - \tfrac{121}{3} f_3, \quad U_{11}(f_2) = -(11)^3 f_1 + \tfrac{308}{3} f_2 - \tfrac{847}{3} f_3, \\ &U_{11}(f_3) = \tfrac{88}{3} f_2 + \tfrac{121}{3} f_3. \end{aligned}$$

3. Two 'allied' modular identities

For Dedekind's η-function, we know that for every $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{Z})$,

$\eta((az+b)(cz+d)^{-1}) = \varepsilon(M)(cz+d)^{\frac{1}{2}}\eta(z)$ with a well-defined branch of $(cz+d)^{\frac{1}{2}}$ and a multiplier $\varepsilon(M)$ which is a 24th root of unity. If M is in $\Gamma_0(11)$, then $M_1 = \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} 1/11 & 0 \\ 0 & 1 \end{pmatrix}$ is in $SL_2(\mathbb{Z})$ and so $\eta_{11}((az+b)(cz+d)^{-1}) = \varepsilon_{11}(M) \times (cz+d)^{\frac{1}{2}}\eta_{11}(z)$ for every $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma_0(11)$, with $\varepsilon_{11}(M) := \varepsilon(M_1)$. Using the fact that η^{24} is of Haupttypus $(-12, 1, 1)$ and $\eta^2\eta_{11}^2$ of Nebentypus $(-2, 11, \varepsilon_{11})$, it is not hard to see that $(\varepsilon^{11}(M_1)/\varepsilon(M))^2 = 1 = (\varepsilon^{11}(M)/\varepsilon(M_1))^2$ for every M in $\Gamma_0(11)$; further, we can verify that η_{11}^{11}/η and η^{11}/η_{11} are both of Nebentypus $(-5, 11, \varepsilon_{11})$.

Lemma. With $q = \exp(2\pi i z)$, *we have*

$$E_{1,3}(z) = \sum_{n=1}^{\infty} \left(\frac{n}{11}\right) \frac{q^n(1+q^n)}{(1-q^n)^3}, \quad E_{2,3}(z) = -3 + \sum_{n=1}^{\infty} \left(\frac{n}{11}\right) n^2 \frac{q^n}{1-q^n}$$

$$E_{1,5}(z) = \sum_{n=1}^{\infty} \left(\frac{n}{11}\right) \frac{q^n(1+11q^n+11q^{2n}+q^{3n})}{(1-q^n)^5},$$

$$E_{2,5}(z) = \frac{1275}{11} + \sum_{n=1}^{\infty} \left(\frac{n}{11}\right) n^4 \frac{q^n}{1-q^n}.$$

Proof. We give only the proof for the assertion involving $E_{1,5}$, since the proofs of the remaining three assertions are the same as that in ([6], Lemma on p. 226). Now $\sum_{j=0}^{3}(m+j)(m+j-1)(m+j-2)(m+j-3) = 4m^4 + 20m^2$ and for $|u| < 1$,

$$\frac{u^{4-j}}{(1-u)^5} = \sum_{m=j+1}^{\infty} \frac{m(m-1)(m-2)(m-3)}{4!} u^{m-j} \quad \text{for} \quad j = 0, 1, 2, 3.$$

Hence

$$\frac{6(u^4+u^3+u^2+u)}{(1-u)^5} = \sum_{m=1}^{\infty} m^4 u^m + 5 \sum_{m=1}^{\infty} m^2 u^m$$

$$= \sum_{m=1}^{\infty} m^4 u^m + 5 \frac{u+u^2}{(1-u)^3}$$

as in ([6], Lemma) again i.e.

$$\frac{u^4+11u^3+11u^2+u}{(1-u)^5} = \sum_{m=1}^{\infty} m^4 u^m, \quad \text{for} \quad |u| < 1.$$

Setting $u = q^r$ $(r = 1, 2, \ldots)$, multiplying both sides of the last identity by the Legendre symbol $\left(\frac{r}{11}\right)$ and summing over r from 1 to ∞, we obtain

$$\sum_{r,m \geq 1} \left(\frac{r}{11}\right) q^{mr} m^4 = \sum_{r=1}^{\infty} \left(\frac{r}{11}\right) \frac{q^r(1+11q^r+11q^{2r}+q^{3r})}{(1-q^r)^5}.$$

Our assertion for $E_{1,5}(z)$ is now immediate.

