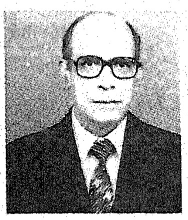
Nonlinearity, Conservation Law and Shocks

2. Stability Consideration and Examples

Phoolan Prasad



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The previous article of this series was:
1. Genuine Nonlinearity and Discontinuous Solutions, April 1997.

In part 1 of this series we explained the concept of genuine nonlinearity, which is responsible for the appearance of discontinuities in a solution which was initially smooth. To include discontinuities in the solution, it became necessary to consider the governing equation in the form of a conservation law. In this part we first discuss an example of a continuous solution satisfying discontinuous initial data. Then we use the stability consideration to fix a unique solution of the conservation law. In the end, we present three examples which show that genuine nonlinearity significantly changes the evolution of the shape of a pulse.

Continuous Solution with Discontinuous Initial Data

In a linear system, discontinuous initial data always leads to a discontinuous solution. For example, consider an initial data

$$u(x,0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \qquad (13)$$

As explained in part I, we consider only a class of piecewise smooth functions with jump discontinuities. For functions of this class, it is not necessary to specify the value of the function at a point of discontinuity.

The solution satisfying the rule (6)[1] is

$$u(x,t) = \begin{cases} 0, & x < ct \\ 1, & x > ct \end{cases} \qquad (14)$$

in which the discontinuity also propagates with the same velocity c.

[1] Refer to part 1 of this series for equations (1) to (12).

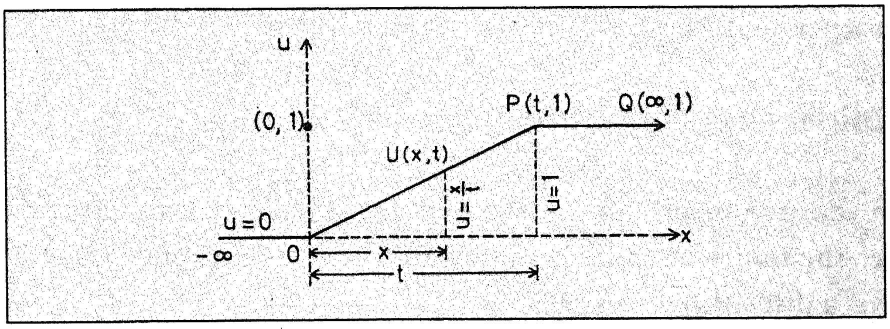


Figure 6a At any time t, the function u is represented by straight line segments $-\infty$ to O, O to P and P to Q.

However, an equation with genuine nonlinearity may give a continuous solution at $t > 0$ even if the initial data is discontinuous. Consider a function $u(x,t)$ which is represented graphically in *Figure 6a*. Let U be a point on the line segment OP having coordinates (x, u). From the properties of similar triangles $u = \frac{x}{t}$. This value of u remains constant when x and t vary in such a way that $\frac{x}{t}$ = constant and this constant is equal to u itself. Thus the point U moving with a velocity u has a constant value u. Therefore, the points on the segment OP satisfy the rule (7). We can easily verify that the rule (7) is true also for the line segments $-\infty$ to O and P to Q. Hence u represented by *Figure 6a* is a solution as per rule (7a). As $t \to 0$, the point P approaches the point (0,1). Thus u is a continuous solution satisfying the rule (7) with discontinuous initial condition (13) and hence from the theorem 2 it is also a solution satisfying the conservation law (10) with initial condition (13). A solution of the type $\frac{x}{t}$, in which the source of all points of the pulse is a single point $x=0$ at $t=0$ is called a *centered wave*. We can also verify that the function

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 < x < t \\ 1, & x > t \end{cases} \qquad (15)$$

has derivatives

$$(u_x, u_t) = \begin{cases} (0,0), & x < 0 \\ (\frac{1}{t}, -\frac{x}{t^2}), & 0 < x < t \\ (0,0), & x > t \end{cases}$$

which satisfy (7b). We also depict the solution in the plane of

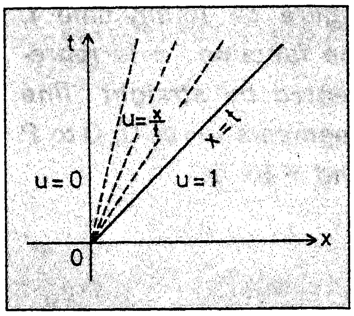


Figure 6b Solution (15) represented in the three different domains in the upper half of the (x, t)-plane.

