SOME THEOREMS ON RESIDUES*

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In this paper an attempt is made to throw more light on the following statement of Hardy and Wright in their book *Introduction to the Theory of Numbers*.

"Fermat's and Wilson's theorems show that $2^{p-1}$ and $(p-1)!$ have the residues 1 and $-1$, mod $p$ ($p$ being prime). Little is known about their residues, mod $p^2$, but they can be transformed in interesting ways."

Theorems 1 to 4, under some restrictions, give such transformations for general values of the modulus $m$. Theorems 5 and 6 are equivalents of Gauss' lemma regarding $\left( \frac{n}{p} \right)$. The remaining theorems give results of these types.

**Theorem 1.** Let $m$, $n$ and $d$ be integers such that $m - 1 = nd$. Then

$$\frac{d\phi(m) - 1}{m} \sum_{d} \frac{\left\langle \frac{x}{n} \right\rangle}{x}, \mod m$$

where $\left\langle x \right\rangle$ denotes the smallest integer greater than or equal to $x$, and the summation is for all $x$ less than and prime to $m$.

*Proof.*—Consider the array

1  $d+1$  $\cdots$  $(n-1)d+1$
2  $d+2$  $(n-1)d+2$
3  $\cdots$
\vdots
\vdots
\cdots
\cdots
d  $2d$  $\cdots$  $nd$

from which all numbers not prime to $m$ [i.e., $nd+1$] are removed. Let the product of the elements of the $i$th row be denoted by $\Pi' (i+jd)$. Then

* I am thankful to the referee for some very helpful suggestions.
we have
\[
P_i' (i + jd) = P_i' \{i (nd + 1) - d (ni - j)\}
\]
\[
= \left\{P_i' (-d) (ni - j)\right\} P_i' \left\{1 - \frac{im}{d (ni - j)}\right\}
\]
\[
= \left\{P_i' (-d) (ni - j)\right\} \left\{1 - \frac{im}{d} \sum_{i} \frac{1}{ni - j}\right\}, \text{ mod } m^2
\]
where, \(\sum\) denotes summation over such values of \(j < n\) for which \(ni - j\) is prime to \(m\). Taking the product for \(i = 1, 2, 3, \ldots d\) it is easily seen that
\[
\Pi \equiv (-d)^{\phi(m)} \Pi \Pi \prod_{i=1}^{d} \left\{1 - \frac{im}{d} \sum_{i} \frac{1}{ni - j}\right\}, \text{ mod } m^2
\]
\[
= (-d)^{\phi(m)} \Pi \left\{1 - \frac{m}{d} \sum_{i} \frac{i}{ni - j}\right\}, \text{ mod } m^2,
\]
where \(\Pi\) denotes the product of all numbers less than \(m\) and prime to it.

Therefore
\[
1 = (-d)^{\phi(m)} \left\{1 - \frac{m}{d} \sum_{i} \frac{i}{ni - j}\right\}, \text{ mod } m^2.
\]

On writing \(d^{\phi(m)} = 1 + \lambda m\), this gives
\[
1 \equiv 1 + m \left\{\lambda - \frac{1}{d} \sum_{i} \frac{i}{ni - j}\right\}, \text{ mod } m^2,
\]
so that
\[
\lambda \equiv \frac{1}{d} \sum_{i} \frac{i}{ni - j}, \text{ mod } m
\]

\[
\frac{d^{\phi(m)} - 1}{m} = \frac{1}{d} \sum_{i} \left\{\frac{x}{n}\right\}, \text{ mod } m.
\]

i.e.,

\[
\text{Corollary 1.} - \text{If } m \text{ is prime and } m - 1 = nd, \text{ then}
\]
\[
\frac{d^{m} - d}{m} \equiv \sum_{i=1}^{d} i \left(\frac{1}{ni} + \frac{1}{ni - 1} + \ldots + \frac{1}{ni - n + 1}\right), \text{ mod } m.
\]

\[
\text{Corollary 2.} - \text{Taking } d = 2 \text{ in Theorem 1, which is possible if } m \text{ is odd, we get}
\]
\[
\frac{2^{\phi(m)} - 1}{m} \equiv \frac{1}{2} \left[\sum_{i} \frac{1}{n - j} + 2 \sum_{i} \frac{1}{2n - j}\right], \text{ mod } m.
\]