The following theorem gives the identities involving the two "allied" functions η_{11}^{11}/η

and η^{11}/η_{11}, which correspond for the modulus 11, to Ramanujan's identities (1)–(2) or (3)–(4) for 5 or 7 respectively.

Theorem. *We have the two "allied" identities:*

(i)
$$x^5 \frac{\prod_{n=1}^{\infty} (1-x^{11n})^{11}}{\prod_{n=1}^{\infty} (1-x^n)} = \frac{3}{3825} \sum_{n=1}^{\infty} \left(\frac{n}{11}\right) \frac{x^n(1+11x^n+11x^{2n}+x^{3n})}{(1-x^n)^5}$$
$$-\left\{\frac{116}{3825} \sum_{n=1}^{\infty} \left(\frac{n}{11}\right) \frac{x^n(1+x^n)}{(1-x^n)^3}\right.$$
$$-\frac{1}{3825}\left(-3+\sum_{n=1}^{\infty}\left(\frac{n}{11}\right) n^2 \frac{x^n}{1-x^n}\right)$$
$$\left.-\frac{32}{3825} \sum_{\mu \in \mathbb{Z}+\mathbb{Z}(1+\sqrt{-11})/2} \mu^2 x^{\mu\bar{\mu}}\right\}$$
$$\times x \prod_{n=1}^{\infty} (1-x^n)^2 (1-x^{11n})^2$$

(ii)
$$\frac{\prod_{n=1}^{\infty} (1-x^n)^{11}}{\prod_{n=1}^{\infty} (1-x^{11n})} = \frac{11}{1275}\left(\frac{1275}{11} + \sum_{n=1}^{\infty} \left(\frac{n}{11}\right) n^4 \frac{x^n}{1-x^n}\right)$$
$$-\left\{\frac{(11)^4}{3825} \sum_{n=1}^{\infty} \left(\frac{n}{11}\right) \frac{x^n(1+x^n)}{(1-x^n)^3}\right.$$
$$-\frac{1276}{3825}\left(-3+\sum_{n=1}^{\infty}\left(\frac{n}{11}\right) n^2 \frac{x^n}{1-x^n}\right)$$
$$\left.-\frac{3872}{3825} \sum_{\mu \in \mathbb{Z}+\mathbb{Z}(1+\sqrt{-11})/2} \mu^2 x^{\mu\bar{\mu}}\right\}.$$
$$\times x \prod_{n=1}^{\infty} (1-x^n)^2 (1-x^{11n})^2.$$

Proof. Since η_{11}^{11}/η and η^{11}/η_{11} are of Nebentypus $(-5, 11, \varepsilon_{11})$ there exist constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ such that

$$\eta_{11}^{11}/\eta = \lambda_1 E_{1,5} + \lambda_2 E_{2,5} + \lambda_3 \eta^2 \eta_{11}^2 E_{1,3} + \lambda_4 \eta^2 \eta_{11}^2 E_{2,3} + \lambda_5 \eta^2 \eta_{11}^2 \vartheta_3,$$
$$\eta^{11}/\eta_{11} = \mu_1 E_{1,5} + \mu_2 E_{2,5} + \mu_3 \eta^2 \eta_{11}^2 E_{1,3} + \mu_4 \eta^2 \eta_{11}^2 E_{2,3} + \mu_5 \eta^2 \eta_{11}^2 \vartheta_3.$$

From (6) and the Fourier expansions

$$\eta_{11}^{11}(z)/\eta(z) = q^5 + \cdots + 11q^{11} + \cdots,$$
$$\eta^{11}(z)/\eta_{11}(z) = 1 - 11q + 44q^2 - 55q^3 - 110q^4 + \cdots - 2068q^{11} + \cdots,$$

we obtain the system of equations for λ_i, μ_j:

$$\lambda_2=0,\quad \lambda_1+\lambda_2-3\lambda_4=0,\quad 15\lambda_1-15\lambda_2+\lambda_3+7\lambda_4+2\lambda_5=0,$$

$$82\lambda_1+82\lambda_2+\lambda_3-2\lambda_4-4\lambda_5=0,\quad 241\lambda_1+241\lambda_2+3\lambda_3+9\lambda_4-12\lambda_5=0,$$