independent variables, in *Figure 6b.*

Uniqueness and Entropy Condition

In addition to the continuous solution represented in *Figure 6* i.e. the solution (15), the initial value problem (10) and (13) also has a discontinuous solution

$$u(x,t) = \begin{cases} 0, & x < \frac{1}{2}t \\ 1, & x > \frac{1}{2}t \end{cases} \tag{16a}$$

The discontinuity at $x = \frac{1}{2}t$ moves with velocity $S = \frac{1}{2}$ and separates a state $u_- = 0$ from a state $u_+ = 1$ so that (10b) is satisfied. It is represented graphically in *Figure 7*. Consider the points (with $u=0$) on the left of the discontinuity at $x=0$ in the initial data. Since these points move with zero velocity, they will influence the solution at any time only upto the point O. Points on the right of $x=0$ in the initial data, move with the velocity 1 and hence give the solution beyond thepoint $R\ (t,1)$ on $u=1$. Therefore, a part of the solution (16) between O and R is not controlled by the initial data. At any time, the point just on the left of the discontinuity moving with zero velocity is left behind by the discontinuity and a point just ahead moving with velocity 1 leaves the discontinuity behind. The failure of the initial data to control the solution between O and R implies that we may be able to construct not only two solutions (15) and (16) but probably infinitely many more. This in fact is the case, but here we just give one more solution.

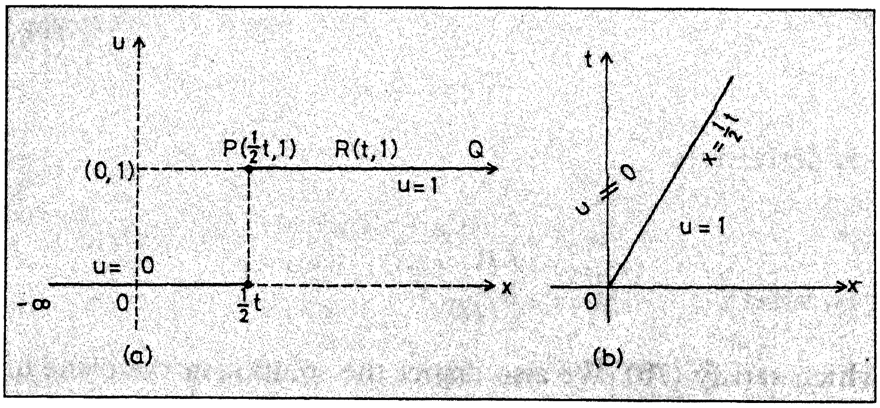


Figure 7(a) Solution (16a) at a time t>0 in (x,u)-plane. (b) Solution (16a) in (x,t)-plane.

$$u(x,t) = \begin{cases} 0, & x < \frac{1}{4}t \\ \frac{1}{2}, & \frac{1}{4}t < x < \frac{3}{4}t \\ 1, & x > \frac{3}{4}t \end{cases} \qquad (16b)$$

It is instructive to draw the graph of this solution at a fixed time t in the (x, u)-plane. The initial discontinuity now breaks into two discontinuities (one at $x = \frac{1}{4}t$ and another at $x = \frac{3}{4}t$) each satisfying the jump relation (10b) derived from the conservation law (10a). In both (16a) and (16b), the initial data has no control on the solution between $O(0,0)$ and $R(t,1)$. A situation like this was first noticed in gas dynamics and Lord Rayleigh in 1910 found that discontinuities like this led to a decrease in the entropy of the system, which was not acceptable from the second law of thermodynamics. It was, of course, known that the entropy of the system remained constant in the continuous solution like (15).

