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On estimates for integral solutions of linear inequalities

S RAGHAVAN

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

Abstract. Recently, Bombieri and Vaaler obtained an interesting adelic formulation of the first and the second theorems of Minkowski in the Geometry of Numbers and derived an effective formulation of the well-known "Siegel's lemma" on the size of integral solutions of linear equations. In a similar context involving linear *inequalities*, this paper is concerned with an analogue of a theorem of Khintchine on integral solutions for inequalities arising from systems of linear forms and also with an analogue of a Kronecker-type theorem with regard to euclidean frames of integral vectors. The proof of the former theorem invokes Bombieri-Vaaler's adelic formulation of Minkowski's theorem.

Keywords. Bounds for integral solutions of linear inequalities; Theorems of Khintchine and of Kronecker-type; Bombieri-Vaaler formulation of Minkowski's theorems in geometry of numbers; Siegel's lemma

1. Introduction

Bombieri and Vaaler have recently in an interesting paper [1], obtained an adelic formulation of the first and second theorems of Minkowski in the Geometry of Numbers and as an application, through a "cube slicing inequality", derived an effective general formulation of the well-known "Siegel's lemma" on the order of magnitude of integral solutions of linear equations. In a similar context involving linear inequalities, one is led naturally to seek an analogue of a theorem of Khintchine's [4] concerning integral solutions for inequalities arising from systems of linear forms and also to look for an analogue of a theorem of Kronecker's with regard to euclidean frames of integral vectors [3]. Our proof of the above mentioned analogue of Khintchine's theorem (which, incidentally, does not seem to be very accessible, according to Lekkerkerker ([7], p. 470)) invokes the adelic formulation of Minkowski's theorem, due to Bombieri and Vaaler, confirming their remark in [1]: "we certainly do not believe that our application of this inequality to Siegel's lemma exhausts its usefulness". In a preliminary version of theorem 1, we had taken into account only the archimedean primes of K; as one may see, its present formulation, however, takes care of a finite number of places of K including all the archimedean ones, as is customary with problems of this category. It is a pleasure to acknowledge here that a connected discussion some time ago with S G Dani prompted us to look for theorem 4 (in the sequel) that actually overlaps (in part) with a result of his in the section "Orbits of euclidean frames" of his interesting paper entitled "Flows in homogeneous spaces and diophantine approximation" wherein, however, different techniques are involved.

2. Terminology

If $\{F_i(x_1, \ldots, x_m, y_1, \ldots, y_r), i = 1, 2, \ldots, p\}$ is a given system of p real-valued functions of the real variables $x_1, \ldots, x_m, y_1, \ldots, y_r$ and if $\{\chi_i(t), i = 1, 2, \ldots, p\}$ is a given set of p positive functions of a real variable $t > 0$, then $\{F_1, \ldots, F_p\}$ *admits the approximation* $\{\chi_1, \ldots, \chi_p\}$ if, for every real number $M > 0$, there exists $u = (u_1, \ldots, u_m) \in \boldsymbol{R}^m$ with $ht\, u := \max_i |u_i| \geqslant M$ and $v = (v_1, \ldots, v_r) \in \boldsymbol{R}^r$ with all the u_i and v_j in $\boldsymbol{Z}$ such that

$$|F_i(u_1, \ldots, u_m, v_1, \ldots, v_r)| < \chi_i(ht\, u), \ i = 1, 2, \ldots, p.$$

For example (see [6]), a homogeneous real linear form $L(x_1, \ldots, x_m, y_1) = a_1x_1 + \ldots + a_mx_m - y_1$ admits the approximation χ with $\chi(t) = 1/t^m$; the "dual" system of m linear forms $L_i(x_1, y_1, \ldots, y_m) = a_ix_1 - y_i$, $1 \leqslant i \leqslant m$, likewise admits the approximation $\{\chi, \ldots, \chi\}$ with $\chi(t) = 1/t^{1/m}$. The index ω'_1 of the linear form L is defined by

$$\omega'_1 = \text{l.u.b. } \{\omega \mid L \text{ admits the approximation } \chi \text{ with } \chi(t) = 1/t^{m+\omega}\}.$$

If, for some constant $c > 0$, the form L does *not* admit the approximation χ_1 with $\chi_1(t) = 1/(ct^{m+\omega'_1})$; then ω'_1 is called the *proper index* of L. Clearly we have $0 \leqslant \omega'_1 \leqslant \infty$. The form L is called *extreme*, if it has proper index 0. Thus, for an extreme form L as above, there exists a constant $c > 0$ such that, for all $u = (u_1, \ldots, u_m) \in \boldsymbol{R}^m$ with integral $u_1, \ldots, u_m$ not all 0 and for all v in $\boldsymbol{Z}$, we have

$$|a_1u_1 + \ldots + a_mu_m - v| \geqslant 1/(c\, ht\, u)^m.$$

For example, the real form $a_1x_1 - y_1$ is extreme if and only if a_1 is irrational with bounded partial quotients in its simple continued fraction expansion; in particular, for any real quadratic irrationality θ, the form $\theta x_1 - y_1$ is extreme. For the "dual" system $\{L_1, \ldots, L_m\}$ of linear forms above, the notions of the index ω'_2, proper index and extreme system are defined in an entirely analogous manner (see [6]). If ω'_1 (respectively ω'_2) is positive, then L (respectively $\{L_1, \ldots, L_m\}$) is said to be "very well approximable", according to Schmidt [9]; the notion "extreme" corresponds to "badly approximable" in the sense of Schmidt [9]. Almost no $(a_1, \ldots, a_m)$ in $\boldsymbol{R}^m$, with regard to Lebesgue measure, has the property that the corresponding form L as above is very well approximable or badly approximable ([9], theorem 6G, p. 219). It is also known [6] that L is extreme if and only if the "dual" system $\{L_1, \ldots, L_m\}$ is extreme.

The following assertion due to Khintchine (cf. [5]) is found in a footnote to page 86 of Koksma's book [6]:

For any given m real numbers $\alpha_1, \ldots, \alpha_m$, there exists a constant $c = c(\alpha_1, \ldots, \alpha_m) > 0$ such that, for all $t \geqslant 1$ and all real numbers $\beta_1, \ldots, \beta_m$, the inequalities

$$0 < x_1 < ct^m, |\alpha_ix_1 - y_i - \beta_i| < 1/t, \ i = 1, 2, \ldots, m$$

are (simultaneously) solvable in integers $x_1, y_1, \ldots, y_m$ if and only if the system $\{L_1, \ldots, L_m\}$ with $L_i := \alpha_ix_1 - y_i (1 \leqslant i \leqslant m)$ is extreme. In §4, we consider a mild generalization of this assertion of Khintchine's and an application to estimate the magnitude of euclidean frames of integral vectors satisfying an 'irrational' system of linear inequalities.

Let K be an algebraic number field of (finite) degree d over the field $\boldsymbol{Q}$ of rational numbers and R, the ring of integers in K with a $\boldsymbol{Z}$-basis $\{\omega_1, \ldots, \omega_d\}$ to be *fixed in the sequel*. Let E be a fixed vector space of dimension l over K and let, for every place v of K, $E_v = E \otimes_K K_v$, where K_v is the completion of K at v. For archimedean v, we write $v \mid \infty$, in symbols and otherwise, we write $v \nmid \infty$. Let, for $v \nmid \infty$, R_v denote the ring of integers in K_v and M_v, a K_v-lattice in E_v (*i.e.* an open compact R_v-module contained in E_v) such that $M_v = R_v^l$ for all but finitely many such v. For each $v \mid \infty$, let $\mathscr{S}_v$ be a non-empty and bounded open convex set in $E_v = K_v^l$ which is, in addition, 0-symmetric *i.e.* for every α in K_v with v-adic value $|\alpha|_v = 1$ and every x in $\mathscr{S}_v$, αx is also in $\mathscr{S}_v$. With M_v given as above for every $v \nmid \infty$, let

$$\mathscr{S} := \prod_{v \mid \infty} \mathscr{S}_v \times \prod_{v \nmid \infty} M_v.$$