$$\tfrac{1275}{11}\mu_2=1,\quad \mu_1+\mu_2-3\mu_4=-11,\quad 15\mu_1-15\mu_2+\mu_3+7\mu_4+2\mu_5=44,$$

$$82\mu_1+82\mu_2+\mu_3-2\mu_4-4\mu_5=-55,$$

$$241\mu_1+241\mu_2+3\mu_3+9\mu_4-12\mu_5=-110.$$

Solving these equations, we get

$$\lambda_1=\tfrac{3}{3825},\quad \lambda_2=0,\quad \lambda_3=-\tfrac{116}{3825},\quad \lambda_4=\tfrac{1}{3825},\quad \lambda_5=\tfrac{32}{3825}$$

$$\mu_1=0,\quad \mu_2=\tfrac{11}{1275},\quad \mu_3=-\frac{(11)^4}{3825},\quad \mu_4=\frac{(11)^2\times 116}{3825},\quad \mu_5=\frac{(11)^3\times 32}{3825}.$$

If we write x in lieu of q, the two "allied" identities are now an immediate consequence, in view of the Lemma above.

Together with (7) and (10), the theorem yields the

COROLLARY.

(i) $$U_{11}(\eta_{11}^{11}/\eta)=(11)^4\eta_{11}^{11}/\eta+\frac{(11)^3}{3}f_1-\tfrac{11}{3}f_2-121f_3$$

(ii) $$U_{11}(\eta^{11}/\eta_{11})=\eta^{11}/\eta_{11}-\frac{(11)^4}{3}f_1+\frac{(11)^2\times 17}{3}f_2-\frac{(11)^3}{3}f_3.$$

Proof. Applying U_{11} to both sides of the identities

$$\eta_{11}^{11}/\eta=\tfrac{1}{3825}(3E_{1,5}-116f_1+f_2+32f_3)$$

$$\eta^{11}/\eta_{11}=\tfrac{1}{3825}(33E_{2,5}-(11)^4f_1+((11)^2\times 116)f_2+((11)^3\times 32)f_3)$$

and using (7) and (10), we obtain, on substituting thereafter for $E_{1,5}$ in terms of η_{11}^{11}/η and for $E_{2,5}$ in terms of η^{11}/η_{11}, assertions (i) and (ii) of the Corollary, after simplification.

Remark 1. The Fourier coefficients in the expansions of $U_{11}(f_j)$, $j=1,2,3$ are all integral clearly. However, it is not obvious why the coefficients of $\frac{11}{3}f_2-\frac{121}{3}f_3$ or $\frac{308}{3}f_2-\frac{847}{3}f_3$ or $\frac{88}{3}f_2+\frac{121}{3}f_3$ are integral or equivalently, why the Fourier coefficients of f_2+f_3 are all in $3\mathbb{Z}$. Actually, the Fourier coefficients of $E_{2,3}+\vartheta_3$ are already in $3\mathbb{Z}$, for purely arithmetical reasons. To see this fact independently, let us note first that the constant Fourier coefficient in $E_{2,3}+\vartheta_3$ is -3 while the Fourier coefficient of q^n for $n\geq 1$ is

$$\sum_{1\leqslant d|n} d^2\left(\frac{d}{11}\right)+\sum_{\substack{\mu\in a\\ \mu\bar{\mu}=n}}\mu^2$$

where $\mathscr{a} := \mathbb{Z} + \mathbb{Z}\cdot\frac{1}{2}(1+\sqrt{-11})$. Let $n = n_1 \cdot 3^r$ with r the highest exponent such that 3^r divides n. Then evidently

$$\sum_{d|n} d^2\left(\frac{d}{11}\right) \equiv \sum_{d|n_1} d^2\left(\frac{d}{11}\right) \pmod 3.$$

When μ_j runs over μ in $\mathscr{a}$ with $\mu\bar{\mu} = n_1$ and ν_k over μ in $\mathscr{a}$ with $\mu\bar{\mu} = 3^r$, then $\mu_j\nu_k$ covers the set of μ in $\mathscr{a}$ with $\mu\bar{\mu} = n$ twice, in view of $\mu_j\nu_k = (-\mu_j)(-\nu_k)$. Hence

$$\sum_{\substack{\mu\in\mathscr{a}\\ \mu\bar{\mu}=n}} \mu^2 = \frac{1}{2} \sum_{\substack{\lambda\in\mathscr{a}\\ \lambda\bar{\lambda}=n_1}} \lambda^2 \sum_{\substack{\mu\in\mathscr{a}\\ \mu\bar{\mu}=3^r}} \mu^2.$$