Consider now another discontinuous initial data

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases} \qquad (17a)$$

In this case, the state $u=1$ on the left of the discontinuity at $x=0$ starts moving with velocity 1 and immediately begins to overtake the state $u=0$ on the right of the discontinuity. Therefore, no continuous state for $t>0$ will ever satisfy the rule (7) and the initial data (17a). Now, we must look for a discontinuous solution satisfying (10). We can easily see that a solution of the initial value problem (10) and (17a) is

$$u(x,t) = \begin{cases} 1, & x < \frac{1}{2}t \\ 0, & x > \frac{1}{2}t \end{cases} \qquad (17b)$$

which is shown in *Figure 8*. The discontinuity in *Figure 8a* moving with the velocity $\frac{1}{2}$ is overtaken from behind by the continuous part $u=1$ of the pulse moving with velocity 1 and it overtakes the continuous part $u=0$ ahead of it. This will always happen for any discontinuity if

$$u_+ < S < u_- . \qquad (18)$$

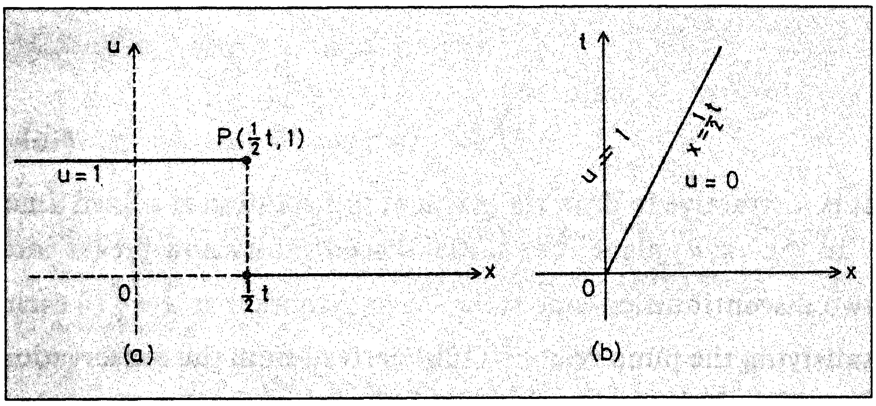


Figure 8(a) Solution (17b) at a time t>0 in (x,u)-plane. (b) Solution (17b) in (x,t)-plane.

In this case the initial data completely determines the solution for all time. The solution so obtained is unique. Appearance of a discontinuity in a solution saves us from a difficult situation when the initial state leads to a multi-valued state under the local law (7) of propagation. A physical interpretation of the condition (18) for gas dynamic waves and its relation to an entropy condition is mentioned later in this section.

Let us consider a *small* perturbation of the solution (15) such that the perturbation vanishes outside a closed interval on the x-axis. Such a perturbation is shown in *Figure 9*.

Since the velocity of a point on the pulse is equal to its amplitude, a point on the perturbation moves with a velocity which is a small addition to the velocity of the corresponding point in the solution (15) with the same x. Hence, a small perturbation of the solution (15) remains small and moves away from the corresponding point on the solution only by a small distance in

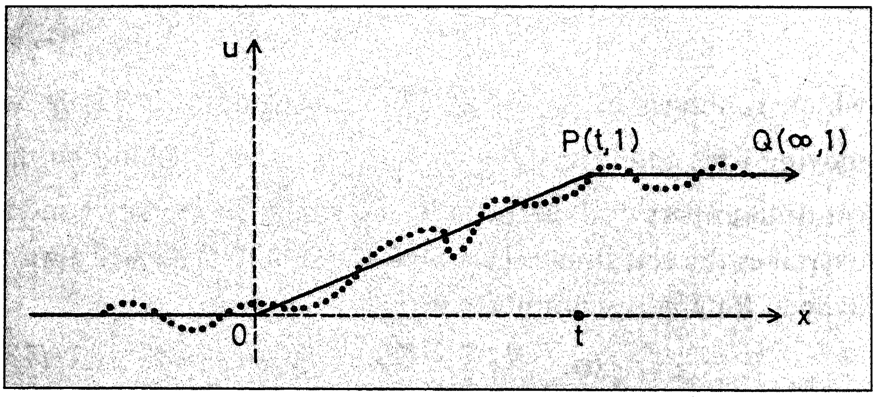


Figure 9 Perturbation of the continuous solution (15) has been shown by dotted lines.