But \(\sum_{i} \frac{1}{n - j} = - \sum_{i} \frac{1}{n - j}, \text{ mod } m,\)
so that, if $m$ is odd,
\[
\frac{2^{\phi(m)} - 1}{m} = -\frac{1}{2} \sum \frac{1}{n-j}, \mod m,
\]
\[
= -\frac{1}{2} \Sigma \frac{1}{x}, \mod m.
\]
where $\Sigma$ denotes summation over all $x$ less than $\frac{m}{2}$ and prime to $m$.

*Corollary 3.*—Taking $d = 3$ in Theorem 1, which is possible if $m$ is of the form $3n + 1$, we get
\[
\frac{3^{\phi(m)} - 1}{m} = \frac{1}{3} \left[ \sum \frac{1}{n-j} + 2 \sum \frac{1}{2n-j} + 3 \sum \frac{1}{3n-j} \right], \mod m.
\]
It is easily seen that
\[
\sum \frac{1}{3n-j} = -\sum \frac{1}{n-j}, \mod m
\]
and
\[
\sum \frac{1}{2n-j} = 0, \mod m.
\]
Therefore
\[
\frac{3^{\phi(m)} - 1}{m} = -\frac{2}{3} \sum \frac{1}{n-j}, \mod m
\]
\[
= -\frac{2}{3} \Sigma \frac{1}{x}, \mod m
\]
where $m$ is of the form $3n + 1$, and $\Sigma$ denotes summation over all $x < \frac{m}{3}$ and prime to $m$.

*Theorem 2.* Let $m + 1 = nd$, and $[x]$ the greatest integer $\leqslant x$. Then
\[
\frac{d^{\phi(m)} - 1}{m} = \frac{1}{d} \sum \left[ \frac{x}{n} \right], \mod m.
\]
This is proved exactly like Theorem 1, by observing that
\[
\Pi'(i + jd) = \Pi'\{ - i (nd - 1) + d (ni + j), (i = 1, 2, \ldots, d - 1)\}
\]
and
\[
\Pi' (1 + j) d = d^{\phi(m,n)} \times \Pi'(j + 1),
\]
where $\phi(m,n)$ denotes the number of numbers not greater than $n$ and prime to $m$.

*Corollary.*—Take $d = 3$. Then
\[
\frac{3^{\phi(m)} - 1}{m} = \frac{1}{3} \left[ \sum \frac{1}{n+j} + 2 \sum \frac{1}{2n+j} \right], \mod m.
\]

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But
\[ \sum_{j} \frac{1}{n+j} \equiv 0, \mod m \]
and
\[ \sum_{j} \frac{1}{2n+j} \equiv -\sum_{j} \frac{1}{n-j-1}, \mod m. \]
Therefore, if \( m \) is of the form \( 3n - 1 \),
\[ \frac{3^{\phi(m)} - 1}{m} = -\frac{2}{3} \sum_{j} \frac{1}{n-j-1}, \mod m \]
\[ = -\frac{2}{3} \sum_{x} 1, \mod m, \]
where \( \Sigma \) denotes summation over all \( x < \frac{m}{3} \) and prime to \( m \).

We therefore see that Corollary 3 of Theorem 1 is true for all values of \( m \) prime to 3.

**Theorem 3.** If \( a \) runs through all numbers less than \( m \) and prime to it and \( \beta \) through all numbers less than \( \frac{m}{2} \) and prime to \( m \), and \( m \) is odd, then
\[ \Pi a = (-1)^{\phi(m)} (a \beta)^{2} 2^{\phi(m)}, \mod m^{2}. \]

This can be proved in the same way as theorem 133 of Hardy and Wright's book, making use of Corollary 2 of Theorem 1 in the place of the more particular result used there.