Then $\mathscr{S}$ is an open relatively compact neighbourhood of 0 in $E_A := E \otimes_K K_A \approx K_A^l$ where K_A is the ring of K-adeles [10]. Since E is a discrete subgroup of E_A, it follows that $\mathscr{S} \cap E$ is finite. For real $\lambda > 0$, let $\lambda\mathscr{S}_v := \{\lambda x \mid x \in \mathscr{S}_v\}$ for $v \mid \infty$ and let

$$\lambda\mathscr{S} := \prod_{v \mid \infty} \lambda\mathscr{S}_v \times \prod_{v \nmid \infty} M_v.$$

The *successive minima* of $\mathscr{S}$ with respect to E (see [1]) are defined, for $1 \leqslant j \leqslant l$, by

$$\lambda_j := \inf \{\lambda > 0 \mid \lambda\mathscr{S} \cap E \text{ contains } j \text{ linearly independent vectors}\},$$

and moreover, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_l < \infty$. For $v \mid \infty$, let $r_v = 1$ or $1/2$ according as $K_v = \boldsymbol{R}$ or $\boldsymbol{C}$ and let $r_v^l \mathrm{d}x_v$ denote Lebesgue measure in K_v^l. For $v \nmid \infty$, let $\mathrm{d}x_v$ denote the normalized measure in K_v^l for which R_v^l has measure $|D_v|_v^{dl/2}$ where $|D_v|_v$ is the (v-adic) value of the local "different" D_v in R_v. If Δ is the discriminant of K, then $|\Delta| = \prod_{v \nmid \infty} |D_v|_v^{-d}$. Let vol $(\mathscr{S})$ denote the volume of $\mathscr{S}$ with respect to the product measure $\prod_v \mathrm{d}x_v$ in E_A. Then we know, from Bombieri and Vaaler [1], that

$$\lambda_1 \ldots \lambda_l \leqslant 2^l / (\mathrm{vol}\,(\mathscr{S}))^{1/d} = \rho_E(\mathscr{S}) = \rho_E, \text{ say.}$$

For a fixed *finite* set V of (mutually inequivalent) valuations (or places) v of K containing all v with $v \mid \infty$, let $K_V := \prod_{v \in V} K_v$. Let us write $V_\infty = \{v \mid v \text{ archimedean}\}$ and $V_f = \{v \in V \mid v \nmid \infty\}$ so that $V = V_\infty \amalg V_f$. The ring $K(V)$ of V-integers in K is, by definition, the subring of x in K integral at all v not in V; it has the 'standard' imbedding in K_V as a discrete subring, with the quotient $F_V = K_V/K(V)$ compact [2]. As a 'fundamental set' F_V', we can take $F_\infty \times \prod_{v \in V_f} R_v$ where $F_\infty := \{\mathrm{a} = (\ldots, \mathrm{a}_v, \ldots) \in \prod_{v \mid \infty} K_v \mid a_v = \sum_{1 \leqslant i \leqslant d} c_i \omega_i^{(v)}$ with $c_1, \ldots, c_d$ in $\boldsymbol{R}$, $\max_i |c_i| \leqslant 1/2$ and for ω_i in the fixed $\boldsymbol{Z}$-basis of R above, $\omega_i^{(v)}$ denotes the image of ω_i under the isomorphism from K to $K_v\}$. When V_f is empty, $V = V_\infty$ and we simply write K_∞ for K_V; then $K(V) = R$. For any x in K_V, the V-integer nearest to x is an element y in $K(V)$, generally unique, such that $x - y$ is in the fundamental set F_V' above; a similar definition applies, if we take a finitely generated submodule M, instead of $K(V)$, in K. For $x = (\ldots, x_v, \ldots)$ in K_V, we write $\|x\|_V$ for $\max_{v \in V} |x_v|_v$. In particular, for $V = V_\infty$, we write $\|x\|_\infty$ instead of $\|x\|_V$, for x in K_∞. For v in V_f, we fix $p_v > 0$ in $\boldsymbol{R}$ such that the corresponding "value group" is just $\{p_v^n \mid n \in \boldsymbol{Z}\}$. We use (small or capital) boldface letters to denote vectors or columns and corresponding letters in italics with subscripts to denote their entries: *e.g.* $\mathbf{x}$ stands for a column with entries say, $x_1, \ldots, x_m$ and $\mathbf{S}$ stands for a column with entries $S_1, \ldots, S_r$, say. With this notation for $\mathbf{x}$, we write $\|\mathbf{x}\|_V$

for $\max_i \|x_i\|_V$ when $x_1, \ldots, x_m$ are in K_V; a similar remark applies to $\|\mathbf{x}\|_v$ when $x_1, \ldots, x_r$ are all in K_v or to $\|\mathbf{S}\|_V$, $\|\mathbf{S}\|_v$ etc. The transpose of a matrix A is denoted by ${}^tA$. If $\varepsilon = (\ldots, \varepsilon_v, \ldots) \in K_V$ with rational $\varepsilon_v > 0$ for every v in V and further, if $\mathbf{a} = {}^t(a_1, \ldots, a_r)$ with all $a_i = (\ldots, a_{i,v}, \ldots)$ in K_V satisfying the condition $\|\mathbf{a}\|_v := \max_{1 \leqslant i \leqslant r} |a_{i,v}|_v \leqslant \varepsilon_v$ for every v in V, then we write

$$\mathbf{a} \underset{V}{\leqslant} \varepsilon.$$

3. An analogue of a theorem of Khintchine's

Let $S_1, \ldots, S_r$ be r linear forms in $m+r$ variables $x_1, \ldots, x_m, y_1, \ldots, y_r$ with coefficients θ_{ij} in K_V given by

$$\mathbf{S} = {}^t\Theta\mathbf{x} - \mathbf{y}$$

where $\mathbf{S} := {}^t(S_1 \ldots S_r)$, $\mathbf{x} = {}^t(x_1 \ldots x_m)$, $\mathbf{y} = {}^t(y_1 \ldots y_r)$ and $\Theta = (\theta_{ij})$ is an (m, r) matrix with θ_{ij} $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant r)$ as entries. Associated to $\mathbf{S}$ we have a 'dual' system of m linear forms $T_i = T_i(\mathbf{a}, \mathbf{b})$ with $\mathbf{a} = {}^t(a_1 \ldots a_r)$ and $\mathbf{b} = {}^t(b_1 \ldots b_m)$ given by

$$\mathbf{T} := {}^t(T_1 \ldots T_m) = \Theta\,\mathbf{a} + \mathbf{b}.$$

If $\mathbf{a}'$ is defined by ${}^t\mathbf{a}' = ({}^t\mathbf{a}\,{}^t\mathbf{b})$, we also write $T_i(\mathbf{a}')$ instead of $T_i(\mathbf{a}, \mathbf{b})$. When $\mathbf{a}' := {}^t(a'_1 \ldots a'_{r+m})$ has all its entries in $K(V)$, we define, for given $\mathbf{u} := {}^t(u_1 \ldots u_r)$ with u_i in K_V, $\mathscr{A}(\mathbf{a}') = \mathscr{A}_{\mathbf{u}}(\mathbf{a}') := (\mathscr{A}_{\mathbf{u}}(\mathbf{a}')_v)_{v \in V} := \left(\left|\sum_{1 \leqslant j \leqslant r} u_j a'_j - c\right|_v\right)_{v \in V}$ where c is "the V-integer nearest to" $\sum_{1 \leqslant j \leqslant r} u_j a'_j$. Then clearly $\max_{v \in V} \left|\sum_j u_j a'_j - c\right|_v \leqslant \max_{v \in V} |\Sigma u_j a'_j - c'|_v$ for every V-integer c' in K. In a similar way, we can also define, for given u and for every finitely generated $K(V)$-module $M \subset K$,

$$\mathscr{A}(\mathbf{a}'; M) = \mathscr{A}_{\mathbf{u}}(\mathbf{a}'; M) := \left(\left|\sum_{1 \leqslant j \leqslant r} u_j a'_j - c^*\right|_v\right)_{v \in V}$$

where now c^* is the element of M "nearest to" $\sum_{1 \leqslant j \leqslant r} u_j a'_j$. Finally, we set, for $v \in V$,

$$\mathscr{T}(\mathbf{a}')_v = \mathscr{T}(\mathbf{a}, \mathbf{b})_v := \max_{1 \leqslant i \leqslant m} |T_i(\mathbf{a}, \mathbf{b})|_v$$