Now

$$\sum_{\substack{\mu\bar{\mu}=n_1\\ \mu\in\mathscr{a}}} \mu^2 \equiv \sum_{\substack{\mu\bar{\mu}=n_1\\ \mu\in\mathscr{a}}} 1 \pmod 3,$$

since $\mu = \frac{1}{2}(x+\sqrt{-11}y)$, $\mu\bar{\mu} = n_1, 3\nmid n_1$ together imply that $3|xy$ but $3\nmid(x,y)$ and

$$\sum_{\substack{x,y\in\mathbb{Z}\\ x^2+11y^2=4n_1}} \tfrac{1}{4}(x+\sqrt{-11}y)^2 = \sum_{\substack{x,y\in\mathbb{Z}\\ x^2+11y^2=4n_1}} \tfrac{1}{4}(x^2-11y^2)$$
$$\equiv \sum_{\substack{x,y\in\mathbb{Z}\\ x^2+11y^2=4n_1}} (x^2+y^2) \equiv \sum_{\substack{\mu\bar{\mu}=n_1\\ \mu\in\mathscr{a}}} 1 \pmod 3.$$

But

$$\sum_{\substack{\mu\bar{\mu}=n_1\\ \mu\in\mathscr{a}}} 1 = 2 \sum_{1\leq d|n_1} \left(\frac{d}{11}\right)$$

since $\mathbb{Q}(\sqrt{-11})$ has class number 1 and has just 2 roots of unity. Also since

$$3\nmid n_1, 2\sum_{1\leq d|n_1}\left(\frac{d}{11}\right) \equiv 2\sum_{1\leq d|n_1} d^2\left(\frac{d}{11}\right) \pmod 3$$

and as a consequence, we have

$$\sum_{\substack{\mu\bar{\mu}=n_1\\ \mu\in\mathscr{a}}} \mu^2 \equiv 2\sum_{1\leq d|n} d^2\left(\frac{d}{11}\right) \pmod 3.$$

On the other hand

$$\sum_{\substack{\mu\bar{\mu}=3^r\\ \mu\in\mathscr{a}}} \mu^2 = \sum_{\substack{x,y\in\mathbb{Z}\\ x^2+11y^2=4\cdot3^r}} \tfrac{1}{4}(x^2-11y^2) \equiv \sum_{\substack{x,y\in\mathbb{Z}\\ x^2+11y^2=4\cdot3^r}} (x^2+y^2)$$
$$\equiv 2\sum_{\substack{\mu\bar{\mu}=3^r\\ \mu\in\mathscr{a}}} 1 \equiv 2\sum_{\substack{\mu\bar{\mu}=3^r\\ 3\nmid\mu,\mu\in\mathscr{a}}} 1 \pmod 3.$$

But the number of μ in $\mathscr{a}$ with $3\nmid\mu$ and $\mu\bar{\mu} = 3^r$ is just 4. Hence

$$\sum_{\substack{\mu\bar{\mu}=3^r\\ \mu\in\mathscr{a}}} \mu^2 \equiv 2 \pmod 3,$$

and as a result,

$$\sum_{\substack{\mu\bar{\mu}=n\\ \mu\in a}} \mu^2 \equiv \sum_{\substack{\mu\in a\\ \mu\bar{\mu}=n_1}} \mu^2 \equiv 2 \sum_{1\leqslant d|n} d^2\left(\frac{d}{11}\right) (\text{mod } 3),$$

proving our assertion above.