**Theorem 4.** If \( a, \beta, \gamma \) run through all numbers prime \( m \) and are such that \( a < m, \beta < \frac{2m}{3} \) and \( \gamma < \frac{m}{3} \) and \( \phi(m, n) \) denotes the number of numbers not greater than \( n \) and prime to \( m \), and \( m \) is prime to 3, then
\[ \Pi a = (-1)^{\phi(m, m/3)} \frac{1}{2} \Pi \beta \cdot \Pi \gamma \{3^{\phi(m)/2} - 1\}, \mod m^{2}. \]

First, let us assume that \( m = 3n + 1 \).

Then, if \( \Pi' \) denotes the product for numbers prime to \( m \), and \( j < n \),
\[ \Pi' (1 + 3j) = \Pi' \{ (3n + 1) - 3 (n-j) \} \]
\[ = \left\{ \Pi' (3-n-j) \right\} \left\{ 1 - \frac{m}{3} \sum_{j} \frac{1}{n-j} \right\}, \mod m \]
and
\[ \Pi' (2 + 3j) = \Pi' \{ (m + 3 (n+j+1)) \} \]
\[ = \left\{ \Pi' 3 (n+j+1) \right\} \left\{ 1 - \frac{m}{3} \sum_{j} \frac{1}{n+j+1} \right\}, \mod m \].
Therefore
\[
\prod_{j} (1 + 3j) \prod_{j} (2 + 3j) \prod_{j} (3 + 3j) (-1)^{\phi(m, n)} 3^{\phi(m)} \prod_{j} (n - j) \prod_{j} (n - j + 1) \times
\]
\[
\Pi'_{j} (1 + j) \left\{ 1 - \frac{m}{3} \left( \sum'_{j} \frac{1}{n-j} + \sum'_{j} \frac{1}{n-j+1} \right) \right\}, \mod m. \]

But \( \sum'_{j} \frac{1}{n-j+1} \equiv 0 \mod m, \)

and hence we get the required result for \( m - 3n - 1 \) by making use of Corollary 3 of Theorem 1 and the fact that \( 3^{\phi(m)} \equiv 1 \mod m. \)

The case \( m = 3n - 1 \) can be treated as above, using corollary of Theorem 2.

**Theorem 5.** If \( m = 2nd \), then \( d^{\phi(2n)} \equiv (-1)^{v} \mod m, \)

where
\[
v = \begin{cases}
\sum_{i=1}^{\frac{d}{2}} \left\{ \phi\left( m, \frac{2i}{2} \right) \phi\left( m, \frac{1}{2} \right) \right\} & \text{if } d \text{ is even} \\
\sum_{i=1}^{d} \left\{ \phi\left( m, 2i \right) \phi\left( m, 2(i-1) \right) \right\} & \text{if } d \text{ is odd.}
\end{cases}
\]

Alternately, \( v \) may be given by the relations
\[
v = \begin{cases}
\sum_{i=1}^{d/2} \left\{ \phi\left( m, 2ni \right) \phi\left( m, 2(i-1)n \right) \right\} & \text{if } d \text{ is even} \\
\sum_{i=1}^{d-1/2} \left\{ \phi\left( m, 2ni \right) \phi\left( m, 2(i-1)n \right) \right\} & \text{if } d \text{ is odd.}
\end{cases}
\]

**Proof.** Consider the array
\[
\begin{array}{cccccccc}
1 & d+1 & 2d+1 & \ldots & (n-1)d+1 & \\
2 & d+2 & 2d+2 & \ldots & (n-1)d+2 & \\
3 & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\end{array}
\]

from which all numbers not prime to \( m \) are removed. If we denote the product of the elements of the \( i \)-th row by \( \Pi'(i+jd) \) it is clear that \( j \) runs through all values from 0 to \( n-1 \) such that \( i+jd \) is prime to \( m \). We have

\[
\Pi'(i+jd) \quad \Pi'(im-d(2ni-j))
\]

\[
\Pi'(i+(-d)(2ni-j)), \mod m.
\]
Similarly
\[ \Pi' \{d - i + jd\} = \Pi' \{d(2in + j + 1) - im\} \]
\[ = \Pi' \{d(2ni + j + 1)\}, \text{ mod } m. \]