We are now in a position to state the following analogue of a result of Khintchine's [4]:

THEOREM 1. Let $S_1, \ldots, S_r$ be linear forms in $l := m + r$ variables $x_1, \ldots, x_m$, $y_1, \ldots, y_r$ as above, with coefficients θ_{ij} in K_V, φ a monotonic increasing function of a positive real variable t and for every v in V_f, $\tau_v \leqslant 0$ be given in $\mathbf{Z}$. Let, further, $\varepsilon = (\varepsilon_v)$ and $\delta = (\delta_v)$ in K_V be given by $\varepsilon_v = c_1/t$, $\delta_v = c_2\varphi(t)$ for $v \mid \infty$, $\varepsilon_w = \delta_w = c_3 p_w^{\tau_w}$ (with positive constants c_1, c_2 and c_3) for $w \in V_f$. Then, for every ${}^t\mathbf{u}$ as above in K_V^r and every such ε and δ, the system of inequalities

$$\mathbf{S} - \mathbf{u} \underset{V}{\leqslant} \varepsilon$$

$$\mathbf{x} \underset{V}{\leqslant} \delta$$

admits as solutions columns $\mathbf{x}$, $\mathbf{y}$ with entries $x_1, \ldots, x_m$ and $y_1, \ldots, y_r$ respectively from a finitely generated $K(V)$-module M contained in K, provided that, for every $\mathbf{a}$ in $K(V)^r$ and $\mathbf{b}$ in $K(V)^m$ as above, the following conditions are fulfilled:

(i) $\mathscr{A}_{\mathbf{u}}(\mathbf{a}, \mathbf{b}) = 0$, whenever $\|\mathbf{a}\|_V \mathscr{T}(\mathbf{a}, \mathbf{b})_V = 0$

(ii) $\mathscr{A}_{\mathbf{u}}(\mathbf{a}, \mathbf{b}) \leqslant \gamma$ with $\gamma = (\gamma_v) \in K_V$ given by

$$\gamma_v := \begin{cases} c_v \|\mathbf{a}\|_\infty / \psi(\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}, \mathbf{b})_\infty) & \text{for } v \mid \infty \\ c_v p_v^{\tau_v} \max(\|\mathbf{a}\|_v, \mathscr{T}(\mathbf{a}, \mathbf{b})_v) & \text{for } v \in V_f \end{cases} \tag{2}$$

for suitable constants $c_v > 0$ and ψ defined by $\psi(t\varphi(t)) = t$.

Proof. Our proof is on the same lines as Khintchine's [4].
For v in V, let $T_{1,v}, \ldots, T_{m,v}$ denote the linear forms in the l variables in $(a'_1 a'_2 \ldots a'_l) = {}^t\mathbf{a}' = ({}^t\mathbf{a}\,{}^t\mathbf{b})$ with the v-adic components $\theta_{ij,v}$ of θ_{ij} in K_V as coefficients in lieu of the coefficients θ_{ij} of $T_1, \ldots, T_m$. With a positive parameter μ, let us define, for every v in V, linear forms $L_{1,v}, \ldots, L_{l,v}$ in $a'_1 = a_1, \ldots, a'_r = a_r, a'_{r+1} = b_1, \ldots, a'_l = b_m$ by

$$L_{i,v} = \mu_v^r T_{i,v}, \; L_{m+j,v} = \mu_v^{-m} a_j \; (1 \leqslant i \leqslant m;\; 1 \leqslant j \leqslant r) \tag{3}$$

where $\mu_v := \mu$ for $v \mid \infty$ and $\mu_v = 1$, otherwise. The determinant of this system of l linear forms is of absolute value 1, for every v in V. Let $\mathscr{S}_v := \{(x_1, \ldots, x_l) \in K_v^l \,\|\, L_{i,v}(x_1, \ldots, x_l)|_v \leqslant 1 \text{ for } 1 \leqslant i \leqslant l\}$ for every v in V and $\mathscr{S} := \prod_{v \in V} \mathscr{S}_v \times \prod_{v \notin V} R_v^l$. If $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_l < \infty$ are the l successive minima of $\mathscr{S}$, there exists a corresponding set $\{\mathbf{a}'(1), \ldots, \mathbf{a}'(l)\}$ of linearly independent vectors in $K(V)^l$ such that $\mathbf{a}'(j)$ is in $\lambda\mathscr{S} \cap K^l$ for all $\lambda > \lambda_j$ and further $\lambda_j = \max_{1 \leqslant i \leqslant l} |L_{i,v}(\mathbf{a}'(j))|_v$ for $v \mid \infty$. From [1], we know that

$$\lambda_1 \ldots \lambda_l \leqslant 2^l / \mathrm{vol}(\mathscr{S})^{1/d} = \rho_E(\mathscr{S}) = \rho_E, \text{ say.}$$

In the sequel, we shall write ${}^t\mathbf{a}'(j) = ({}^t\mathbf{a}(j)\,{}^t\mathbf{b}(j))$ with ${}^t\mathbf{a}(j) \in K^r$ and ${}^t\mathbf{b}(j) \in K^m$ for $1 \leqslant j \leqslant l$. From (3) and the construction of $\mathbf{a}'(j)$, it is immediate that

$$\|\mathbf{a}(j)\|_v \leqslant \begin{cases} \mu^m \lambda_j & \text{for } v \mid \infty \\ 1 & \text{for } v \in V_f \end{cases} \tag{4}$$

and

$$\mathscr{T}(\mathbf{a}'(j))_v \leqslant \begin{cases} \mu^{-r} \lambda_j & \text{for } v \mid \infty \\ 1 & \text{for } v \in V_f. \end{cases} \tag{5}$$

We now claim that, as a consequence of conditions (i)–(ii) in (2), we have, for $1 \leqslant j \leqslant l$,

$$\mathscr{A}_{\mathbf{u}}(\mathbf{a}'(j)) \underset{V}{\leqslant} \gamma' \text{ with } \gamma' = (\gamma'_v) \text{ and } \gamma'_v = \begin{cases} c_v \lambda_j \mu^m / \psi(\mu^l) & \text{for } v \mid \infty \\ c_v p_v^{\tau_v} & \text{for } v \in V_f. \end{cases} \tag{6}$$

In view of (4) and (5), only the part of the claim pertaining to $v \mid \infty$ in (6) needs to be established. If $\|\mathbf{a}(j)\|_\infty \mathscr{T}(\mathbf{a}'(j))_\infty = 0$, then there is nothing to prove. Let then condition (ii) in (2) hold, with $\|\mathbf{a}(j)\|_\infty \mathscr{T}(\mathbf{a}'(j))_\infty > 0$. If $\sigma_j := \|\mathbf{a}(j)\|_\infty / \mathscr{T}(\mathbf{a}'(j)) \geqslant \mu^l$, then the required inequality follows at once from (4) and condition (ii), in view of the monotonicity of ψ. On the other hand, if $\sigma_j < \mu^l$, then $\sigma_j / \psi(\sigma_j) = \varphi(\psi(\sigma_j))$

$\leqslant \varphi(\psi(\mu^l)) = \mu^l/\psi(\mu^l)$. Again, it follows from (ii) and (5) that for $v|\infty$,

$$\gamma_v \leqslant c_v \mathscr{T}(\mathbf{a}'(j))_\infty \mu^l/\psi(\mu^l) \leqslant \gamma'_v$$

proving our claim.