Likewise, from the simple congruence relation

$$\begin{aligned}\sum_{1\leqslant d|n} d^2\left\{\left(\frac{d}{11}\right)-\left(\frac{n/d}{11}\right)\right\} &\equiv \sum_{1\leqslant d|n_1} d^2\left\{\left(\frac{d}{11}\right)-\left(\frac{n_1/d}{11}\right)\right\}\\ &\equiv \sum_{1\leqslant d|n_1}\left(\frac{d}{11}\right)-\sum_{1\leqslant d|n_1}\left(\frac{n_1/d}{11}\right)\equiv 0 (\text{mod } 3),\end{aligned}$$

we see that $E_{1,3}-E_{2,3}$ and hence f_1-f_2 has all its Fourier coefficients in $3\mathbb{Z}$. It is no more a mystery as to why all the Fourier coefficients of the sum of the last three terms on the right hand side of assertion (ii) in the Corollary above are integral, since the Fourier coefficients of $f_1-2f_2-f_3$ are all in $3\mathbb{Z}$! In connection with congruences for Fourier coefficients of modular forms, we should refer to the recent interesting results due to J Sturm [9].

Remark 2. From the definition of U_{11}, it is easily seen that

$$\begin{aligned}U_{11}\left(\frac{q^5\prod_{n=1}^{\infty}(1-q^{11n})^{11}}{\prod_{n=1}^{\infty}(1-q^n)}\right) &= U_{11}\left(\prod_{n=1}^{\infty}(1-q^{11n})^{11}\right)\cdot U_{11}\left(\frac{q^5}{\prod_{n=1}^{\infty}(1-q^n)}\right)\\ &= \prod_{n=1}^{\infty}(1-q^n)^{11}\cdot U_{11}\left(q^5\sum_{n=0}^{\infty}p(n)q^n\right)\\ &= \prod_{n=1}^{\infty}(1-q^n)^{11}\sum_{m=0}^{\infty}p(11m+6)q^{m+1}.\end{aligned}$$

Hence, by assertion (i) of the Corollary above, we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}p(11m+6)q^{m+1} &= q^5\frac{\prod_{m=1}^{\infty}(1-q^{11m})^{11}}{\prod_{m=1}^{\infty}(1-q^m)^{12}}\\ &\times\left\{(11)^4+\frac{(11)^3E_{1,3}-11E_{2,3}-363\vartheta_3}{3q^4\prod_{n=1}^{\infty}(1-q^{11n})^9\Big/\prod_{n=1}^{\infty}(1-q^n)^3}\right\}.\end{aligned}$$

This may be seen to be equivalent to identity (3.25) derived in ([3], p. 158) after "lengthy but elementary" calculations (that have, however, been suppressed), as soon as we check that α, β can be identified respectively with $(2E_{1,3}-\vartheta_3)/6(\eta_{11}^9/\eta^3)$,

$E_{1,3}/(\eta_{11}^9/\eta^3)$. The infinite products occurring on the right hand side of the identity just stated above have expansions as power-series in q with constant term 1 and other coefficients in $\mathbb{Z}$. Hence the right hand side of this identity is a power-series in q with coefficients in $11\mathbb{Z}$. As a result, the well-known partition congruences $p(11m+6) \equiv 0 \pmod{11}$ due to Ramanujan become evident. Partition congruences for a higher modulus, say $(11)^2$ may also be obtained therefrom; for example, if a_m is the coefficient of q^{m-1} in

$$E_{2,3} \prod_{n=1}^{\infty} \frac{(1-q^{11n})^2}{(1-q^n)^9},$$

then $p(11m+6) + 44a_m \equiv 0 \pmod{(11)^2}$.

Remark 3. Following Newman [5], let $p_r(n)$ denote the coefficient of q^n in $\prod_{m=1}^{\infty}(1-q^m)^r$. Now

$$U_{11}(\eta^{11}/\eta_{11}) = U_{11}\left(\sum_{n=0}^{\infty} p_{11}(n)q^n\right)\Big/\prod_{n=1}^{\infty}(1-q^n)$$
$$= \left(\sum_{n=0}^{\infty} p_{11}(11n)q^n\right)\Big/\prod_{n=1}^{\infty}(1-q^n).$$

On the other hand, we know that

$$U_{11}(\eta^{11}/\eta_{11}) = \tfrac{11}{1275} U_{11}(E_{2,5}) - \frac{(11)^2}{3825}((11)^2 U_{11}(f_1) - 116 U_{11}(f_2) - 352 U_{11}(f_3)).$$

Therefore

$$\sum_{n=0}^{\infty} p_{11}(11n)q^n = \prod_{n=1}^{\infty}(1-q^n) \times \left\{\tfrac{11}{1275}E_{2,5} - \frac{11^2}{3825}((11)^2 U_{11}(f_1) - 116U_{11}(f_2) - 352U_{11}(f_3))\right\}$$

giving us the congruence

$$p_{11}(11n) \equiv p_1(n) - 11 \sum_{r=1}^{n} p_1(n-r) \sum_{1 \leqslant d \mid r} d^4 \left(\frac{d}{11}\right) \pmod{(11)^2}$$

and in particular, the classical congruence $p_{11}(11n) \equiv p_1(n) \pmod{11}$.