a finite time. This is true even if a discontinuity appears in the perturbation. Thus, the solution (15) is stable with respect to small perturbations which are nonzero only over finite intervals on the x-axis. The discontinuous solution (17b) is also stable with respect to such perturbations. A positive or negative perturbation δu created to the left of the initial discontinuity at $x=0$ moves with a velocity $1+\delta u > \frac{1}{2}$ since δu is small and hence eventually interacts with the discontinuity at $x = \frac{1}{2}t$. During this interaction the velocity of the discontinuity also changes by a small quantity and hence the discontinuity is displaced from the position $\frac{1}{2}t$ by a small distance. A small perturbation created on the right of $x=0$ moves with a velocity $\delta u < \frac{1}{2}$ and hence is overtaken by the discontinuity. During the interaction, velocity of the discontinuity changes by a small quantity. We note that in all cases, a small perturbation of the initial data leads to a small change in the solution. The situation is different for a solution having a discontinuity which does not satisfy (18). For example, the solution (16a) is not stable with respect to small perturbations. This is obvious when we note that the solutions (16a) and (16b) arise from the same initial data (13) i.e. there is another solution (16b) which is not near (16a) even if a perturbation in the initial data is not introduced. As it happens at $x=0$, $t=0$, the solution (16a) can break up at any future time into solutions like (16b) i.e. the solution (16a) is unstable. In nature, an unstable solution does not represent a physically realizable state of a system. Hence we must reject such solutions. Thus we conclude that only discontinuities which satisfy (18) are acceptable in a solution.

The entropy of an element of a gas increases as it crosses a shock.

We define a discontinuity satisfying (18) to be a *shock*.

In gas dynamics, the condition (18) means that the shock velocity is supersonic relative to the state ahead of the shock (i.e. $S > u_+$) and it is subsonic relative to the state behind (i.e. $S < u$). It has been shown in standard books on gas dynamics that the entropy of an element of a gas increases as it crosses a shock.

A shock wave is destructive, but it is also useful. For example in medicine it can be used for breaking and removing kidney stones.

Hence in mathematics the abstract condition (18) is called an *entropy condition*.

We have discussed in this section two types of admissible solutions starting from a discontinuity at $x=0$, $t=0$: a continuous solution (15) in $0<x<t$, called *centered wave* and a *shock* at $x=\frac{1}{2}t$ as in (17b). Anyone of these is complementary to the other and the two together can solve any problem in which the initial data consists of two constant states separated by a single point of discontinuity at $x=0$. This initial value problem is called a *Riemann problem*. When initially $u_- < u_+$, we get a centered wave and when $u_- > u_+$ we get a shock wave. We notice that the solution of a Riemann problem is very easy in a genuinely nonlinear system expressed by a single conservation law. It becomes difficult when the number of equations in a system of conservation laws is more than one but its solution is of vital importance for a mathematical development of the subject and for numerical computation of solutions.

It has been observed in a gas that a shock wave carries a jump in the pressure, density and fluid velocity. Pressure and density always rise immediately behind a shock. Hence if a shock wave hits any object, it gives an impact which can be very large for a strong shock. Thus a shock wave is destructive, but it is also useful. For example, in medicine it can be used for breaking and removing kidney stones.

An extremely elegant theory on the stability of steady states of a quite general system near sonic type of barriers was proposed by two Russian scientists Kulikovskii and Slobodkina (1968). This theory, important from the point of view of applications, involves study of an equation obtained by replacing the right hand side of (7b) by a linear function of x and u.

Examples of Three Solutions

We consider here examples in which genuine nonlinearity significantly changes the evolution of wave profiles.

Example 1 Consider an initial data

$$u(x,0)=\begin{cases}1, & x<-\frac{1}{2}\\ \frac{1}{2}-x, & -\frac{1}{2}<x<\frac{1}{2}\\ 0, & x>\frac{1}{2}\end{cases}. \qquad (19)$$

The solution has different representations in different time intervals.