Since $\mathbf{a}'(1), \ldots, \mathbf{a}'(l)$ are linearly independent,

$$\mathscr{B} := \det(\mathbf{a}'(1) \ldots \mathbf{a}'(l)) \in K \setminus \{0\}.$$

Assuming all valuations v to be suitably normalized, as in [1], we have $\prod_v |\mathscr{B}|_v = 1$, by the product formula. The linear forms $L_{i,v}$ (for $1 \leqslant i \leqslant l$) have determinant 1 in absolute value, for every v in V and therefore

$$|\mathscr{B}|_v = |\det(L_{i,v}(\mathbf{a}'(j)))_{1 \leqslant i,j \leqslant l})|_v$$

In the light of the definition of the vectors for successive minima, this gives immediately the inequalities

$$|\mathscr{B}|_v \leqslant \begin{cases} l!\lambda_1 \ldots \lambda_l \leqslant l!2^l/\mathrm{vol}(\mathscr{S})^{1/d} = l!\rho_E \text{ for } v|\infty \\ 1 \quad \text{for} \quad v \nmid \infty. \end{cases} \tag{7}$$

For v-adic measures normalized as in [1], we note that $\mathrm{vol}(\mathscr{S}) = 2^{ld}(\pi/2)^{ls}|\Delta|^{-l/2}$ where s is the number of complex places of K. From the product formula and (7), we derive at once the following estimates:

$$1 = \prod_v |\mathscr{B}|_v \leqslant |\mathscr{B}|_{v_1} \times \begin{cases} (l!\rho_E)^{d-1} & \text{for} \quad v_1|\infty \\ (l!\rho_E)^d & \text{for} \quad v_1 \nmid \infty \end{cases}$$

i.e.

$$|\mathscr{B}|_v \geqslant \theta_v := \begin{cases} (l!\rho_E)^{1-d} & \text{for} \quad v|\infty \\ (l!\rho_E)^{-d} & \text{for} \quad v \nmid \infty. \end{cases} \tag{8}$$

By (7), $\mathscr{B}$ is in $R\setminus\{0\}$ and there exists a possibly non-empty *finite* set W of valuations v of K with $v \nmid \infty$, depending only on K, l and V, such that $|\mathscr{B}|_v = 1$ for all $v \nmid \infty$ with $v \notin W$.

Let us next consider the expansion

$$\det(L_{j,v}(\mathbf{a}'(i))) = \sum_\sigma \pm L_{1,v}(\mathbf{a}'(l_1)) \ldots L_{m,v}(\mathbf{a}'(l_m)) D(\sigma)_v$$

where the summation is over all the distinct subsets $\sigma = \{l_1, \ldots, l_m\}$ consisting of m (distinct) elements in $\{1, \ldots, l\}$ and for any such σ, $D(\sigma)_v$ is the determinant of the r-rowed square matrix obtained by deleting the rows with indices $i = l_1, \ldots, l_m$ from the matrix constituted by the last r columns of $(L_{j,v}(\mathbf{a}'(i))$ corresponding to the column indices $j = m+1, m+2, \ldots, m+r(=l)$. The number of such subsets σ being $l!/(r!)$, the (v-adic) value of the term corresponding to *at least one* of these subsets is $\geqslant r!\theta_v/(l!)$ for $v|\infty$ and $\geqslant \theta_v$ for $v \in V_f$, in view of (8). Let us fix one of these subsets, say, $\sigma_0 = \{q_1, \ldots, q_m\}$ which, of course, depends on v and let $\{1, \ldots, l\}\setminus\sigma_0 = \{s_1, \ldots, s_r\}$. Thus, if

$$\mathscr{M}_v := |\det(L_{m+j,v}(\mathbf{a}'(i)))_{\substack{1 \leqslant j \leqslant r \\ i \notin \sigma_0}}|_v,$$

then

$$\mathscr{M}_v \prod_{1 \leqslant i \leqslant m} |L_{i,v}(\mathbf{a}'(q_i))|_v \geqslant \begin{cases} r!\theta_v/(l!) & \text{for} \quad v|\infty \\ \theta_v & \text{for} \quad v \in V_f. \end{cases}$$

From the nature of the construction of $\mathbf{a}'(q_i)$, this gives us

$$\mathcal{M}_v \geqslant \begin{cases} r!\theta_v/(l!\lambda_{q_1}\ldots\lambda_{q_m}) & \text{for} \quad v\mid\infty \\ \theta_v & \text{for} \quad v\in V_f. \end{cases}$$

Writing

$$\mathbf{a}'(i) = {}^t(\alpha_{i,1}\ldots\alpha_{i,l})\,(1\leqslant i\leqslant l),\ \mathcal{D}_v := |\det(\alpha_{ij})_{\substack{1\leqslant j\leqslant r \\ i\notin\sigma_0}}|_v,$$

we see from (3) that $\mathcal{D}_v = \mu_v^{mr}\mathcal{M}_v$ for every v in V. Therefore, we have the lower bounds

$$\mathcal{D}_v \geqslant \begin{cases} r!\theta_v\mu^{mr}/(l!\lambda_{q_1}\ldots\lambda_{q_m}) & \text{for} \quad v\mid\infty \\ \theta_v & \qquad v\in V_f. \end{cases} \tag{9}$$

In order to obtain the required bounds for $\mathbf{S}-\mathbf{u}$ and $\mathbf{x}$ under the assumption of conditions (i)–(ii) in (2), let us consider

$$\begin{aligned} \eta_i :&= {}^t\mathbf{a}'(i)(\mathbf{S}-\mathbf{u})\,(1\leqslant i\leqslant l) \\ &= \sum_{1\leqslant j\leqslant r}\alpha_{i,j}\left(\sum_{1\leqslant k\leqslant m}(\theta_{kj}x_k - y_j - u_j)\right) \\ &= \sum_{1\leqslant k\leqslant m} x_k T_k(\mathbf{a}'(i)) - \sum_{1\leqslant k\leqslant m}\alpha_{i,r+k}x_k - \sum_{1\leqslant j\leqslant r}\alpha_{i,j}y_j \\ &\qquad - \sum_{1\leqslant j\leqslant r}\alpha_{i,j}u_j. \end{aligned}$$

Observe now that it is possible to find $x_1,\ldots,x_m, y_1,\ldots,y_r$ in K (uniquely, in view of $\mathcal{B}\neq 0$) such that

$$\sum_{1\leqslant k\leqslant m}\alpha_{i,r+k}x_k + \sum_{1\leqslant j\leqslant r}\alpha_{i,j}y_j = v_i\ (1\leqslant i\leqslant l) \tag{10}$$

where, for $1\leqslant i\leqslant l$, v_i is the V-integer nearest to $-\sum_{1\leqslant j\leqslant r}\alpha_{i,j}u_j$. The elements $x_1,\ldots,x_m, y_1,\ldots,y_r$ constituting the solution belong to the fixed $K(V)$-module $\mathcal{B}^{-1}K(V)$ (independent of $\mathbf{u}$) and are integral at all the non-archimedean v outside the finite set W (described after the derivation of (8)). Then, for $1\leqslant i\leqslant l$, we have

$$\sum_{1\leqslant j\leqslant r}\alpha_{i,j}(S_j-u_j) = \eta_i = \sum_{1\leqslant k\leqslant m}x_kT_k(\mathbf{a}'(i)) - \sum_{1\leqslant j\leqslant r}\alpha_{i,j}u_j - v_i. \tag{11}$$