4. Ramanujan's proof for partition congruences modulo 11

In this section, we present the proof for the congruences $p(11n+6) \equiv 0 \pmod{11}$ given tersely by Ramanujan in [7], along with proofs for the case of the moduli $5, 7, 49, 121, 23$ etc.

Writing

$$P=1-24\sum_{n=1}^{\infty}\left(\sum_{1\leqslant d|n} d\right)\exp(2\pi inz),\quad Q=1+240\sum_{n=1}^{\infty}\left(\sum_{1\leqslant d|n} d^3\right)\exp(2\pi inz),$$

$$R=1-504\sum_{n=1}^{\infty}\left(\sum_{1\leqslant d|n} d^5\right)\exp(2\pi inz),$$

we know that QR is just the normalized Eisenstein series of weight 10 for the modular group, namely

$$QR=1-264\sum_{n=1}^{\infty}\left(\sum_{1\leqslant d|n} d^9\right)\exp(2\pi inz).$$

Also, the normalized Eisenstein series E_{12} of weight 12, given by

$$E_{12}(z)=1+\frac{65520}{691}\sum_{n=1}\left(\sum_{1\leqslant d|n} d^{11}\right)\exp(2\pi inz)$$

is a linear combination of Q^3 and R^2 which generate the space of modular forms of weight 12 for the modular group; indeed, with $q:=\exp(2\pi iz)$,

$$691+65520\sum_{n=1}^{\infty} q^n\left(\sum_{1\leqslant d|n} d^{11}\right)=441Q^3+250R^2. \tag{11}$$

Given any natural number m and two power series

$$R_1=\sum_{n=0}^{\infty} a_n q^n,\quad R_2=\sum_{n=0}^{\infty} b_n q^n,$$

with all a_n, b_n integral, we write $R_1\equiv R_2 \pmod m$ whenever $a_n\equiv b_n \pmod m$ for all $n\geqslant 0$. Using the congruence $d^{11}\equiv d \pmod{11}$ for every integer d, we note that the left hand side of (11) is $\equiv -2P \pmod{11}$ while the right hand side is $\equiv Q^3-3R^2 \pmod{11}$. Thus we have

$$Q^3-3R^2\equiv -2P \pmod{11}. \tag{12}$$

It is also clear that

$$QR\equiv 1 \pmod{11}. \tag{13}$$

As a consequence,

$$\begin{aligned}(Q^3-R^2)^5&=(Q^3-3R^2)^5+10(Q^3-3R^2)^4R^2+40(Q^3-3R^2)^3R^4\\&\quad+80(Q^3-3R^2)^2R^6+80(Q^3-3R^2)R^8+32R^{10}\\&\equiv(Q^3-3R^2)^5-(Q^3-3R^2)^4R^2+7(Q^3-3R^2)^3R^4\\&\quad+3(Q^3-3R^2)^2R^6+3(Q^3-3R^2)R^8-R^{10} \pmod{11}\\&\equiv(Q^3-3R^2)^5+(Q^3-3R^2)^3R^2(-Q^3+10R^2)\\&\quad+3(Q^3-3R^2)^2R^6+3(Q^3-3R^2)R^8-R^{10} \pmod{11}\end{aligned}$$