(i) When $0 \le t < 1$, the solution is represented by

$$u(x,t)=\begin{cases}1, & x<-\frac{1}{2}+t\\ \frac{(1/2)-x}{1-t}, & -\frac{1}{2}+t<x<\frac{1}{2}\\ 0, & x>\frac{1}{2}\end{cases} \qquad (20)$$

which is shown in *Figure 10.* The nonconstant part of u is linear in x and as t increases it becomes steeper. As $t \to 1-0$, this part develops into a discontinuity of amplitude 1 at $x = \frac{1}{2}$.

(ii) For $t \ge 1$ the solution has a shock moving along the path $x = X(t) \equiv \frac{1}{2}t$. Thus, for $t > 1$, the solution of this problem is same as the solution (17).

We observe a property of genuine nonlinearity, which is very important from the point of view of physical interpretation. Two initial data (17a) and (19) lead to the same unique solution for $t > 1$. Thus a phenomenon represented by a discontinuous solution of a system having genuine nonlinearity is irreversible in time, since the past cannot be traced back uniquely. If there is a shock in the solution at $x = X(t)$ which separates two constant regions, then the initial state between the points $X(t) - tu_-(t)$ and

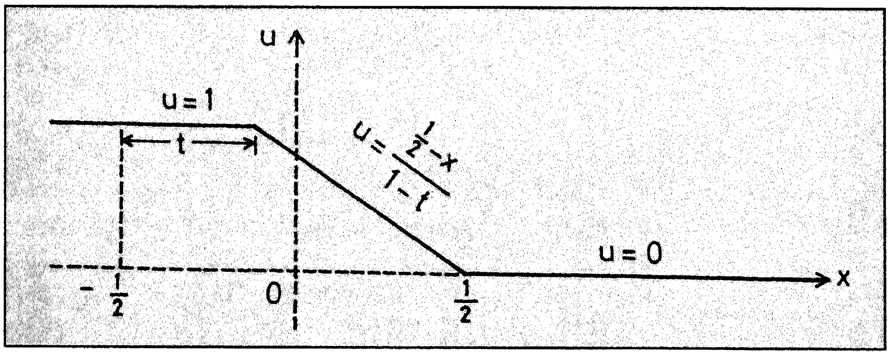


***Figure 10** The solution (20) remains continuous upto t=1 but the middle part becomes steeper as t increases.*

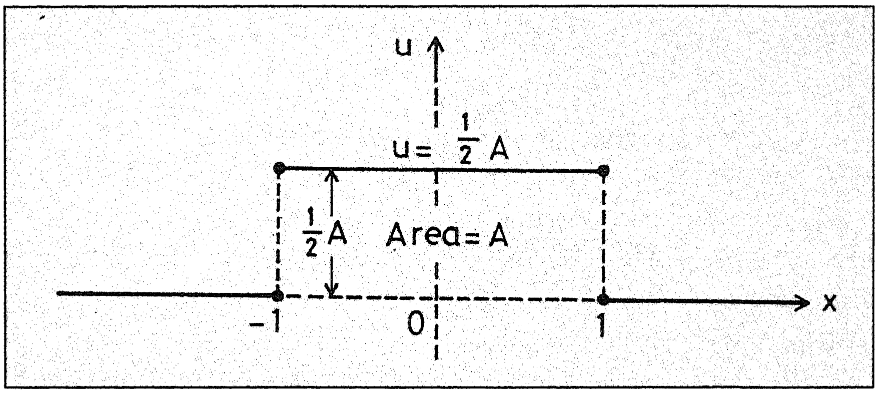


***Figure 11** Initial pulse (21) is represented in (x,u)-plane. The pulse occupies an area A above the x-axis.*

$X(t) + tu_+(t)$ is irreversibly lost due to interaction with the shock.

Example 2 Consider an initial data

$$u(x,0) = \begin{cases} \frac{1}{2}A, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \qquad (21)$$

where we take $A > 0$. The data is represented in the *Figure 11*. According to the linear law of evolution (6), the rectangular pulse as a whole propagates with a constant velocity without a change in shape and amplitude.