Solving for $x_1,\ldots,x_m$ from (10) and working in the field K_v for a fixed v in V, we have

$$x_j\cdot\mathcal{B} = \det{}^t(\boldsymbol{\beta}_1\ldots\boldsymbol{\beta}_l) = \det(\boldsymbol{\beta}_1\ldots\boldsymbol{\beta}_l) = \det(\boldsymbol{\beta}'_1\ldots\boldsymbol{\beta}'_l) = \mathcal{C},\ \text{say},$$

where the columns $\boldsymbol{\beta}_i\cdot\ \boldsymbol{\beta}'_i$ are respectively given by

$$\begin{aligned} {}^t\boldsymbol{\beta}_i &= (\alpha_{i,1}\ldots\alpha_{i,r}\alpha_{i,r+1}\ldots\alpha_{i,r+j-1}\ v_i\ \alpha_{i,r+j+1}\ldots\alpha_{i,r+m}), \\ {}^t\boldsymbol{\beta}'_i &= (\alpha_{i,1}\ldots\alpha_{i,r}T_1(\mathbf{a}'(i))\ldots T_{j-1}(\mathbf{a}'(i)) \\ &\qquad v_i + \sum_k \alpha_{i,k}u_kT_{j+1}(\mathbf{a}'(i))\ldots T_m(\mathbf{a}'(i))), \end{aligned}$$

taking the liberty of omitting the subscript v from $T_{j,v}$ $(1 \leqslant j \leqslant m)$ and $u_{k,v}$ $(1 \leqslant k \leqslant r)$. The general term in the expansion of det $(\boldsymbol{\beta}'_1 \ldots \boldsymbol{\beta}'_l)$ is of the form

$$\pm \alpha_{t_1,1} \ldots \alpha_{t_r,r} \Big(\prod_{1 \leqslant n \leqslant m} T_n(\mathbf{a}'(t_{r+n})) \Big) / T_j(\mathbf{a}'(t_{r+j}))(v_{t_{r+j}} + \sum_k \alpha_{t_{r+j},k} u_k)$$

where $(t_1, \ldots, t_l)$ is a permutation of $(1, 2, \ldots, l)$. Because of our special choice of v_i above, we have

$$\Big(\Big| v_i + \sum_{1 \leqslant k \leqslant r} \alpha_{i,k} u_k \Big|_v \Big)_{v \in V} = \mathscr{A}(\mathbf{a}'(i)) \quad \text{for} \quad 1 \leqslant i \leqslant l.$$

Using now the inequalities in (4)–(6), we obtain

$$|\mathscr{C}|_v \leqslant \begin{cases} l!\lambda_1 \ldots \lambda_l \mu^r c_v \mu^m / \psi(\mu^l) & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} & \text{for} \quad v \in V_f \end{cases}$$

and in view of (8), we can deduce that, for $1 \leqslant k \leqslant m$,

$$|x_k|_v \leqslant \begin{cases} l!\lambda_1 \ldots \lambda_l c_v \mu^l / (\theta_v \psi(\mu^l)) & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} / \theta_v & \text{for} \quad v \in V_f. \end{cases} \tag{12}$$

From (11), (12), (5) and (6), it then follows easily for $1 \leqslant i \leqslant l$ that

$$\begin{aligned} |\eta_i|_v &\leqslant \begin{cases} \sum_{1 \leqslant k \leqslant m} |x_k|_v \mathscr{T}(\mathbf{a}'(i))_v + \mathscr{A}(\mathbf{a}'(i))_v & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} \max(1, 1/\theta_v) & \text{for} \quad v \in V_f \end{cases} \\ &\leqslant \begin{cases} l!m\lambda_1 \ldots \lambda_l c_v (\mu^l / (\theta_v \psi(\mu^l)))(\lambda_i/\mu^r) + c_v \lambda_i \mu^m / \psi(\mu^l) & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} \max(1, 1/\theta_v) & \text{for} \quad v \in V_f \end{cases} \\ &\leqslant \begin{cases} c_v (l!m\rho_E/\theta_v + 1)\mu^m \lambda_i / \psi(\mu^l) & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} \max(1, 1/\theta_v) & \text{for} \quad v \in V_f. \end{cases} \end{aligned} \tag{13}$$

Reading off the values of $S_j - u_j$ $(1 \leqslant j \leqslant r)$ from the equations on the left half of (11) with $i = s_1, \ldots, s_r, \ldots$, we obtain the estimates

$$\mathscr{D}_v |S_j - u_j|_v \leqslant \begin{cases} (r-1)! \sum_{1 \leqslant i \leqslant r} |\eta_{s_i}|_v \Big(\prod_{1 \leqslant k \leqslant r} \|\mathbf{a}(s_k)\|_v \Big) \Big/ \|\mathbf{a}(s_i)\|_v & \text{for} \quad v|\infty \\ \max_{1 \leqslant i \leqslant r} |\eta_{s_i}|_v \prod_{1 \leqslant k \leqslant r} \|\mathbf{a}(s_k)\|_v / \|\mathbf{a}(s_i)\|_v & \text{for} \quad v \in V_f. \end{cases}$$

Applying (13), (4) and (9) next, we have

$$|S_j - u_j|_v \leqslant \begin{cases} c_v \rho_E (l!/\theta_v)(1 + m\rho_E(l!)/\theta_v)/\psi(\mu^l) & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} \max(1, 1/\theta_v) & \text{for} \quad v \in V_f \end{cases}$$

and

$$|x_k|_v \leqslant \begin{cases} c_v \rho_E (l!/\theta_v) \mu^l / \psi(\mu^l) & \text{for} \quad v|\infty \\ (c_v/\theta_v) p_v^{\tau_v} & \text{for} \quad v \in V_f \end{cases}$$

for $1 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant m$. Let us now take

$$c_1 = c_v \rho_E (l!/\theta_v)(1 + m\rho_E(l!)/\theta_v), \; c_2 = c_v \rho_E(l!)/\theta_v, \; c_3 = c_v \max(1, 1/\theta_v).$$

We may note further that for any given $t > 0$, there exists $\mu > 0$ with $\psi(\mu^l) = t$. Then the existence of $x_1, \ldots, x_m, y_1, \ldots, y_r$ in the module $\mathscr{B}^{-1}K(V)$ satisfying the given system of inequalities for $\mathbf{S}-\mathbf{u}$ and $\mathbf{x}$ is established as a consequence of conditions (i)–(ii) in (2).

Towards a converse (to theorem 1), we have the following

THEOREM 2. With notation as in theorem 1, let for $\mathbf{S}$, $\mathbf{u}$ and every ε and δ, the inequalities $\mathbf{S}-\mathbf{u} \underset{V}{\leqslant} \varepsilon$, $\mathbf{x} \underset{V}{\leqslant} \delta$ be solvable with columns $\mathbf{x}$, $\mathbf{y}$ having entries in a finitely generated $K(V)$-module $M \subset K$. Then, for every ${}^t\mathbf{a}' = ({}^t\mathbf{a}\,{}^t\mathbf{b})$ with ${}^t\mathbf{a}$ in $K(V)^r$ and ${}^t\mathbf{b}$ in $K(V)^m$, we have

(i) $\mathscr{A}_{\mathbf{u}}(\mathbf{a}'; M) = 0$, whenever $\|\mathbf{a}\|_V \mathscr{T}(\mathbf{a}, \mathbf{b})_V = 0$ and

(ii) $\|\mathscr{A}_{\mathbf{u}}(\mathbf{a}'; M)\|_V \leqslant \|\gamma\|_V$ for $\gamma = (\gamma_v) \in K_V$ where

$$\gamma_v := \begin{cases} c_v \|\mathbf{a}\|_\infty / \psi(\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}, \mathbf{b})_\infty) & \text{for } v \mid \infty \\ c_v p_v^{\tau_v} \max(\|\mathbf{a}\|_v, \mathscr{T}(\mathbf{a}, \mathbf{b})_v) & \text{for } v \in V_f \end{cases}$$

with positive constants c_v for every v in V.