Therefore
\[ \prod_{i=0}^{d-1} \Pi' (i + jd) \prod_{i=0}^{d-1} \Pi' (d - i + jd) \]
\[ = \prod_{i=1}^{d/2} \Pi' (-d)(2ni - j) \prod_{i=0}^{d-1} \Pi' (d)(2ni + j + 1), \text{ mod } m. \]

Taking \( k = d/2 \) or \( (d - 1)/2 \) according as \( d \) is even or odd we get
\[ \Pi\alpha = (-1)^r d^{\phi(m)} \Pi\alpha, \text{ mod } m, \]
where \( \alpha \) runs through the numbers prime to \( m \) and not greater than \( m/2 \).

Dividing both sides by \( \Pi\alpha \) we get the required result.

In a similar manner we may prove

**Theorem 6.** If \( m + 1 = 2nd \),
\[ d^{\phi(m)} \equiv (-1)^\mu \text{ mod } m, \]
where \( \mu = \left\{ \sum_{i=1}^{\phi(m)} \phi(m, id - 1) - \phi\left(m, \frac{2i-1}{2} d - 1\right) \right\}, \text{ if } d \text{ is even} \]
\[ \left\{ \sum_{i=2}^{\phi(m)} \phi(m, id - 1) - \phi\left(m, \frac{2i-1}{2} d - 1\right) \right\}, \text{ if } d \text{ is odd.} \]

As corollaries of Theorems 5 and 6 we get,

**Theorem 7.** If \( m = 2nd + 1 \) be prime, then to the modulus \( m \)
\[ \frac{m-1}{2} \equiv (-1)^r \frac{m-1}{4}, \text{ if } d \text{ is even} \]
\[ \equiv (-1)^{\mu(d-1)/2}, \text{ if } d \text{ is odd}; \]
and

**Theorem 8.** If \( m = 2nd - 1 \) be prime, then to the modulus \( m \)
\[ \frac{m-1}{2} \equiv (-1)^r \frac{m+1}{4}, \text{ if } d \text{ is even} \]
\[ \equiv (-1)^{\mu(d-1)/2}, \text{ if } d \text{ is odd}. \]

By combining the methods of Theorems 1 and 5 we get

**Theorem 9.** If residues are taken to the modulus \( m = 2nd + 1 \), and
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\[ \langle x \rangle \text{ and } \lfloor x \rfloor \text{ are as in Theorems 1 and 2.} \]

\[ (-1)^\nu \frac{d^{\phi(m)}}{m} - 1 = \sum_d \frac{\{x\}}{2n} \text{ mod } m, \]

where the summation is for all \( x < \frac{m}{2} \) and prime to \( m \), and \( \{x\} \) denotes \( \left\langle \frac{x}{2n} \right\rangle \) or \( \left\lfloor \frac{x}{2n} \right\rfloor \) according as \( \left\langle \frac{x}{n} \right\rangle \) is even or odd, and \( \nu \) has the same meaning as in Theorem 5;

and

**THEOREM 10.** If residues are taken to the modulus \( m \cdot 2nd - 1 \), then

\[ (-1)^\mu \frac{d^{\phi(m)}}{m} - 1 = \sum_d \frac{\left\lceil \frac{x}{2n} \right\rceil}{x} \text{ mod } m, \]

where the summation is for all \( x < \frac{m}{2} \) and prime to \( m \), and \( \left\lceil \frac{x}{2n} \right\rceil \) denotes \( \left\lceil \frac{x}{2n} \right\rceil \) or \( \left\lceil \frac{x}{2n} \right\rceil \) according as \( \left\lceil \frac{x}{n} \right\rceil \) is even or odd, and \( \mu \) has the same meaning as in Theorem 6.