If the evolution of the initial data is described according to the conservation law (10), then the solution has two distinct time intervals describing two types of states:

(i) $0<t<\frac{8}{A}$: (*Figure 12*). In this interval a shock starts from the point $x=1$ and moves with a uniform velocity $\frac{1}{4}$. At the time

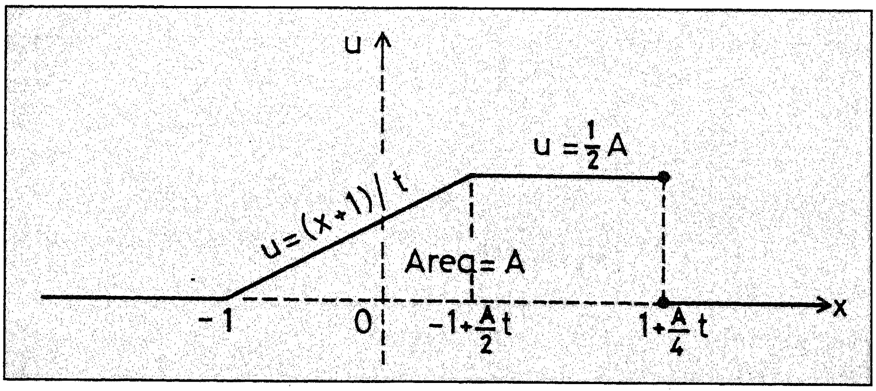


***Figure 12** Graph of the solution with initial value (21) valid in the time interval $0< t < \frac{8}{A}$.*

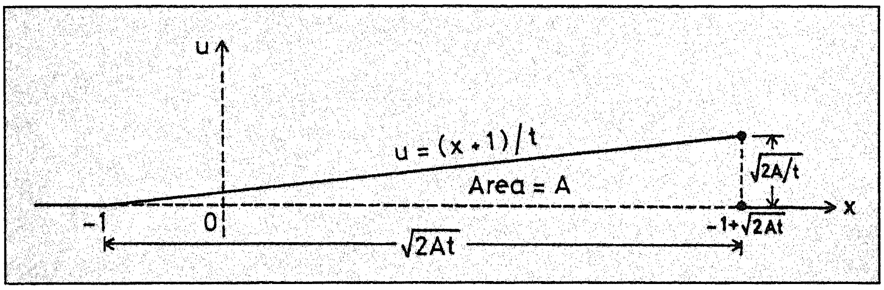


Figure 13 Graph of the solution with initial condition (21) valid for $t > \frac{8}{A}$.

$t = \frac{8}{A}$, the shock reaches the point $x=3$. There is a centered wave $u = \frac{x+1}{t}$ between $-1 < x < -1 + \frac{A}{2}\, t$ all points of which originate from $x=-1$ at $t=0$. In the interval $-1 + \frac{A}{2}\, t < x < 1 + \frac{A}{4} t$, there is a constant state $u = \frac{A}{2}$. At the time $t = \frac{8}{A}$, the centered wave overtakes the shock at $x=3$ and the pulse attains a triangular shape.

(ii) $t > \frac{8}{A}$: (*Figure 13*) During this period, the centered wave interacts with the shock.

Since the shock velocity is given by $S = \frac{1}{2}(u_- + u_+) = \frac{1}{2}\, u_-$ and u_- is given by the centered wave, the shock path $x=X(t)$ is obtained by solving an ordinary differential equation with an initial condition:

$$\frac{dX}{dt} = \frac{X+1}{2t}, \qquad X\left(t = \frac{8}{A}\right) = 3 \tag{22}$$

which gives

$$X(t) = -1 + \sqrt{2At} \ . \tag{23}$$

At $x = X(t)$, the amplitude of the pulse (just behind the shock) is given by

$$u = \left.\frac{x+1}{t}\right|_{x=X(t)} \quad \sqrt{\frac{2A}{t}} \ . \tag{24}$$

The pulse now takes a triangular shape, the base of which spreads over a distance $\sqrt{2At}$ and whose height is $\sqrt{\frac{2A}{t}}$, the total area of the pulse being A.

We can easily verify that the total area of the pulse remains A even during the time period $0<t< \frac{8}{A}$. As mentioned in part 1, this is a general property of the original conservation law (8). If the initial data is such that $\lim_{x \to -\infty} F(u(x,0))$ and $\lim_{x \to \infty} F(u(x,0))$ are both equal to zero (which is the case when $F(0)=0$ and u vanishes initially outside a closed boundedinterval on the x -axis), then the total quantity u i.e. the area between the solution curve and x-axis remains constant. We make another important remark. The genuine nonlinearity present in conservation laws produces dissipation of energy through shocks. Hence, all kinetic energy is ultimately converted into heat and though the pulse continues to spread over a larger region (as it should happen when dissipation is present), it ultimately dies with its amplitude decaying as inversely proportional to the square root of time (a result true for any pulse which is initially nonzero only in a bounded interval).