Proof. For a given $\mathbf{x} = {}^t(x_1 \ldots x_m)$, $\mathbf{y} = {}^t(y_1 \ldots y_r)$ satisfying the inequalities $\mathbf{S}-\mathbf{u} \leqslant \varepsilon$, $\mathbf{x} \leqslant \delta$ corresponding to the given ε, δ and $\mathbf{u}$ and for the given $\mathbf{a} = {}^t(a_1 \ldots a_r)$, $\mathbf{b} = {}^t(b_1 \ldots b_m)$, let us define $v = \sum_{1 \leqslant j \leqslant r} a_j y_j + \sum_{1 \leqslant i \leqslant m} b_i x_i$ which is clearly in M. Then, as in (11), we have

$$\sum_{1 \leqslant j \leqslant r} a_j (S_j - u_j) = \sum_{1 \leqslant k \leqslant m} x_k T_k(\mathbf{a}') - \sum_{1 \leqslant j \leqslant r} a_j u_j - v$$

and

$$\begin{aligned} \mathscr{A}(\mathbf{a}'; M)_V &\leqslant \Big\| \sum_{1 \leqslant j \leqslant r} a_j u_j + v \Big\|_V \\ &= \Big\| \sum_{1 \leqslant j \leqslant r} a_j (S_j - u_j) - \sum_{1 \leqslant k \leqslant m} x_k T_k(\mathbf{a}') \Big\|_V \\ &\leqslant \underset{\substack{v \mid \infty \\ w \in V_f}}{\mathrm{Max}} \begin{Bmatrix} r \|\mathbf{a}\|_v c_1 / t + m c_2 \varphi(t) \mathscr{T}(\mathbf{a}')_v \\ \|\mathbf{a}\|_w c_3 p_w^{\tau_w}, c_3 p_w^{\tau_w} \mathscr{T}(\mathbf{a}')_w \end{Bmatrix} \end{aligned}$$

If $\|\mathbf{a}\|_V = 0$, then $\mathbf{a} = 0$ and so $\mathscr{A}(\mathbf{a}'; M)_V = 0$. On the other hand, if $\mathscr{T}(\mathbf{a}')_V = 0$, then the validity of

$$\mathscr{A}(\mathbf{a}'; M)_V \leqslant \mathrm{Max}\,(r \|\mathbf{a}\|_v c_1 / t, \|\mathbf{a}\|_w c_3 p_w^{\tau_w})$$

for every $t > 0$ and every $\tau_w \leqslant 0$ in $\mathbf{Z}$ forces $\mathscr{A}(\mathbf{a}'; M)_V$ to be 0. If therefore $\|\mathbf{a}\|_\infty \mathscr{T}(\mathbf{a}')_\infty > 0$, we can find $t > 0$ such that $t\varphi(t) = \|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}')_\infty$ i.e. $t = \psi(\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}')_\infty)$ and then $\mathscr{T}(\mathbf{a}')_\infty \varphi(t) = \|\mathbf{a}\|_\infty / \psi(\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}')_\infty)$. Thus, in any case,

$$\mathscr{A}(\mathbf{a}'; M)_\infty \leqslant (rc_1 + mc_2) \|\mathbf{a}\|_\infty / \psi(\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}')_\infty)$$

implying (i)–(ii) for $v \mid \infty$. Consequently, for every v in V,

$$\mathscr{A}(\mathbf{a}'; M)_v \leqslant \mathscr{A}(\mathbf{a}'; M)_V \leqslant \underset{v \in V}{\mathrm{Max}} |\gamma_v|$$

with $c_v = rc_1 + mc_2$ for $v \mid \infty$ and $c_w = c_3$ for $w \in V_f$.

4. Applications

We apply the foregoing to obtain a mild generalization of an assertion of Khintchine ([6], page 86) referred to earlier and a quantitative version of Kronecker's theorem concerning bounds for the size of frames of integral vectors in euclidean space which arise as solutions of linear inequalities given by an 'irrational' system of linear forms.

Let $\vartheta_1, \ldots, \vartheta_r$ in K_A be linearly independent over K. The homogeneous form $L = \vartheta_1 x_1 + \ldots + \vartheta_m x_m - y$ has K-index $\geqslant 0$ in a sense quite analogous to that of [6], in view of [2]. Let us now suppose that L has *finite* proper K-index $v \geqslant 0$, in the following sense: namely, that for any given $\varepsilon > 0$ and for any $t > 0$, there exist $u'_1, \ldots, u'_r, v'$ not all 0 in R such that

$$\|\vartheta_1 u'_1 + \ldots + \vartheta_r u'_r - v'\|_\infty < 1/t^{r+v-\varepsilon}, \quad \max_{1 \leqslant i \leqslant r} \|u'_i\|_\infty \leqslant t$$

but there exists $c = c(\vartheta_1, \ldots, \vartheta_r, K)$ such that

$$\|\vartheta_1 a_1 + \ldots + \vartheta_r a_r - b\|_\infty \geqslant 1/\left(c \max_{1 \leqslant i \leqslant r} \|a_i\|_\infty\right)^{r+v} \tag{14}$$

for all $a_1, \ldots, a_r, b$ (not simultaneously 0) in R. We are then in a position to appeal to theorem 1, taking

$$S_j = \vartheta_j x_1 - y_j \, (1 \leqslant j \leqslant r), \; m = 1, \; V = V_\infty \text{ and } \varphi(t) = t^{r+v}.$$

It is clear that $\mathscr{A}(\mathbf{a}, b)_\infty = 0$ for $\mathbf{a} = {}^t(a_1 \ldots a_r)$, whenever $\|\mathbf{a}\|_\infty \cdot \mathscr{T}(\mathbf{a}, b)_\infty = 0$, in view of the linear independence of $\vartheta_1, \ldots, \vartheta_r$ over K. Since $\mathscr{A}(\mathbf{a}, b)_v \leqslant 1/2 \sum_{1 \leqslant i \leqslant d} \|\omega_i\|_\infty$, the first inequality in (ii) for $v | \infty$ can certainly be satisfied with $c_v = 1/2 \sum_{1 \leqslant i \leqslant d} \|\omega_i\|_\infty c^{(r+v)/(r+v+1)}$; indeed, (14) ensures that $\mathscr{T}(\mathbf{a}, b)_\infty \geqslant 1/(c\|\mathbf{a}\|_\infty)^{r+v}$ for $\|\mathbf{a}\|_\infty \geqslant 1$ and therefore

$$\begin{aligned} 1/2 \sum_{1 \leqslant i \leqslant d} \|\omega_i\|_\infty &\leqslant c_v \|\mathbf{a}\|_\infty / (\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}, b))_\infty^{1/(r+1+v)} \\ &= c_v \|\mathbf{a}\|_\infty^{(r+v)/(r+1+v)} \mathscr{T}(\mathbf{a}, b)_\infty^{1/(r+1+v)}. \end{aligned}$$

Thus, for *every* given $u_1, \ldots, u_r$ in K_∞, the given system of inequalities for $\mathbf{S} - \mathbf{u}$ and $\mathbf{x}$ is satisfied by $x_1, y_1, \ldots, y_r$ in $\mathscr{B}^{-1}R$. Taking $\mathscr{B}^{-1}u_1, \ldots, \mathscr{B}^{-1}u_r$ in place of $u_1, \ldots, u_r$ and using constants $c'_1 = c_1 \|\mathscr{B}\|_\infty$, $c'_2 = c_2 \|\mathscr{B}\|_\infty$, we have

THEOREM 3. *Let $\vartheta_1, \ldots, \vartheta_r$ in K_A be linearly independent over K and let the linear form $\vartheta_1 x_1 + \ldots + \vartheta_r x_r - y$ in $x_1, \ldots, x_r, y$ have (finite) proper K-index $v \geqslant 0$ in the sense described above. Then, for any $t \geqslant 1$ and any $u_1, \ldots, u_r$ in K_A, we have positive constants c'_1, c'_2 depending on $\vartheta_1, \ldots, \vartheta_r$ and K and further $u'_1, v'_1, \ldots, v'_r$ not all 0 in R such that*

$$\|\vartheta_i u'_1 - v'_i - u_i\|_\infty \leqslant c'_1/t, \; 1 \leqslant i \leqslant r \text{ and}$$

$$\|u'_1\|_\infty \leqslant c'_2 t^{r+v}.$$

Remark. For $K = \mathbf{Q}$ and $v = 0$, this is just Khintchine's assertion ([6], p. 86) for extreme systems.