$$\equiv (Q^3-3R^2)^5 - Q(Q^3-3R^2)^3 - (Q^3-3R^2)^3 R^4$$
$$+3(Q^3-3R^2)^2R^6 + 3(Q^3-3R^2)R^8 - R^{10} \pmod{11}, \text{by (13)}$$
$$\equiv (Q^3-3R^2)^5 - Q(Q^3-3R^2)^3 - (Q^3-3R^2)^2R^4(Q^3-3R^2)$$
$$+3(Q^3-3R^2)R^8 - R^{10} \pmod{11}$$
$$\equiv (Q^3-3R^2)^5 - Q(Q^3-3R^2)^3 - R(Q^3-3R^2)^2$$
$$+(Q^3-3R^2)(R^6(6Q^3-18R^2)+3R^8) - R^{10} \pmod{11}$$
$$\equiv (Q^3-3R^2)^5 - Q(Q^3-3R^2)^3 - R(Q^3-3R^2)^2$$
$$+(Q^3-3R^2)(6Q^3+7R^2)R^6 - R^{10} \pmod{11}$$
$$\equiv (Q^3-3R^2)^5 - Q(Q^3-3R^2)^3 - R(Q^3-3R^2)^2$$
$$+6Q^6R^6 - 11Q^3R^8 - 22R^{10} \pmod{11}$$
$$\equiv (Q^3-3R^2)^5 - Q(Q^3-3R^2)^3 - R(Q^3-3R^2)^2$$
$$-5QR \pmod{11} \tag{14}$$

since $6Q^6R^6 \equiv 6 \equiv -5 \equiv -5QR \pmod{11}$. By (12) and (14), we have

$$(Q^3-R^2)^5 \equiv -32P^5 + 8QP^3 - 4RP^2 - 5QR \pmod{11}$$
$$\equiv P^5 - 3P^3Q - 4P^2R - 5QR \pmod{11}. \tag{15}$$

We have, on the other hand, the well-known Ramanujan differential equations

$$q\frac{dP}{dq} = \tfrac{1}{12}(P^2-Q), \quad q\frac{dQ}{dq} = \tfrac{1}{3}(PQ-R), \quad q\frac{dR}{dq} = \tfrac{1}{2}(PR-Q^2) \tag{16}$$

which are of fundamental importance for the theory of p-adic modular forms due to Serre [8]. From (16), we have immediately the congruences

$$q\frac{dP}{dq} \equiv P^2-Q, \quad q\frac{dQ}{dq} \equiv 4(PQ-R), \quad q\frac{dR}{dq} \equiv 6(PR-Q^2) \pmod{11}. \tag{17}$$

Now (15) and (17) imply without difficulty that

$$(Q^3-R^2)^5 \equiv q\frac{d}{dq}(3P^4+7P^2Q+5PR) \pmod{11}. \tag{18}$$

But

$$(Q^3-R^2) = 1728q\prod_{n=1}^{\infty}(1-q^n)^{24} \equiv q\prod_{n=1}^{\infty}(1-q^n)^{24} \pmod{11}$$

so that

$$(Q^3-R^2)^5 \equiv q^5\prod_{n=1}^{\infty}(1-q^n)^{120} \pmod{11}. \tag{19}$$

Since

$$q^5\prod_{n=1}^{\infty}(1-q^n)^{120}=q^5\prod_{n=1}^{\infty}(1-q^n)^{121}\Big/\prod_{n=1}^{\infty}(1-q^n)$$

$$=\prod_{n=1}^{\infty}(1-q^n)^{121}\sum_{n=0}^{\infty}p(n)q^{n+5}$$

$$\equiv\prod_{n=1}^{\infty}(1-q^{121n})\sum_{n=0}^{\infty}p(n)q^{n+5}\,(\mathrm{mod}\ 11)$$

and since, further, the coefficient of q^{11m} in $q(\mathrm{d}/\mathrm{d}q)(\sum_{n=0}^{\infty}c_nq^n)$ with c_n in $\mathbb{Z}$ is $11mc_{11m}\equiv 0\,(\mathrm{mod}\ 11)$, we obtain from (18) and (19), the required congruence $p(11m-5)\equiv 0\,(\mathrm{mod}\ 11)$ for $m\geqslant 1$.

Without using the differential equations (16), Ramanujan derives directly from (15) and four other congruences, the general congruence relation

$$p(n-5)-p(n-126)-p(n-247)+p(n-610)+p(n-852)-\cdots$$
$$+n^4\sum_{d|n}d-3n^3\sum_{d|n}d^3-3n^2\sum_{d|n}d^5+5n\sum_{d|n}d^7\equiv 0\,(\mathrm{mod}\ 11)$$

which, of course, implies the simpler congruence $p(11n-5)\equiv 0\,(\mathrm{mod}\ 11)$ for $n\geqslant 1$.

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