Example 3. An equation, governing the propagation of small perturbations trapped at a point on the sonic line of a steady gas flow, is given by (first derived by Prasad, 1973)

$$u_t + (u - Kx)\, u_x = Ku \; . \tag{25}$$

The dependent and independent variables have been properly scaled. The constant K is proportional to the deceleration of the fluid element at the sonic point in the steady flow. When the fluid is passing from a supersonic state to a subsonic state, $K>0$.

Had the genuine nonlinearity not been present, the approximate equation would have been

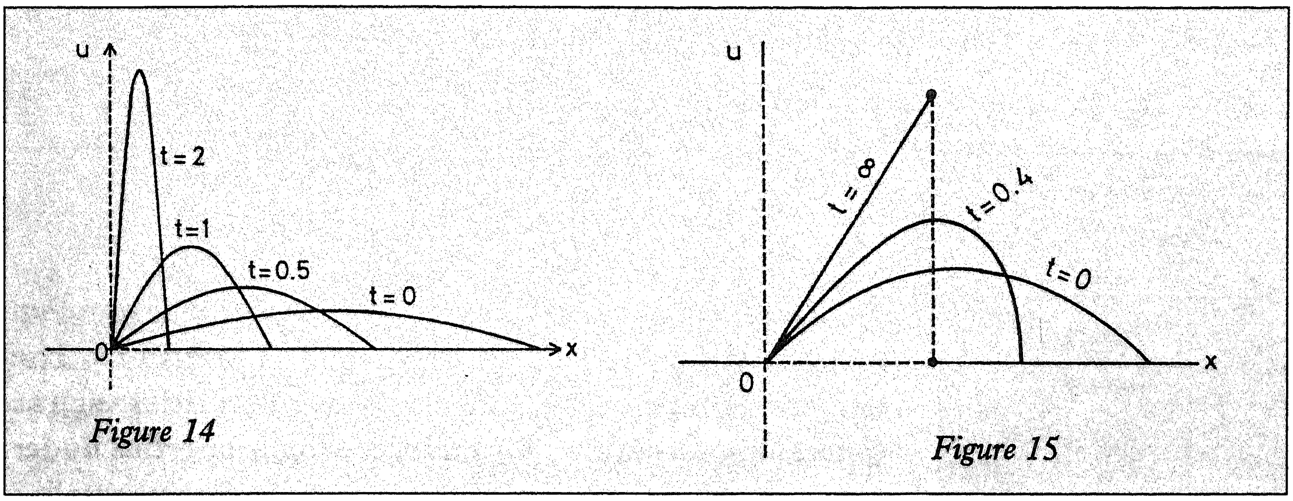
$$u_t - Kx\, u_x = Ku. \tag{26}$$

We can easily verify that the solution of the initial value problem

$$u\,(x,0) = u_0(x) \tag{27}$$

for (26) is

$$u = u_0(\,xe^{Kt}\,)\, e^{Kt} \; . \tag{28}$$



Figures 14 and 15. (left) Solution of (26) and (27) with K=1, $u_0(x)$ = sin x, 0< x < π and zero elsewhere . The amplitude of the pulse tends to infinity as $t \to \infty$. (right) Shape of the successive positions of a pulse in a transonic region. The pulse of a positive area attains a triangular form as $t \to \infty$ and gets trapped in the subsonic region (Prasad, 1973). Here K > 0.

The solution shows that for $K>0$ the amplitude u would tend to infinity as t increases to infinity. However, the perturbation u will get concentrated near the sonic point $x=0$ as shown in *Figure 14*.

However, genuine nonlinearity is always present in an ideal gas. The conservation form of the equation (25) brings in shocks which cut off the growing part of the amplitude as shown in *Figure 15*.

Acknowledgements

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Suggested Reading

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