Suppose now that $B = (b_{ij})$ is an $(l-1, l)$ matrix with elements in K_∞ such that for every non-zero row ${}^t\mathbf{a} = (a_1, \ldots, a_l)$ with all a_i in K, the (l, l) matrix $A := \begin{pmatrix} B \\ {}^t\mathbf{a} \end{pmatrix}$ is invertible. Then the same property holds as well for BV, where V is an arbitrary permutation matrix. We may therefore assume, without loss of generality, that the first $l-1$ columns of B form an invertible matrix, say B_1. Writing $B = B_1(E\,\vartheta)$ with E equal to the $(l-1)$-rowed identity matrix and ${}^t\vartheta = (\vartheta_1, \ldots, \vartheta_{l-1})$, we say that $\vartheta_1, \ldots, \vartheta_{l-1}$ are "associated to" B. For every row ${}^t\mathbf{a} \neq 0$ over K, $\begin{pmatrix} B_1^{-1} & {}^tO \\ O & 1 \end{pmatrix} A$ is invertible, whenever A is invertible and therefore the above-mentioned property of B is equivalent to the condition that

$$1, \vartheta_1, \ldots, \vartheta_{l-1} \text{ are linearly independent over } K, \tag{15}$$

for $\vartheta_1, \ldots, \vartheta_{l-1}$ associated to B. Indeed, ${}^t\mathbf{a}$ is independent of the $l-1$ rows of $(E\,\vartheta)$ if and only if the relations

$$\begin{aligned} a_1 + \lambda'_1 = a_2 + \lambda'_2 = \ldots &= a_{l-1} + \lambda'_{l-1} \\ &= a_l + \vartheta_1\lambda'_1 + \vartheta_2\lambda'_2 + \ldots + \vartheta_{l-1}\lambda'_{l-1} = 0 \end{aligned}$$

hold for *no* $\lambda'_1, \ldots, \lambda'_{l-1}$ in K_∞; this condition is evidently equivalent to the condition in (15). For $K = \mathbf{Q}$, the above-mentioned property of B is the same as the corresponding $l-1$ linear forms having rationality rank $l-1$ in the sense of Kronecker [8]. If, in this case, B has all its entries in $\mathbf{R} \cap \overline{\mathbf{Q}}$ then $\vartheta_1, \ldots, \vartheta_{l-1}$ are real algebraic numbers satisfying condition (15). For general K, we say that the $l-1$ linear forms arising from B have K-rationality rank $l-1$, whenever the associated $\vartheta_1, \ldots, \vartheta_{l-1}$ satisfy condition (15).

Corollary to theorem 3. Let $F_j = \sum_{1 \leqslant p \leqslant l} b_{j,p} x_p$, $1 \leqslant j \leqslant l-1$, be $l-1$ linear forms in variables $x_1, \ldots, x_l$ and coefficients $b_{j,p}$ from K_∞, having K-rationality rank equal to $l-1$ and let, further, the linear form $\vartheta_1 x_1 + \ldots + \vartheta_{l-1}x_{l-1} - y$ for $\vartheta_1, \ldots, \vartheta_{l-1}$ associated to $(b_{j,p})$ have (finite) proper index $v \geqslant 0$. Then for any $u_1^*, \ldots, u_{l-1}^*$ in K_∞ and $t > 0$, there exists constants $c_1'', c_2'' > 0$ depending, in general, only on $F_1, \ldots, F_{l-1}$ and K and integers $u'_1, \ldots, u'_l$ not all 0 in K such that

$$\Big\| \sum_{1 \leqslant p \leqslant l} b_{j,p} u'_p - u_j^* \Big\|_\infty \leqslant c_1''/t,\ 1 \leqslant j \leqslant l-1 \text{ and}$$

$$\|u'_i\|_\infty \leqslant c_2'' t^{l-1+v},\ 1 \leqslant i \leqslant l.$$

Proof. After an appropriate permutation of the variables $x_1, \ldots, x_l$, if necessary and taking $l-1$ suitable linear combinations of $F_1, \ldots, F_{l-1}$, we may reduce ourselves to a system of linear forms of the type $\vartheta_1 x_1 - x_2, \ldots, \vartheta_{l-1}x_1 - x_l$ and suitable corresponding values for $u_1^*, \ldots, u_{l-1}^*$. This process being evidently reversible, the corollary is immediate from theorem 3.

Remark. The constant c_1'' in the corollary may, without loss of generality, be taken as 1, after suitably modifying t and c_2''.

As another application, we derive now a quantitative version of a classical result of

Kronecker type (Satz 62 [8], p. 148) on the existence of independent lattice points in $\boldsymbol{R}^l$ (or what is the same, of integral frames) satisfying linear inequalities arising from an "irrational" system of linear forms (*i.e.* of maximal rationality rank in the sense of Kronecker [8]).

THEOREM 4. Let $F_j = F_j(\mathbf{x}) = \sum_{1 \leqslant p \leqslant l} a_{j,p} x_p$, $1 \leqslant j \leqslant l-1$, be $l-1$ linear forms in l variables $x_1, \ldots, x_l$ with coefficients $a_{j,p}$ in K_∞, K-rationality rank $l-1$ and (finite) proper K-index $\nu \geqslant 0$ as described above. With $\tau_{j,m} = \tau_{j,m}(\mathbf{u}_m) := \sum_{1 \leqslant p \leqslant l} a_{j,p} u_{p,m}$ for $1 \leqslant j, m \leqslant l-1$, we can find, for any $T > 0$, independent vectors ${}^t\mathbf{u}_i = (u_{1,i} \ldots u_{l,i}) \in R^l$, $1 \leqslant i \leqslant l-1$ such that

$$\|\tau_{j,m}\|_\infty < 1/T^{1/(l-1)}, \ 1 \leqslant j, m \leqslant l-1, \text{ and}$$

$$\|\mathbf{u}_q\|_\infty := \max_{1 \leqslant p \leqslant l} \|u_{p,1}\| < c_3' T^{l+\nu-1/(l-1)}, \ 1 \leqslant q \leqslant l-1$$

for a constant $c_3' > 0$ depending only on $F_1, \ldots, F_{l-1}$ and K, in general and further such that the $(l-1, l-1)$ matrix $(\tau_{j,m})$ is invertible.

Proof. Applying the corollary to theorem 3 (with $c_1'' = 1$, as mentioned in the subsequent remark, $(u_1^*)_v = \ldots = (u_{l-1}^*)_v = 2/t$ for $v | \infty$ and $t = 4T^{1/(l-1)}$), there exists ${}^t\mathbf{u}_1 = (u_{1,1} \ldots u_{l,1}) \in R^l$ such that

$$1/(4T^{1/(l-1)}) \leqslant \left\| \sum_{1 \leqslant p \leqslant l} a_{j,p} u_{p,1} \right\|_\infty < 3/(4T^{1/(l-1)}), \ 1 \leqslant j \leqslant l-1$$

$$\|u_1\|_\infty \leqslant c_2^* T^{1+\nu/(l-1)} \tag{16}$$

for some constant c_2^*. Clearly then, every $\tau_{j,1} = \sum_{1 \leqslant p \leqslant l} a_{j,p} u_{p,1}$ is invertible in K_∞. Let us suppose, without loss of generality, that $\|\tau_{l-1,1}\|_\infty \geqslant \|\tau_{j,1}\|_\infty$ for $1 \leqslant j \leqslant l-1$. The independence of $F_1, \ldots, F_{l-1}$ implies that of $G_1, \ldots, G_{l-2}$ where, for $1 \leqslant i \leqslant l-2$,

$$G_i := F_i - (\tau_{i,1}/\tau_{l-1,1}) F_{l-1} = \sum_{1 \leqslant p \leqslant l} (a_{i,p} - (\tau_{i,1}/\tau_{l-1,1}) a_{l-1,p}) x_p.$$

Moreover, $G_1, \ldots, G_{l-2}$ have K-rationality rank $l-2$; otherwise, any non-zero linear form with coefficients in K belonging to the K_∞-linear span of $G_1, \ldots, G_{l-2}$ will also lie in the K_∞-linear span of $F_1, \ldots, F_{l-1}$, giving a contradiction. By induction on the rationality rank, there exist already $l-2$ independent vectors ${}^t\mathbf{v}_j = (v_{1,j}, \ldots, v_{l,j})$ in R^l, $1 \leqslant j \leqslant l-2$ such that, with

$$\tau_{i,j}' := \sum_{1 \leqslant p \leqslant l} (a_{i,p} - (\tau_{i,1}/\tau_{l-1,1}) a_{l-1,p}) v_{p,j}, \ 1 \leqslant i \leqslant l-2,$$

we have

$$\|\tau_{i,j}'\|_\infty < 1/(4T^{1/(l-1)}) \ 1 \leqslant i, j \leqslant l-2,$$

$$\|v_{p,j}\|_\infty \leqslant c_3'' T^{\nu_{l-2}} \quad (1 \leqslant j \leqslant l-2; \ 1 \leqslant p \leqslant l), \tag{17}$$

$$\det(\tau_{i,j}') \neq 0$$

for an exponent ν_{l-2} and a suitable constant $c_3'' > 0$. Evidently

$$\sum_{1 \leqslant p \leqslant l} (a_{i,p} - (\tau_{i,1} | \tau_{l-1,1}) a_{l-1,p}) u_{p,1} = 0, \; 1 \leqslant i \leqslant l-1,$$

implying that for *any* w_j in R, we have, for $1 \leqslant i, j \leqslant l-2$,

$$\tau'_{i,j} = \sum_{1 \leqslant p \leqslant l} (a_{i,p} - (\tau_{i,1} | \tau_{l-1,1}) a_{l-1,p}) (v_{p,j} - w_j u_{p,1}).$$

Since $\|\tau_{l-1,1}\|_\infty \geqslant 1/(4T^{1/(l-1)})$ by (16), we can choose w_j in R to satisfy the condition

$$\left\| \sum_{1 \leqslant p \leqslant l} a_{l-1,p} (v_{p,j} - w_j u_{p,1}) \right\|_\infty = \left\| \sum_{1 \leqslant p \leqslant l} a_{l-1,p} v_{p,j} - w_j \tau_{l-1,1} \right\|_\infty < c_4/(4T^{1/(l-1)}) \quad (18)$$

where $c_4 = \max_{1 \leqslant i \leqslant d} \|\omega_i\|_\infty$. From this together with (16) and (17), we deduce that, for a constant $c_5 > 0$,

$$\|w_j\|_\infty \leqslant c_5 T^{\nu_{l-2} + 1/(l-1)} (1 \leqslant j \leqslant l-2).$$

Further, on setting

$$u_{p,j+1} := v_{p,j} - w_j u_{p,1} \; (1 \leqslant p \leqslant l; \; 1 \leqslant j \leqslant l-2),$$

we also obtain

$$\begin{aligned} \|u_{p,j+1}\|_\infty &\leqslant c_6 (T^{\nu_{l-2}} + T^{\nu_{l-2} + 1/(l-1) + (1+\nu/(l-1))}) \\ &\leqslant 2c_6 T^{\nu_{l-1}} \end{aligned} \quad (19)$$

for a constant c_6, with $\nu_{l-1} := \nu_{l-2} + 1 + (1+\nu)/(l-1)$. Setting $\nu_1 = 1 + \nu/(l-1)$, we have inductively $\nu_{l-1} = l + \nu - 1/(l-1)$. Since $\|\tau_{l-1,1}\|_\infty \geqslant \|\tau_{i,1}\|_\infty$ for $1 \leqslant i \leqslant l-1$, we have

$$\begin{aligned} \Big\| \sum_{1 \leqslant p \leqslant l} a_{i,p} u_{p,j+1} \Big\|_\infty &\leqslant \Big\| \sum_{1 \leqslant p \leqslant l} (a_{i,p} - (\tau_{i,1}/\tau_{l-1,1}) a_{l-1,p}) u_{p,j+1} \Big\|_\infty \\ &\quad + \|\tau_{i,1}\|_\infty / \|\tau_{l-1,1}\|_\infty \Big\| \sum_{1 \leqslant p \leqslant l} a_{l-1,p} u_{p,j+1} \Big\|_\infty \\ &< c_4/(2T^{1/(l-1)}) \end{aligned}$$

in view of (17) and (18). Taking ${}^t\mathbf{u}_1$ and ${}^t\mathbf{u}_j = (u_{1,j} \ldots u_{l,j})$ from above for $2 \leqslant j \leqslant l-1$, the assertions of theorem 4 follow from the preceding inequality together with (16), (17) and (19), except for $(\tau_{j,p})$ being invertible which we verify in a moment. Indeed, for $1 \leqslant i, j \leqslant l-2$, we have

$$\tau_{i,j+1} = \sum_{1 \leqslant p \leqslant l} a_{i,p} u_{p,j+1} = \sum_{1 \leqslant p \leqslant l} a_{i,p} (v_{p,j} - w_j u_{p,1}) = \tau'_{i,j} + \rho_j \tau_{i,1}$$

where $\rho_j = (1/\tau_{l-1,1}) \sum_{1 \leqslant p \leqslant l} a_{l-1,p} v_{p,j} - w_j$ and further $\tau_{l-1,j+1} = \rho_j \tau_{l-1,1}$. The $l-1$ columns of $(\tau_{i,j})$ thus form the matrix with $(\tau_{1,1} \ldots \tau_{l-1,1})$ as its first row and $(\tau'_{1,j} + \rho_j \tau_{1,1} \ldots \tau'_{l-2,j} + \rho_j \tau_{l-2,1} \; \rho_j \tau_{l-1,1})$ for $1 \leqslant j \leqslant l-2$ as its next $l-2$ rows. The determinant of this matrix is, verified easily to be the same as $\pm \tau_{l-1,1} \times \det(\tau'_{i,j}) \neq 0$.

Remarks. (i) In view of corollary 7D of Schmidt [9], theorem 4 above certainly applies to the system $\vartheta_i x_1 - y_i$, $1 \leqslant i \leqslant l-1$ where $\vartheta_1, \ldots, \vartheta_{l-1}$ are real algebraic numbers such that $1, \vartheta_1, \ldots, \vartheta_{l-1}$ are linearly independent over $\boldsymbol{Q}$.

(ii) For general systems of $l-1$ real linear forms in l variables with rationality rank $l-1$, it does not seem to be possible, in general, to obtain independent integral vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{l-1}$ satisfying the required linear inequalities, along with a *polynomial* bound in T for $\|\mathbf{u}_1\|_\infty, \ldots, \|\mathbf{u}_{l-1}\|_\infty$, unless some assumption on the proper index as in theorem 4 were to be imposed. It is a question of obtaining explicit estimates for the quantity ω occurring in the proof of Kronecker's theorem (theorem 1 on page 458 of Lekkerkerker's interesting book [7]) in terms of the given ε. (For example, in the 2-dimensional case, Minkowski's theorem leads to $\omega \leqslant c/\varepsilon$ for a constant c. For higher dimensions, one can perhaps assert only the *existence* of a *finite* number $\omega = \omega(\varepsilon)$ depending on the given system of linear forms and the given ε. In this sense, one can get a version of theorem 4 with $\|\mathbf{u}_p\|_\infty \leqslant T^{1-1(l-1)} (\omega(T^{-1/(l-1)}))^{l-1}$ for $1 \leqslant p \leqslant l-1$. For a system $\vartheta_i x_1 - x_{i+1}$, $1 \leqslant i \leqslant l-1$ as above and given ε, there can be seen to exist, in the notation of theorem 1, integral $a_1, \ldots, a_l$ not all 0, such that $|\vartheta_i a_1 - a_{i+1}| \leqslant c_1 \varepsilon (1 \leqslant i \leqslant l-1)$ and $|a_1| \leqslant c_2 \varepsilon / h(\varepsilon^{-1})$ with $h(u) := \sup_{|a_1| \leqslant u} (|a_1| / \mathscr{T}(\mathbf{a}'))$. Using the result of Bombieri and Vaaler, one can thus obtain some adelic version of theorem 1 in [7] (p. 458).

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